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ON A CLASS OF REPRESENTATION-FINITE OF-3 ALGEBRAS

(Dedicated to Professor Hisao Tominaga on his 60th birthday)

By

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Let k be an algebraically closed field, and A be a finite dimensional k-algebra. Then $DA=\mathrm{Hom}_k(A, k)$ has a canonical A-A-bimodule structure. In Hughes-Waschbüsch's work [13] on trivial extensions of tilted algebras, an infinite matrix algebra:

$$\hat{A}=\begin{pmatrix} \ddots & & & \\ & A & & 0 \\ & DA & A & \\ & & DA & A \\ 0 & & & \ddots \end{pmatrix}$$

plays an important role. So, it seems interesting to consider a 'finite' dimensional k-algebra:

$$A^{(t)}=\begin{pmatrix} A_0 & & & & \\ D_1 & A_1 & & 0 & \\ & D_2 & A_2 & & \\ 0 & & \ddots & \ddots & \\ & & & D_t & A_t \end{pmatrix} \quad \text{for } t\geqq 1$$

(where $A_i=A$, $D_i={}_ADA_A$ for all i) with the ordinary matrix addition and the multiplication induced from the canonical A-A-bimodule structure of DA and the map $DA\otimes_A DA\rightarrow 0$.

When A is hereditary, $A^{(t)}$ is representation-finite for some t iff so is $A^{(t)}$ for all t, iff so is the trivial extension $T(A)=A\ltimes DA$ of A and iff A is of Dynkin type. However, these equivalences no longer hold for iterated tilted algebras. Actually, in Assem [1], $A^{(1)}$ was studied in the case where A is an (iterated) tilted algebra of Dynkin type, and in particular, it was proved that, if A is iterated tilted of Dynkin type, then $A^{(1)}$ is simply connected, and consequently, so is $A^{(t)}$ for any $t\geqq 1$.

If the trivial extension $T(A)$ is representation-finite, then, by Yamagata [22], the (ordinary) quiver Q_A of A contains no oriented cycle, and hence A must be a homomorphic image of a hereditary algebra. However, an algebra A with an

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oriented cycle or a loop might have a representation-finite $A^{(1)}$. For instance, let A be given by the quiver:

$$\begin{tikzcd}
& & \circ \arrow[drr, "\beta"] & & \\
\circ \arrow[urr, "\alpha"] & & & & \circ \arrow[dl] \\
& \circ \arrow[ul] & & \circ \arrow[ll] &
\end{tikzcd} \quad \text{with } \beta\alpha=0,$$

then $A^{(1)}$ is representation-finite. Thus, a different class of algebras of finite type from a trivial extension case is included in our consideration.

Finally, what we'd like to point out is that $A^{(t)}$ is a so-called QF-3 algebra, namely the injective hull $\mathrm{E}(A^{(t)})$ of $A^{(t)}$ is projective. Moreover, $A^{(1)}$ has the (left and right) maximal quotient ring:

$$Q(A^{(1)})=\begin{bmatrix} A & \mathrm{Hom}_A(DA,\ A) \\ DA & A \end{bmatrix}$$

which is also QF-3 and, if the Auslander algebra of A is representation-finite, then so is $Q(A^{(1)})$.

In § 1, we shall give bounds for the global and the dominant dimensions of $A^{(t)}$ in terms of the global dimension of A:

$1 \leqq \mathrm{dom.\ dim}\ A^{(t)} \leqq \mathrm{gl.\ dim}\ A+1 \leqq \mathrm{gl.\ dim}\ A^{(t)} \leqq (t+1)\ \mathrm{gl.\ dim}\ A+t.$

In § 2, we assume that the algebra A has square-zero radical and give an effective criterion to determine when $A^{(t)}$ is representation-finite.

In § 3, we consider the case where A is a Nakayama algebra, and, using the lists of [12], [4], we give a criterion for deciding when $A^{(1)}$ is representation-finite. As a particular case, we shall determine the representation type of $A^{(1)}$ when A belongs to a class of Nakayama algebras studied by Marmaridis in [15]. We shall use most of the notations of [1], especially, if the ordinary quiver Q_A of A has as vertices $1, 2, \cdots, n$ corresponding to an admissible ordered complete set of orthogonal primitive idempotents $\{e_1, e_2, \cdots, e_n\}$ of A, then the ordinary quiver of $A^{(1)}$ contains as full connected subquivers two copies of Q_A, denoted respectively by Q_A and Q'_A. Also, there is an arrow $i' \to j$ whenever $\mathrm{rad}\ (e'_i A^{(1)} e_j)/\mathrm{rad}^2\ (e'_i A^{(1)} e_j) \cong D(e_j (\mathrm{rad}\ A) e_i) \neq 0$ (here i' and e'_j denote respectively the vertex of Q'_A and the corresponding idempotent associated to the vertex i of Q_A).

§ 1. Homological dimensions of the algebra $A^{(t)}$:

PROPOSITION (1.1)

$\mathrm{gl.\ dim}\ A+t \leqq \mathrm{gl.\ dim}\ A^{(t)} \leqq (t+1)\mathrm{gl.\ dim}\ A+t.$

Thus, the global dimension of $A^{(t)}$ *is finite if and only if the global dimension of* A *is finite.*

PROOF. Since

$$A^{(t)}=\begin{bmatrix} A^{(t-1)} & 0 \\ M^{(t)} & A \end{bmatrix} \quad \text{for } t\geqq 1$$

(where $M^{(t)}=[0, \cdots, 0, DA]$ is an $A-A^{(t-1)}$-bimodule and $A^{(0)}=A$), it follows from [17, Corollary (8.4′)] that:

$$\begin{aligned} &\max\{\mathrm{gl.dim}\ A^{(t-1)},\ \mathrm{gl.dim}\ A,\ \max(\mathrm{pd}_A DA,\ \mathrm{pd}(DA)_A)+1\} \\ &\leqq \mathrm{gl.dim}\ A^{(t)} \\ &\leqq \max\{\mathrm{gl.dim}\ A^{(t-1)},\ \mathrm{gl.dim}\ A,\ \min(\mathrm{gl.dim}\ A^{(t-1)}+\mathrm{pd}(DA)_A, \\ &\qquad \mathrm{gl.dim}\ A+\mathrm{pd}\ {}_A DA)+1\} \end{aligned}$$

for $t\geqq 1$. Thus, if $\mathrm{gl.dim}\ A=\infty$, then $\mathrm{gl.dim}\ A^{(t)}=\infty$ and the result holds. Assume now that $\mathrm{gl.dim}\ A:=d<\infty$. We claim that:

$$\mathrm{pd}\ {}_A DA=\mathrm{pd}\ (DA)_A=d.$$

Since A is of finite global dimension d, there exists a module M_A such that $\mathrm{Ext}_A^d(M, -)\neq 0$ but $\mathrm{Ext}_A^{d+1}(M, -)=0$. Let X_A be an A-module such that $\mathrm{Ext}_A^d(M, X)\neq 0$, and consider a short exact sequence of A-modules:

$$0\to K_A\to L_A\to X_A\to 0$$

with L_A free. It induces an exact sequence:

$$\mathrm{Ext}_A^d(M, L)\to \mathrm{Ext}_A^d(M, X)\to \mathrm{Ext}_A^{d+1}(M, K)=0.$$

Since $\mathrm{Ext}_A^d(M, X)\neq 0$, we have $\mathrm{Ext}_A^d(M, L)\neq 0$, and so $\mathrm{id}\ L_A\geqq d$. Since L_A is a free module, $\mathrm{id}\ L_A=\mathrm{id}\ A_A$. Hence $\mathrm{id}\ A_A\geqq d$, and so $\mathrm{id}\ A_A=d$. However, by [14, Proposition (1)],

$$\mathrm{id}\ A_A=\sup\{\mathrm{pd}({}_A I)\mid {}_A I \text{ injective}\}=\mathrm{pd}_A DA.$$

This shows that $\mathrm{pd}_A(DA)=d$. Similarly, $\mathrm{pd}(DA)_A=d$. Substituting in the first inquality yields:

$$\begin{aligned} \mathrm{gl.dim}\ A+1 &\leqq \mathrm{gl.dim}\ A^{(t)} \\ &\leqq \mathrm{gl.dim}\ A^{(t-1)}+\mathrm{gl.dim}\ A+1 \end{aligned}$$

for $t\geqq 1$.

Now, it remains to prove

$$\mathrm{gl.dim}\ A^{(t-1)}<\mathrm{gl.dim}\ A^{(t)} \quad \text{for } t\geqq 1.$$

In order to do that, let F: $\mathrm{mod}\ A^{(t-1)}\to \mathrm{mod}\ A^{(t)}$ be the canonical functor defined by:

$$F(X)=(X\otimes_{A^{(t-1)}} M^{(t)}\to 0)$$

for an $A^{(t-1)}$-module $X_{A^{(t-1)}}$. Assume $\mathrm{gl.dim}\ A^{(t-1)}=d$ then there exists a simple $A^{(t-1)}$-module S with $\mathrm{pd}(S_{A^{(t-1)}})=d$. Thus, let

$$0 \to P_d \xrightarrow{f_d} P_{d-1} \to \cdots \to P_0 \xrightarrow{f_0} S \to 0$$

be a minimal projective resolution for S, then

$$F(P_d) \xrightarrow{F(f_d)} F(P_{d-1}) \to \cdots \to F(P_0) \to F(S) \to 0$$

is also a minimal projective resolution for $A^{(t)}$-module $F(S)$, and moreover, all $F(P_d)$, $F(P_{d-1}), \cdots, F(P_0)$ are projective-injective $A^{(t)}$-modules by the construction of $A^{(t)}$ from $A^{(t-1)}$ and A. Hence, $F(f_d)$ is never injective and so $\mathrm{pd}F(S)_{A^{(t)}} > d$. As a consequence,

$$\mathrm{gl.\,dim}\, A^{(t-1)} < \mathrm{gl.\,dim}\, A^{(t)} \quad \text{for any } t \geqq 1.$$

Combining this with the second inequality above, we have the derired result.

REMARK (1.2) (a) The above Proposition and proof are actually valid if A is only assumed to be a left and right noether ring by using a minimal injective cogenerator and a cyclic module instead of DA and a simple module S, respectively.

(b) The bounds given in Proposition are the best bounds possible. The lower bound is attained for instance when $A=k$, then $A^{(1)}$ is the hereditary algebra $T_2(k)$ of all two-by-two lower triangular matrices with coefficients from k. To see a situation where the upper bound is attained, let A be given by the quiver:

$$1 \underset{\beta}{\overset{\alpha}{\rightleftarrows}} 2$$

bound by $\alpha\beta=0$. Then $\mathrm{gl.\,dim}\, A=2$, and $A^{(1)}$ is given by the quiver:

$$2 \underset{\alpha}{\overset{\beta}{\rightleftarrows}} 1 \xleftarrow{\gamma} 1' \underset{\alpha'}{\overset{\beta'}{\leftrightarrows}} 2'$$

bound by $\alpha\beta=0=\alpha'\beta'$, $\gamma\beta'\alpha'=\beta\alpha\gamma$. It is easily shown that $\mathrm{gl.\,dim}\, A^{(1)}=5$. Observe that $A^{(1)}$ is representation-finite (this indeed will follow from the results of § 3).

Moreover, we claimed in the proof above that $\mathrm{gl.\,dim}\, A^{(t)} - \mathrm{gl.\,dim}\, A^{(t-1)} \leqq \mathrm{gl.\,dim}\, A+1$ and the equality actually holds. For example, let A be given by the quiver:

$$\begin{tikzcd}[row sep=small] & 2 \arrow[dr] & \\ 1 \arrow[ur] \arrow[dr] & & 4 \\ & 3 \arrow[ur] & \end{tikzcd}$$

with the commutativity relation, then $\mathrm{gl.\,dim}\, A=2$, $\mathrm{gl.\,dim}\, A^{(2)}=4$ and $\mathrm{gl.\,dim}\, A^{(3)}=7$.

PROPOSITION (1.3) $1 \leqq \mathrm{dom.\,dim}\, A^{(t)} \leqq \mathrm{gl.\,dim}\, A+1$.

PROOF. Since $\mathrm{dom.\,dim}\, A^{(t)}=\mathrm{dom.\,dim}\, A^{(1)}$, we only prove the case $t=1$. That $\mathrm{dom.\,dim}\, A^{(1)}\geqq 1$ follows directly from the fact that $A^{(1)}$ is a QF-3 algebra. We thus only have to show that $\mathrm{dom.\,dim}\, A^{(1)}\leqq \mathrm{gl.\,dim}\, A+1$. If $\mathrm{gl.\,dim}\, A=\infty$, there is nothing to prove. Assume that A has finite global dimension, let 1 denote its identity, and set:

$$e=\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}$$

then the unique minimal faithful left $A^{(1)}$-module of the QF-3 algebra $A^{(1)}$ is $A^{(1)}e$. It follows from a result of Tachikawa [20, Theorem (2.8)] that

$$\mathrm{dom.\,dim}\, A^{(1)}\leqq \mathrm{id}\,(A^{(1)}e)_{eA^{(1)}e}+1.$$

Now, $eA^{(1)}e=A$ and, as an A-module, $A^{(1)}e=A_A\oplus(DA)_A$. Therefore $\mathrm{id}\,(A^{(1)}e)_{eA^{(1)}e}=\mathrm{id}\,A_A=\mathrm{gl.\,dim}\,A$ (by the proof of Proposition (1.1)).

REMARKS (1.4) (a) Propositions (1.1) and (1.3) allow us to find those algebras $A^{(t)}$ which are Auslander algebras. Indeed, if $A^{(t)}$ is the Auslander algebra of an algebra B, then $\mathrm{gl.\,dim}\, A^{(t)}\leqq 2$ and $\mathrm{dom.\,dim}\, A^{(t)}\geqq 2$ [2]. Therefore $\mathrm{gl.\,dim}\, A=1$ and so A is hereditary and also representation-finite. Let $n=\mathrm{Card}\,((Q_A)_0)$, then the ordinary quiver of $A^{(t)}$ is the Auslander-Reiten quiver of B, which has thus $2n$ vertices, of which n are projective. This is only possible if A is of type A_3 [6]. Finally, it is easily seen that the orientation $\circ\leftarrow\circ\leftarrow\circ$ of Q_A does not give an Auslander Algebra, while each of the other two $\circ\leftarrow\circ\rightarrow\circ$ and $\circ\rightarrow\circ\leftarrow\circ$ does. Observe that, in each of these cases, $A^{(1)}$ is in fact the Auslander algebra of B and $\mathrm{dom.\,dim}\, A^{(1)}=2$.

(b) In the two examples just constructed, the upper bound in Propositon (1.3) is actually attained. The lower bound is attained for instance when $A=\mathrm{T}_2(k)$ is the 2×2 lower triangular matrix algebra with coefficients from k. Then $A^{(1)}$ is given by the quiver:

$$1\xleftarrow{\alpha}2\xleftarrow{\beta}1'\xleftarrow{\alpha'}2'$$

bound by $\alpha\beta\alpha'=0$, and $\mathrm{dom.\,dim}\, A^{(1)}=1$. Thus the bounds given are again the best possible.

§ 2. Algebras with square-zero radicals

In this section, we consider the case when A is an algebra with a square-zero radical, and derive a criterion for deciding when $A^{(t)}$ is representation-finite. We start by generalizing to QF-3 rings a result already obtained by W. Müller [16] and E. L. Green and W. H. Gustafson [10] in the case of QF rings.

PROPOSITION (2.1) *Let R be a* QF-3 *artin ring, J its radical, and n be such that $J^n \neq 0$ but $J^{n+1}=0$. Then:*

(i) *If M_R is an indecomposable module such that $MJ^n \neq 0$, then M_R is projective-injective.*

(ii) *R is representation-finite if and only if so is R/J^n.*

PROOF. (i) Let e be a primitive idempotent of R such that $MJ^n e \neq 0$ and choose an element $ae \in J^n e$ such that $ae \neq 0$. Let us define an R-linear map f: $eR \to aeR$ by $f(ex)=aex$ for $x \in R$, then we have

$$f(eJ)=aeJ \subseteq J^n J=0$$

that is, $eJ \subseteq \ker(f)$. However, eJ is the unique maximal submodule of the projective indecomposable eR, and $\ker(f) \neq eR$ since $f(e)=ae \neq 0$. Therefore, $eJ= \ker(f)$. Hence the simple module $S_R=eR/eJ$ is embedded in $aeR \subseteq R$, that is to say, S_R is a minimal right ideal of R. Since R is QF-3, the injective hull $\mathrm{E}(S)$ of S is also embedded in R, and is projective-injective.

Let us now take an element $m \in MJ^n$ such that $me \neq 0$, and define an R-linear map g: $S \to M$ by

$$g(ex+eJ)=mex \quad \text{for } x \in R.$$

Clearly g is well-defined and injective because it is nonzero and S is simple. Then there exists an R-linear map h: $M \to \mathrm{E}(S)$ such that $hg=j$ (where j: $S \to \mathrm{E}(S)$ denotes the canonical inclusion).

$$\begin{tikzcd} 0 \arrow[r] & S \arrow[d, "j"'] & M \arrow[l, "g"'] \arrow[dl, dotted, "h"] \\ & \mathrm{E}(S) & \end{tikzcd}$$

Assume now that $h(M)$ is strictly contained in $\mathrm{E}(S)$. Since $\mathrm{E}(S)$ is indecomposable projective with radical $\mathrm{E}(S)J$, $\mathrm{h}(M)$ must in fact be contained in $\mathrm{E}(S)J$. But this implies that

$$0 \neq i(S)=hg(S) \subseteq h(MJ^n)=h(M)J^n \subseteq \mathrm{E}(S)J \cdot J^n=0$$

an absurdity. Therefore, $h(M)=\mathrm{E}(S)$, and so h: $M \to \mathrm{E}(S)$ is an epimorphism. Now, $\mathrm{E}(S)$ is projective and M is indecomposable, therefore h is actually an isomorphism, and $M \cong \mathrm{E}(S)$ is indeed projective-injective.

(ii) Follows directly from (i) : indeed, an indecomposable R-module which is not annihilated by J^n is projective-injective. Hence, if R/J^n is representation-finite, the same is true for R. The converse is trivial.

In order to state our criterion, we recall the notion of a separated quiver due to Gabriel [9]. Let A be a finite dimentional algebra over an algebraically closed field k, and $\{e_1, \cdots, e_n\}$ denote a complete set of primitive orthogonal idempotents of A. The separated quiver $\Delta(A)$ of A is defined to have as vertices the elements of the set $\{1, 2, \cdots, n\} \times \{0, 1\}$, the number of arrows from $(i, 0)$ to $(j, 1)$ is equal to $\dim_k(e_j(\mathrm{rad}\, A)e_i)$, and these are all the arrows of $\Delta(A)$. Gabriel has proved that, if $\mathrm{rad}^2 A = 0$, then A is representation-finite if and only if the underlying graph of $\Delta(A)$ is a disjoint union of Dynkin diagrams. We deduce the following Corollary:

COROLLARY (2.2) *If A has square-zero radical, then $A^{(t)}$ is representation-finite if and only if the underlying graph of $\Delta(A^{(t)}/\mathrm{rad}^2 A^{(t)})$ is a disjoint union of Dynkin diagrams.*

PROOF. Indeed, if A has square-zero radical, the radical of $A^{(t)}$ is such that $\mathrm{rad}^3 A^{(t)} = 0$. Thus, it follows from Proposition (2.1) that $A^{(t)}$ is representation-finite if and only if the square-zero radical algebra $A^{(t)}/\mathrm{rad}^2 A^{(t)}$ is representation-finite, that is to say, if and only if the underlying graph of $\Delta(A^{(t)}/\mathrm{rad}^2 A^{(t)})$ is a disjoint union of Dynkin diagrams.

The separated quiver of $A^{(t)}/\mathrm{rad}^2 A^{(t)}$ is easily constructed as follows: let $\{e_1, e_2, \cdots, e_n\}$ be a complete set of primitive orthogonal idempotento of A and $e_i^{(\kappa)}$ primitive idempotents of $A^{(t)}$ with e_i on the $(\kappa+1, \kappa+1)$-component and 0 elsewhere for $i=1, \cdots, n$, then $\{e_1, \cdots, e_n, e_1^{(1)}, \cdots, e_n^{(1)}, \cdots, e_1^{(t)}, \cdots, e_n^{(t)}\}$ is a complete set of primitive orthogonal idempotents of $A^{(t)}$, and the set of vertices of $\Delta(A^{(t)}/\mathrm{rad}^2 A^{(t)})$ is, by denoting $i^{(0)} = i$, $\{(i^{(\kappa)}, l) \mid \kappa = 0, 1, \cdots, t\,;\ i = 1, \cdots, n\,;\ l = 0, 1\}$, also, for any arrow $(i, 0) \to (j, 1)$ in $\Delta(A)$, there are arrows $(i, 0) \to (j, 1)$, $(i^{(\kappa)}, 0) \to (j^{(\kappa)}, 1)$ and $(j^{(\kappa-1)}, 0) \to (i^{(\kappa)}, 1)$, $1 \leqq \kappa \leqq t$, in $\Delta(A^{(t)}/\mathrm{rad}^2 A^{(t)})$ (for, $e_i^{(1)}$ $(\mathrm{rad}\, A^{(t)}/\mathrm{rad}^2 A^{(t)})e_j = D(e_j(\mathrm{rad}\, A)e_i) \neq 0$) and these are all the arrows in $\Delta(A^{(t)}/\mathrm{rad}^2 A^{(t)})$.

In contrast to the case where A is iterated tilted of Dynkin type [1], we may have $A^{(1)}$ representation-finite but $A^{(t)}$ representation-infinite for some $t \geqq 2$. For example, if A is the algebra with square-zero radical given by the quiver:

$$\circ \rightleftarrows \circ \longrightarrow \circ$$

then $A^{(1)}$ is representation-finite but $A^{(2)}$ is not. On the other hand, if A is the algebra with square-zero radical given by the quiver:

$$\begin{tikzcd} & \circ \arrow[dl] & \\ \circ & & \circ \arrow[ul] \arrow[ll] \end{tikzcd}$$

thus A is a quotient of a hereditary algebra, but is not iterated tilted, then $A^{(t)}$ is representation-finite for any $t \geqq 1$.

§ 3. Nakayama algebras

In this section, we consider the case when A is a Nakayama (generalized uniserial) algebra, describe an easy construction for the bound quiver of $A^{(1)}$, and deduce an effective criterion for deciding whether $A^{(1)}$ is representation-finite. In the following, we denote $A^{(1)}$ by $\bar{A}$.

We shall first assume that A has a simple projective module. In this case, A is given by the quiver:

$$1 \longleftarrow 2 \longleftarrow 3 \longleftarrow \cdots \longleftarrow n$$

bound by certain zero-relations which we can encode in the Kupisch series of A. Here, for the right module category, we shall consider the left Kupisch series of A, $(c_i)_{1 \leqq i \leqq n}$, thus

$$c_i = l(Ae_i) = l(\mathrm{I}(i)_A).$$

We have $c_n = 1$ and $2 \leqq c_i \leqq c_{i-1} + 1$ for any $i \neq n$. Observe that the left Kupisch series of A is closely related to the right, in fact, one may be deduced from the other [8]. We associate to $(c_i)_{1 \leqq i \leqq n}$ a new series $(a_i)_{1 \leqq i \leqq n}$ which we define by:

$$a_i = i + c_i - 1 \quad (1 \leqq i \leqq n).$$

By adding $i-2$ to the three members of the inequality $2 \leqq c_{i+1} \leqq c_i + 1$ $(i \neq 1)$, we see that $i \leqq a_{i-1} \leqq a_i$ for all $i \neq 1$, and also $a_1 = c_1$, $a_n = n$. In fact, since $c_i = l(\mathrm{I}(i))$ and $\mathrm{S}(i) = \mathrm{soc}\, \mathrm{I}(i)$, we have $\mathrm{S}(a_i) = \mathrm{top}\, \mathrm{I}(i)$.

Recall that the ordinary quiver $\mathrm{Q}_{\bar{A}}$ of $\bar{A}$ consists of two copies Q_A and Q'_A of the ordinary quiver of A, fully embedded in $\mathrm{Q}_{\bar{A}}$ and such that every vertex of $\mathrm{Q}_{\bar{A}}$ belongs to either Q_A or Q'_A, together with some additional arrows of the form $i' \to j$ with $i' \in (\mathrm{Q}'_A)_0$ and $j \in (\mathrm{Q}_A)_0$. To construct $\mathrm{Q}_{\bar{A}}$, it suffices thus to characterize these extra arrows:

LEMMA (3.1) *There is an arrow $i' \to j$ if and only if* $a_i = j$ and $a_{i-1} \neq j$ for $i \neq 1$, and for $i = 1$ *if and only if* $a_1 = j$.

PROOF. By definition, the number of arrows from i' to j is given by $\dim_k \mathrm{Hom}_{\bar{A}}(\mathrm{P}(j), \mathrm{rad}\, \mathrm{P}(i')/\mathrm{rad}^2\, \mathrm{P}(i'))$. Thus, there is an arrow $i' \to j$ ifand only if $\mathrm{S}(j)$ is a direct summand of $\mathrm{rad}\, \mathrm{P}(i')/\mathrm{rad}^2\, \mathrm{P}(i')$. Recall that $\mathrm{I}(i)$ is the maximal submodule of $\mathrm{P}(i')$ whose composition factors lie in Q_A (even $\mathrm{Supp}\, \mathrm{P}(i') \cap \mathrm{Q}_A = \mathrm{Supp}\, \mathrm{I}(i)$).

Assume that there is an arrow $i' \to j$. $\mathrm{S}(j)$ must then be a composition factor of the uniserial module $\mathrm{I}(i)$, thus $j \in [i, a_i]$ and, since $i' \to j$ is an arrow, we even have $j=a_i$. On the other hand, we have rad $\mathrm{P}(i')/\mathrm{rad}^2\mathrm{P}(i')=\mathrm{S}(j)\oplus\mathrm{S}(i-1)'$ and so

$$\begin{aligned}\mathrm{Hom}_A(\mathrm{I}(i-1),\mathrm{I}(j))&\cong D(e_jAe_{i-1})\\&\cong e'_{i-1}\bar{A}e_j\\&\cong\mathrm{Hom}_{\bar{A}}(\mathrm{P}(j),\ \mathrm{P}(i-1)')=0.\end{aligned}$$

Therefore $j\notin[i-1,a_{i-1}]$ and in particular $j\neq a_{i-1}$.

Conversely, suppose that $a_i=j\neq a_{i-1}$. Then $j=a_i>a_{i-1}$ hence $\mathrm{Hom}_A(\mathrm{I}(i-1),\mathrm{I}(j))=0$ and so there is no nonzero path from $(i-1)'$ to j. Since $a_i=j$, we have a nonzero map from $\mathrm{I}(i)$ to $\mathrm{I}(j)$ and hence a nonzero path from i' to j. Since this path cannot factor through $(i-1)'$ and cannot factor either through any $l>j$ (for, otherwise, $\mathrm{S}(l)$ would be a composition factor of $\mathrm{I}(i)$, and this contradicts $a_i=j$), there must be an arrow $i'\to j$.

Observe that the binding relations on $Q_{\bar{A}}$ are the original zero-relations on Q_A, the corresponding zero-relations on Q'_A, all possible commutativity relations together with some zero-relations of the form $w=0$ where w is a path from i' to $i-1$, for $2\leq i\leq n$, or from $2'$ to 1, arising from the fact that $\bar{A}$ is constructed from A using a sequence of one-point extensions. We have thus completely determined the bound quiver of $\bar{A}$.

Next, we shall consider the case where the Nakayama algebra A has no simple projective module. In this case, the quiver of A is an oriented cycle with (say) n vertices. To describe these vertices, we shall adopt the following convention: for an integer i, we shall let $\bar{i}$ be the least strictly positive remainder of i modulo n, thus, if i is a multiple of n, we have $\bar{i}=n$. We shall also order the cycle in such a way that decreasing circular order corresponds to the counter-clockwise direction:

$$\begin{tikzcd}\bar{1} & \bar{2}\arrow[l] & \bar{3}\arrow[l] & \cdots\arrow[l] & \bar{n}\arrow[l] \arrow[from=1-1, to=1-5, bend right]\end{tikzcd}$$

Let $(c_{\bar{i}})_{1\leq i\leq n}$ denote the left Kusisch series of A, and put, as before:

$$a_{\bar{i}}=\overline{i+c_{\bar{i}}-1}\qquad(1\leq i\leq n).$$

Again, we have $\mathrm{S}(a_{\bar{i}})=\mathrm{top}\,\mathrm{I}(\bar{i})$, and also:

LEMMA (3.2) *There is an arrow* $\bar{i}'\to j$ *if and only if* $a_{\bar{i}}=\bar{j}$ and $a_{\overline{i-1}}\neq\bar{j}$.

PROOF. Similar to the proof of Lemma (3.1).

The binding relations on $Q_{\bar{A}}$ are constructed just as before: they are the original zero- relations on $Q_{\bar{A}}$, the corresponding zero-relations on Q'_A, all possible commutativity relations on $Q_{\bar{A}}$, together with some extra zero-relations of the form $w=0$, where w is a path from $\overline{i'}$ to $\overline{i-1}$ (for $i \in (Q_A)_0$).

In order to state our criterion for the representation type of $\bar{A}$, let us consider again the case where A has a simple projective module. It follows easily from the previous construction that $\bar{A}$ is a Schurian and directed algebra which is $\tilde{\mathbf{A}}$-free in the sense of [5]. Also, A is a tree algebra, and so is simply connected. Therefore, as in [1], Proposition (1.4), $\bar{A}$ satisfies the condition (S) of [3]. Consequently, $\bar{A}$ is representation-finite if and only if its bound quiver does not contain as a full convex subquiver the bound quiver of a critical simply connected algebra from the list of [12], [4]. On the other hand, if A has no simple projective module, and so its ordinary quiver is an oriented cycle, a Galois covering $\tilde{A} \rightarrow \bar{A}$ with $\tilde{A}$ simply connected is constructed as follows: the set of vertices of $Q_{\tilde{A}}$ is the set $\{(a, t) \mid a \in (Q_{\bar{A}})_0,\ t \in \boldsymbol{Z}\}$ and there is an arrow $(a, t) \rightarrow (b, s)$ in one of the following cases:

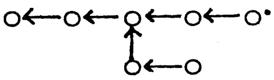
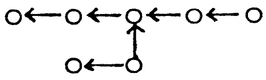
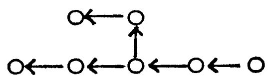
(i) $t=s$, $a=\bar{i}$, $b=\overline{i-1}$ or $a=\overline{i'}$, $b=\overline{(i-1)'}$ with $i \neq 1$

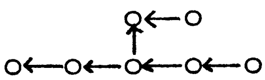
(ii) $t=s+1$, $a=\bar{1}$, $b=\bar{n}$ or $a=\overline{1'}$, $b=\overline{n'}$

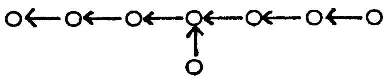
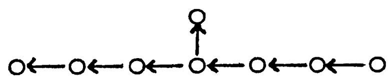
(iii) $t=s$, $a=\overline{i'}$, $b=\bar{j}$ and there is an arrow $\overline{i'} \rightarrow \bar{j}$ in $Q_{\bar{A}}$.

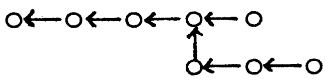
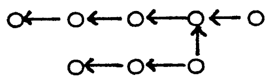
The binding relations on $\tilde{A}$ are taken to be the lifted relations, thus $\tilde{A}$ is locally bounded and Schurian, its quiver is connected, directed and interval-finite. The group of the Galois coverling $\tilde{A} \rightarrow \bar{A}$ is infinite cyclic. Finally, every finite full convex bound subquiver of $Q_{\tilde{A}}$ satisfies condition (S) and so $\tilde{A}$ satisfies the conditions of [5]. Therefore $\bar{A}$ is representation-finite if and only if $\tilde{A}$ does not contain (as a full convex bound subquiver) the bound quiver of a critical simply connected algebra. We thus have:

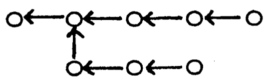
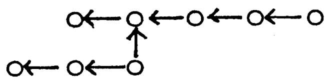
THEOREM (3.2) *Let A be a Nakayama algebra with (respectively, without) a simple projective module. Then $\bar{A}$ is representation-finite if and only if its bound quiver (respectively, the bound quiver of its Galois covering $\tilde{A}$) does not contain as a full convex bound subquiver one of the following bound quivers:*

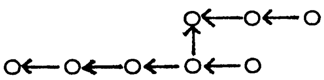
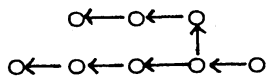
1.a)  b)  c) 

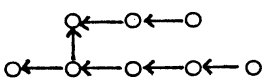
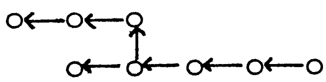
d) 

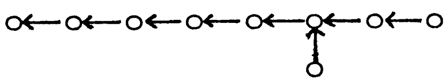
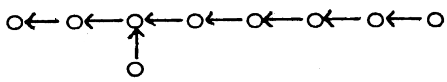
2.a)  b) 

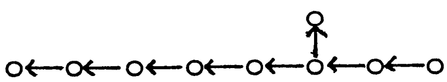
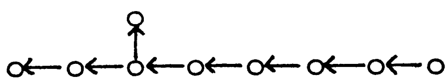
c)  d) 

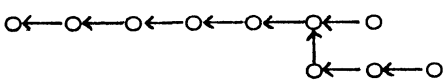
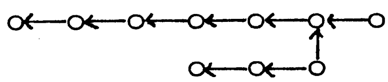
e)  f) 

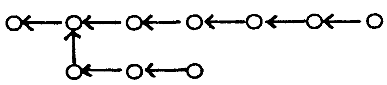
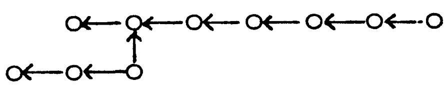
g)  h) 

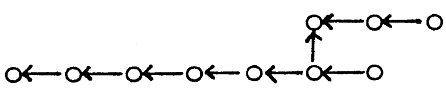
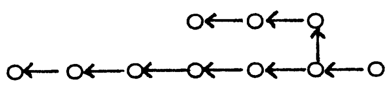
i)  j) 

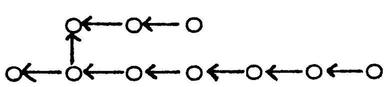
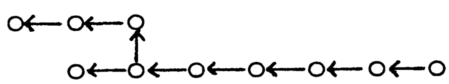
4.a)  b) 

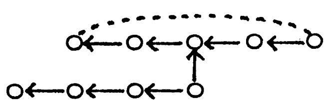
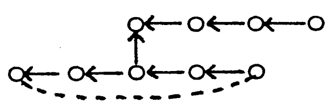
c)  d) 

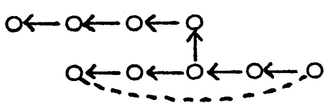
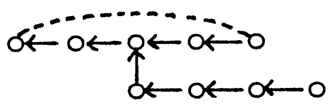
e)  f) 

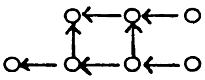
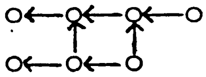
g)  h) 

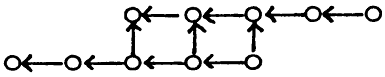
i)  j) 

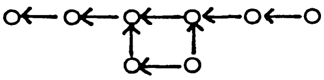
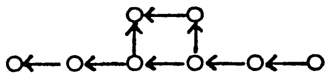
k)  ℓ) 

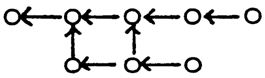
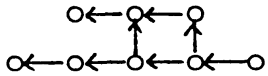
7.a)  b) 

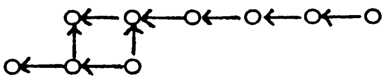
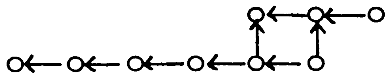
c)  d) 

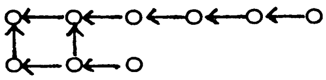
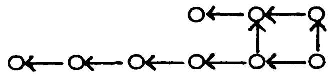
11.a)  b) 

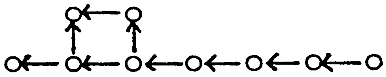
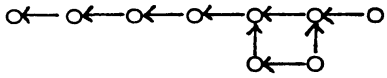
12. 

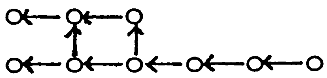
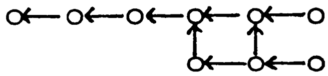
13.a)  b) 

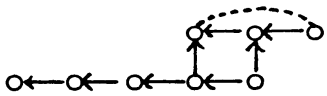
14.a)  b) 

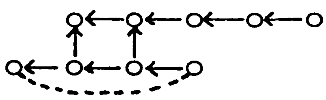
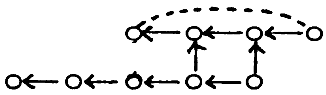
17.a)  b) 

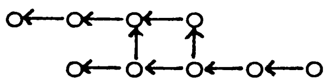
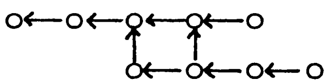
18.a)  b) 

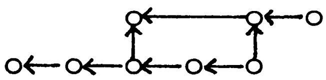
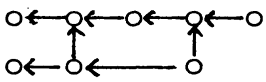
19.a)  b) 

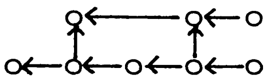
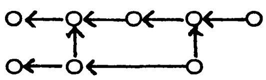
21.a)  b) 

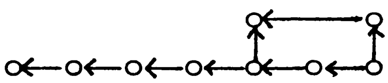
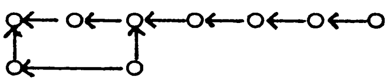
23) 

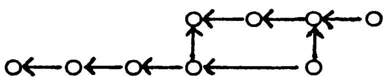
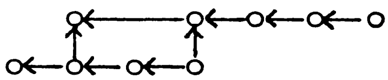
25.a)  b) 

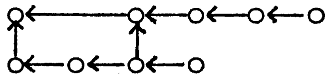
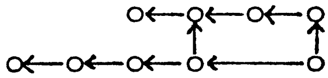
29.a)  b) 

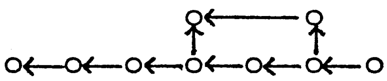
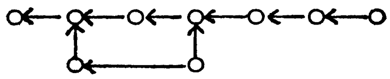
36.a)  b) 

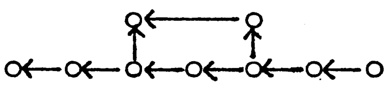
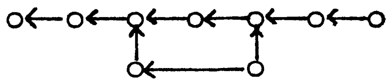
37.a)  b) 

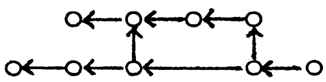
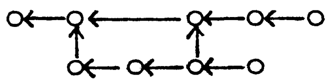
39.a)  b) 

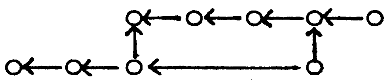
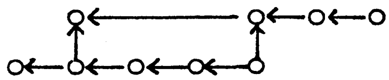
40.a)  b) 

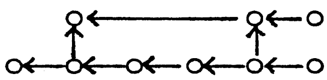
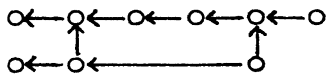
41.a)  b) 

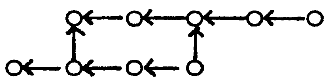
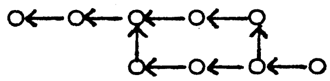
42.a)  b) 

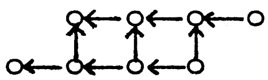
44.a)  b) 

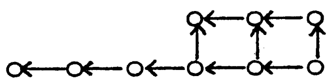
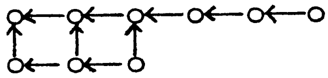
45.a)  b) 

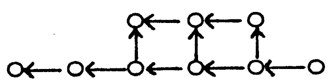
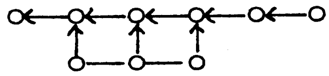
64.a)  b) 

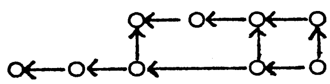
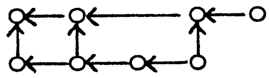
65.a)  b) 

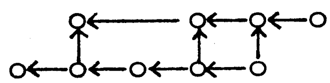
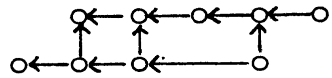
76.a)  b) 

86. 

89.a)  b) 

93.a)  b) 

106.a)  b) 

114.a)  b) 

(All the previous quivers are assuumed to be fully commutative. Dotted lines indicate zero-relations. The numbert are those of [4].)

PROOF. The previous list was established by going through the list of [12], [4] and observing that most of the critical algebras cannot in our case. Also, the possible orientations of the arrows are prescribed by the shape of the quiver $Q_{\bar{A}}$. Thus, eliminating the impossible cases and imposing all possible orientations, we obtain our list. There only remains to prove that the listed critical bound quivers may actually occur as full convex bound subquivers of $Q_{\tilde{A}}$ for some Nakayama algebra A. We give below a list of algebras A realising the corresponding listed critical bound quivers. All the algebras listed have a simple projective module, n is the number of vertices of Q_A, a pair (a, b) means that there is a zero-relations from b to a:

1. a) $n=9$, $(3,7)$ b) $n=10$, $(2,5)$, $(3,8)$
   c) $n=9$, $(3,8)$ d) $n=9$, $(3,7)$, $(6,8)$
2. a) $n=8$, $(1,6)$ b) $n=9$, $(4,9)$
   c) $n=9$, $(3,8)$ d) $n=10$, $(3,5)$, $(4,9)$
   e) $n=9$, $(3,6)$ f) $n=11$, $(3,6)$, $(4,8)$
   g) $n=10$, $(3,7)$, $(5,8)$ h) $n=9$, $(4,7)$
   i) $n=11$, $(2,8)$, $(6,9)$ j) $n=10$, $(2,8)$
4. a) $n=9$, $(1,8)$ b) $n=9$, $(1,5)$
   c) $n=10$, $(6,10)$ d) $n=10$, $(3,10)$
   e) $n=11$, $(3,10)$ f) $n=14$, $(3,7)$, $(5,13)$
   g) $n=12$, $(3,7)$ h) $n=13$, $(3,6)$, $(4,8)$
   i) $n=13$, $(6,10)$, $(8,11)$ j) $n=11$, $(6,9)$
   k) $n=13$, $(2,10)$, $(8,11)$ l) $n=11$, $(2,9)$
7. a) $n=11$, $(4,7)$, $(5,10)$, $(7,11)$ b) $n=11$, $(1,5)$, $(2,6)$, $(5,8)$
   c) $n=8$, $(2,5)$ d) $n=10$, $(5,9)$, $(6,10)$
11. a) $n=10$, $(2,7)$, $(3,8)$, $(5,9)$ b) $n=10$, $(2,6)$, $(3,8)$, $(4,9)$
12. $n=10$, $(1,6)$, $(2,7)$, $(3,8)$, $(4,9)$, $(5,10)$
13. a) $n=9$, $(1,7)$, $(2,8)$, $(3,9)$
    b) $n=11$, $(1,7)$, $(2,8)$, $(3,9)$, $(4,10)$, $(5,11)$
14. a) $n=10$, $(1,7)$, $(3,8)$ b) $n=10$, $(3,8)$, $(4,10)$
17. a) and b) $n=12$, $(1,8)$, $(2,9)$, $(3,10)$, $(4,11)$, $(5,12)$
18. a) $n=10$, $(1,5)$, $(3,6)$ b) $n=10$, $(5,8)$, $(6,10)$
19. a) and b) $n=9$, $(1,8)$, $(2,9)$
21. a) $n=11$, $(2,8)$, $(3,10)$, $(7,11)$ b) $n=12$, $(1,6)$, $(2,10)$, $(4,11)$
23. $n=11$, $(4,9)$, $(5,10)$, $(9,11)$
26. a) $n=9$, $(1,4)$, $(2,6)$, $(4,7)$ b) $n=12$, $(4,9)$, $(5,11)$, $(9,12)$
29. a) $n=11$, $(2,8)$, $(3,11)$ b) $n=12$, $(1,7)$, $(2,10)$, $(5,11)$
36. a) $n=11$, $(3,9)$, $(5,10)$, $(6,11)$ b) $n=11$, $(2,7)$, $(3,8)$, $(4,10)$
37. a) $n=11$, $(2,8)$, $(4,9)$, $(6,10)$ b) $n=11$, $(2,6)$, $(3,8)$, $(4,10)$
39. a) $n=10$, $(5,9)$, $(7,10)$ b) $n=12$, $(1,6)$, $(2,8)$, $(3,12)$
40. a) $n=12$, $(4,9)$, $(5,10)$, $(6,12)$ b) $n=11$, $(2,7)$, $(4,8)$, $(5,9)$

41. a)	$n=10$, (2,6), (4,7)	b) $n=10$, (4,7), (5,9)
42. a)	$n=11$, (4,10), (6,11)	b) $n=9$, (1,5), (2,7)
44. a)	$n=10$, (3,9), (5,10)	b) $n=9$, (1,6), (2,8)
45. a)	$n=10$, (3,7), (4,9)	b) $n=12$, (1,7), (3,9), (5,10)
64. a)	$n=11$, (2,7), (3,8), (4,11)	b) $n=11$, (2,8), (5,9), (6,10)
65. a)	$n=12$, (2,9), (5,10), (7,11)	b) $n=12$, (2,6), (3,8), (4,11)
76. a)	$n=11$, (2,7), (4,8), (5,10)	b) $n=8$, (2,7)
86.	$n=9$, (1,6), (2,7), (3,8), (4,9)	
89. a) and b)	$n=11$, (1,7), (2,8), (3,9), (4,10), (5,11)	
93. a) and b)	$n=10$, (1,7), (2,8), (3,9), (4,10)	
106. a)	$n=10$, (3,7), (4,8), (5,10)	b) $n=9$, (1,6), (3,7), (4,8)
114. a)	$n=10$, (2,7), (4,8), (5,9), (6,10)	
b)	$n=11$, (2,7), (3,8), (4,9), (5,11)	

The following generalizes an old result of S. Brenner [7]:

COROLLARY (3.4) *Let A be a self-injective Nakayama algebra of Loewy length s. Then $\bar{A}$ is representation-finite if and only if $s\leqq 3$.*

PROOF. In this case, the ordinary quiver of A is an oriented cycle bound by the relation ideal generated by all paths of constant length s. Clearly, if $s\geqq 4$, the bound quiver of $\tilde{A}$ contains the number (86) of the list, while if $s\leqq 3$, it contains no critical full convex subquiver.

§ 4. The case of the algebras $\Lambda(n, s)$

In this section, we shall investigate the representation type of $\bar{A}$ for A being the algebra of the quiver:

$$1 \xleftarrow{\alpha} 2 \xleftarrow{\alpha} 3 \leftarrow \cdots \leftarrow n-1 \xleftarrow{\alpha_{n-1}} n$$

bound by the ideal generated by the set of all paths of constant length $s\geqq 2$ [15]. We shall denote this algebra by $\Lambda(n, s)$. If there are no relations on the quiver of A, we shall denote it by $\Lambda(n, 0)$ (thus, with this convention, we always have $n>s$). For each algebra $\Lambda(n, s)$, the representation type of the two-by-two lower triangular matrix algebra $T_2(\Lambda(n, s))$ is known:

THEOREM (4.1) [15] (i) *$T_2(\Lambda(n, 0))$ is representation-finite if and only if $n\leqq 4$.*

(ii) *$T_2(\Lambda(n, s))$, $n>s>0$, is representation-finite if and only if* $s=2$, or 3.

We shall now prove:

THEOREM (4.2) (i) *If* $s=0$, *2 or 3, then* $\bar{\Lambda}(n, s)$ *is representation-finite for any value of* $n>s$.

(ii) *If* $s \geqq 4$, *then* $\bar{\Lambda}(n, s)$ *is representation-finite if and only if* $n=s+1$ *or if* (n, s) *is one of the five pairs* $(6,4)$, $(7,4)$, $(7,5)$, $(8,5)$ *and* $(8,6)$.

($\bar{\Lambda}(n, s)$ is an algebra $\bar{A}$ for $A=\Lambda(n, s)$.)

Observe that the ordinary quiver of $\bar{\Lambda}(n, s)$ is:

$$\begin{tikzcd}
1 & 2 \arrow[l, "\alpha_1"'] & \cdots \arrow[l, "\alpha_2"'] & s \arrow[l] & s+1 \arrow[l, "\alpha_s"'] & \cdots \arrow[l] & n \arrow[l, "\alpha_{n-1}"'] & & \\
 & & & 1' \arrow[u, "\beta_1"] & 2' \arrow[l, "\alpha'_1"] \arrow[u, "\beta_2"] & \cdots \arrow[l, "\alpha'_2"] & (s-1)' \arrow[l] \arrow[u, "\beta_{n-s}"] & \cdots \arrow[l] & n' \arrow[l, "\alpha'_{n-1}"]
\end{tikzcd}$$

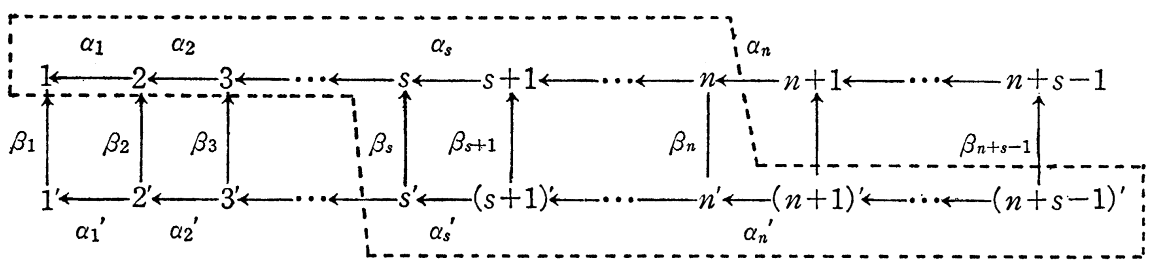
bound by the zero-relations $\alpha_i \alpha_{i+1} \cdots \alpha_{i+s-1}=0=\alpha'_i \alpha'_{i+1} \cdots \alpha'_{i+s-1}$ $(1 \leqq i \leqq n-s)$, the commutativity relations $\alpha_i \alpha'_i=\alpha_{i+s-1} \beta_{i+1}$ $(1 \leqq i \leqq n-s)$, and relations of the form $\alpha_i w \alpha'_i=0$, where w is a path from i' to $i+1$. Observe also that $\Lambda(n, s)$ satisfies the condition (S) of [3], and hence is simply connected whenever it is representation-finite. In the proof of Theorem (4.2), we shall use the fact (already mentioned in the Introduction) that if A is an iterated tilted algebra of Dynkin type, then $\bar{A}$ is representation-finite ([1, Proposition (1.4)]). We shall also need the following lemma;

LEMMA (4.3) (i) *For any pair* (n, s), $s \neq 0$, *there is a full exact embedding of mod* $\bar{\Lambda}(n, s)$ *into mod* $\mathrm{T}_2(\Lambda(n+s-1, s))$. *Thus, if the latter algebra is representation-finite, so is the former.*

(ii) *For any pair* (n, s), $s \neq 0$, $n>2s-1$, *there is a full exact embedding of mod* $\mathrm{T}_2(\Lambda(n-s+1, s))$ *into mod* $\bar{\Lambda}(n, s)$. *Thus, if the former algebra is representation-infinite, so is the latter.*

(iii) *For any pair* (n, s), $n>s \geqq 3$, *there is a full exact embedding of mod* $\bar{\Lambda}(n, s)$ *into mod* $\bar{\Lambda}(n+1, s+1)$. *Thus, if the former algebra is representation-infinite, so is the latter.*

PROOF. (i) Consider the specialization [18] of the bound quiver of $\mathrm{T}_2(\Lambda(n+s-1, s))$ obtained by shrinking the arrows $\beta_1, \beta_2, \cdots, \beta_{s-1}$ and $\beta_{n+1}, \beta_{n+2}, \cdots, \beta_{n+s-1}$. We clearly obtain in this way the bound quiver of $\bar{\Lambda}(n, s)$. The statement then follows from [18].



(ii) The bound quiver of $T_2(A(n-s+1, s))$, $n>2s-1$, is obtained from that of $\bar{A}(n, s)$ by deleting the vertices $1, 2, \cdots, s-1$ and $(n-s+1)', \cdots, n'$. Hence the result follows.

(iii) The bound quiver of $\bar{A}(n, s)$ is obtained from that of $\bar{A}(n+1, s+1)$ by deleting the vertices 1 and $(n+1)'$, and adding all possible zero-relations of constant length s on the two paths from $n+1$ to 2 and from n' to $1'$.

PROOF OF THEOREM (4.2): (i) Consider first the case $s=0$. In this case, $A(n, 0)$ is a hereditary algebra of type $\boldsymbol{A}_n$, and hence $\bar{A}(n, 0)$ is representation-finite. If $s=2$ or 3, it follows from Theorem (4.1) and Lemma (4.3) (i) that $\bar{A}(n, s)$ is representation-finite for any value of $n>s$.

(ii) Assume now that $s \geqq 4$. Observe first that, for any value of $s \neq 0$, $A(s+1, s)$ is a tilted algebra of Dynkin type $\boldsymbol{D}_{s+1}$, thus $\bar{A}(s+1, s)$ is representation-finite. On the other hand, if $n>2s-1$, the algebra $T_2(A(n-s+1, s))$ is representation-infinite by Theorem (4.1), hence so is $\bar{A}(n, s)$ by Lemma (4.3) (ii). We thus only have to consider the case of the pairs $(s+t, s)$ for $2 \leqq t \leqq s-1$. By Lemma (4.3) (iii), there exists a full exact embedding of mod $\bar{A}(4+t, 4)$ into mod $\bar{A}(s+t, s)$ for any values of t and $s \geqq 4$. Now, if $t>3$, then $4+t>7$ implies by the above remarks that $\bar{A}(4+t, 4)$ (and hence $\bar{A}(s+t, s)$) are representation-infinite.

Consider now the algebras $\bar{A}(s+2, 3)$ and $\bar{A}(s+3, s)$ for $s \geqq 4$. By Lemma (4.3) (iii) there exist full exact embeddings of mod $\bar{A}(9, 6)$ into mod $\bar{A}(s+3, s)$ for any value of $s \geqq 6$ and of mod $\bar{A}(9, 7)$ into mod $\bar{A}(s+2, s)$ for any value of $s \geqq 7$. Now the bound quiver of $\bar{A}(9, 6)$ (respectively, $\bar{A}(9, 7)$) contains as a full convex subquiver the critical subquiver number (13. b) (respectively, (19. b)) of the list of Theorem (3.3). It follows that $\bar{A}(9, 6)$ and $\bar{A}(9, 7)$ are representation-infinite, and consequently so are $\bar{A}(s+3, s)$ for $s \geqq 6$ and $\bar{A}(s+2, s)$ for $s \geqq 7$.

There remains to consider the cases of the five pairs $(6, 4)$, $(7, 4)$, $(7, 5)$, $(8, 5)$ and $(8, 6)$. It is easily verified that $A(6, 4)$ is iterated tilted of type $\boldsymbol{E}_6$, $A(7, 4)$

and $\Lambda(7,5)$ are iterated tilted of type $\boldsymbol{E}_7$, while $\Lambda(8,5)$ and $\Lambda(8,6)$ are iterated tilted of type $\boldsymbol{E}_8$. Therefore $\bar{\Lambda}(6,4)$, $\bar{\Lambda}(7,4)$, $\bar{\Lambda}(7,5)$, $\bar{\Lambda}(8,5)$ and $\bar{\Lambda}(8,6)$ are representation-finite. The proof of the theorem is now complete.

NOTE (1) After the completion of this work, the authors learned that A. Skowronski has also studied the case of the algebras with square-zero radical in [19], obtaining the same criterion for the representation-finiteness of $A^{(1)}$ described in § 2 by the different method.

(2) This paper is the complete and revised version of the talk at ICRA IV (1984) in Ottawa.

(3) This paper was written while the first author was an Alexander von Humboldt fellow at the University of Bielefeld.

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