TSUKUBA J. MATH.
Vol. 11 No. 1 (1987). 177-197

A NOTE ON STRONG HOMOLOGY OF INVERSE SYSTEMS

By

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1. Introduction.

Ju. T. Lisica and the author have defined in [4] strong homology groups $H_p(\boldsymbol{X};G)$ of inverse systems of spaces $\boldsymbol{X}=(X_\lambda, p_{\lambda\lambda'}, \Lambda)$ over directed cofinite sets Λ (every element $\lambda\in\Lambda$ has only finitely many predecessors). It was shown in [5] that these groups are functors on the coherent prohomotopy category CPHTop, introduced in [2] and [3]. The notion of strong or Steenrod homology $H_p^s(X;G)$ of an arbitrary space X was then defined [1], [6] and shown to be a functor on the strong shape category SSh [2], [3]. The procedure consisted in choosing a cofinite ANR-resolution $\boldsymbol{p}:X\to\boldsymbol{X}$ of X ([7], [8], [9]) and of defining $H_p^s(X;G)$ as $H_p(\boldsymbol{X};G)$. That the group $H_p^s(X;G)$ does not depend on the choice of the resolution is a consequence of the following factorization theorem ([3], Theorem II.2.3). If $\boldsymbol{p}:X\to\boldsymbol{X}$ is a resolution and $f:X\to\boldsymbol{Y}$ is a coherent map into a cofinite ANR-system, then there exists a unique coherent homotopy class of coherent maps $g:\boldsymbol{X}\to\boldsymbol{Y}$ such that gp and f are coherently homotopic.

The definition of composition in CPHTop and the proof of the factorization theorem essentially used the assumption that the index sets Λ be cofinite. On the other hand, the construction of the homology groups $H_p(\boldsymbol{X};G)$ did not require this assumption. Therefore, it remained unclear whether one can use also non-cofinite ANR-resolutions to determine the homology groups $H_p^s(X;G)$ of the space X. To prove that this is indeed the case is the main purpose of this paper. Such an information can prove useful in situations where a non-cofinite ANR-resolution naturally arises.

The main idea of the proof is to replace a given ANR-resolution $\boldsymbol{p}:X\to\boldsymbol{X}$ by a cofinite ANR-resolution $\boldsymbol{p}^*:X\to\boldsymbol{X}^*$ using the "trick" described in ([9], Theorem I, 1.2). What remains to be done is to exhibit a natural isomorphism $u_*:H_p(\boldsymbol{X};G)\to H_p(\boldsymbol{X}^*;G)$. The correct formula for u_* is easily found. However, the formula for the inverse v_* of u_* is less obvious. Even more complicated is the verification of the two equalities $u_*v_*=1$, $v_*u_*=1$.

In order to simplify notations throughout the paper we omit the coefficient groups G, although all results hold for an arbitrary G.

Received January 13, 1986, Revised May 13, 1986

2. Coherent maps and homotopy of systems.

Let $\boldsymbol{X}=(X_\lambda, p_{\lambda\lambda'}, \Lambda)$ and $\boldsymbol{Y}=(Y_\mu, q_{\mu\mu'}, M)$ be inverse systems of spaces over directed sets Λ and M respectively, i.e., $\boldsymbol{X}$ and $\boldsymbol{Y}$ are assumed to be objects of the category pro-Top (see I. 1 of [9]). In all but the last section we will assume that Λ and M are antisymmetric, i.e. $\lambda\leqq\lambda'$ and $\lambda'\leqq\lambda$ implies $\lambda=\lambda'$.

We denote by Λ^n, $n\geqq 0$, the set of all increasing sequences $\boldsymbol{l}=(\lambda_0, \cdots, \lambda_n)$ in Λ of length n, $\lambda_0\leqq\cdots\leqq\lambda_n$. If $0\leqq j\leqq n$, $\boldsymbol{l}_j=(\lambda_0, \cdots, \hat{\lambda}_j, \cdots, \lambda_n)$ denotes the sequence of length $n-1$ obtained from $\boldsymbol{l}$ by deleting λ_j. Dually, $\boldsymbol{l}^j=(\lambda_0, \cdots \lambda_j, \lambda_j, \cdots, \lambda_n)$ is obtained from $\boldsymbol{l}$ by repeating λ_j. The standard n-simplex is denoted by Δ^n and $\partial_j^n:\Delta^{n-1}\to\Delta^n$, $\sigma_j^n:\Delta^{n+1}\to\Delta^n$ denote the face and degeneracy operators.

According to [2], [3], a coherent map $f:\boldsymbol{X}\to\boldsymbol{Y}$ consists of the following:

(i) a function $\varphi:\bigcup_{n\geqq 0} M^n\to\Lambda$ such that

$$\varphi(\boldsymbol{m})\geqq\varphi(\boldsymbol{m}_j),\quad 0\leqq j\leqq n,\ n>0; \tag{1}$$

(ii) maps $f_{\boldsymbol{m}}:\Delta^n\times X_{\varphi(\boldsymbol{m})}\to Y_{\mu_0}$, $\boldsymbol{m}\in M^n$, $n\geqq 0$, such that

$$f_{\boldsymbol{m}}(\partial_j^n t, x)=\begin{cases} q_{\mu_0\mu_1}f_{\boldsymbol{m}_0}(t, p_{\varphi(\boldsymbol{m}_0)\varphi(\boldsymbol{m})}(x)), & j=0,\\ f_{\boldsymbol{m}_j}(t, p_{\varphi(\boldsymbol{m}_j)\varphi(\boldsymbol{m})}(x)), & 0<j\leqq n.\end{cases} \tag{2}$$

$$f_{\boldsymbol{m}}(\sigma_j^n t, p_{\varphi(\boldsymbol{m})\varphi(\boldsymbol{m}^j)}(x))=f_{\boldsymbol{m}^j}(t, x),\quad 0\leqq j\leqq n. \tag{3}$$

A coherent homotopy $F:I\times\boldsymbol{X}\to\boldsymbol{Y}$, connecting coherent maps $f, f':\boldsymbol{X}\to\boldsymbol{Y}$, is a coherent map, given by a function Φ and by maps $F_{\boldsymbol{m}}:\Delta^n\times I\times X_{\Phi(\boldsymbol{m})}\to Y_{\mu_0}$, such that

$$\Phi(\boldsymbol{m})\geqq\varphi(\boldsymbol{m}), \varphi'(\boldsymbol{m}), \tag{4}$$

$$F_{\boldsymbol{m}}(t, 0, x)=f_{\boldsymbol{m}}(t, p_{\varphi(\boldsymbol{m})\Phi(\boldsymbol{m})}(x)), \tag{5}$$

$$F_{\boldsymbol{m}}(t, 1, x)=f'_{\boldsymbol{m}}(t, p_{\varphi'(\boldsymbol{m})\Phi(\boldsymbol{m})}(x)). \tag{6}$$

In [3] a coherent map $f:\boldsymbol{X}\to\boldsymbol{Y}$ was called *special* provided there existed an increasing function $\chi:M\to\Lambda$ satisfying

$$\varphi(\boldsymbol{m})=\chi(\mu_n),\quad \boldsymbol{m}=(\mu_0, \cdots, \mu_n). \tag{7}$$

If M is cofinite, then every coherent map $f:\boldsymbol{X}\to\boldsymbol{Y}$ admits a special coherent map $f':\boldsymbol{X}\to\boldsymbol{Y}$ such that f and f' are coherently homotopic, i.e. $f\cong f'$. Moreover, if $f', f'':\boldsymbol{X}\to\boldsymbol{Y}$ are special coherent maps and $f'\cong f''$, then there exists a special coherent homotopy connecting f' and f'' (see [3], Lemmas I. 6.5 and I. 6.6). In [2] and [3] compostion of special coherent maps $f:\boldsymbol{X}\to\boldsymbol{Y}$, $g:\boldsymbol{Y}\to\boldsymbol{Z}$ was defined. In the case of cofinite systems, it induces a composition of coherently homotopic classes of coherent maps, yielding thus the category CPHTop.

3. Strong homology groups of systems.

Let $X=(X_\lambda, p_{\lambda\lambda'}, \Lambda)$ be an inverse system over a directed set Λ. Following [4], we define a chain complex $C_\#(X)$ as follows. For $p\geqq 0$, a strong p-chain of X is a function x which assigns to every $l\in\Lambda^n$ a singular $(p+n)$-chain x_l of X_{λ_0}, $x_l\in C_{p+n}(X_{\lambda_0})$. The group $C_p(X)$ consists of all strong p-chains of X. The boundary operator $d: C_{p+1}(X)\to C_p(X)$ is given by

$$(1)\qquad (dx)_{\lambda_0}=\partial(x_{\lambda_0}),$$

$$(2)\qquad (-1)^n(dx)_l=\partial(x_l)-p_{\lambda_0\lambda_1\#}x_{l_0}-\sum_{j=1}^{n}(-1)^j x_{l_j},\quad n\geqq 1.$$

Strong homology groups $H_p(X)$ of a system X are defined as the homology groups of the chain complex $C_\#(X)$.

If $f: X\to Y$ is a special coherent map, given by an increasing function $\varphi: M\to\Lambda$ and by maps $f_m: \Delta^n\times X_{\varphi(\mu_n)}\to Y_{\mu_0}$, then f induces a chain mapping $f_\#: C_\#(X)\to C_\#(Y)$, defined in (4.1) of [5] by

$$(3)\qquad (f_\# x)_{\mu_0\cdots\mu_n}=\sum_{i=0}^{n} f_{\mu_0\cdots\mu_i\#}(\Delta^i\times x_{\varphi(\mu_i)\cdots\varphi(\mu_n)}).$$

Special coherently homotopic maps $f, f': X\to Y$ induce chain homotopic chain maps $f_\#\cong f'_\#$. Moreover, $(gf)_\#\cong g_\# f_\#$ (see [5]). Consequently, special coherent maps $f: X\to Y$ induce homomorphisms $f_*: H_p(X)\to H_p(Y)$ satisfying $(gf)_*=g_* f_*$. By definition, f_* is the homomorphism induced by the chain mapping $f_\#: C_\#(X)\to C_\#(Y)$ and depends only on the coherent homotopy class of f. How to define f_* for coherent maps, which are not special, will be shown in 8.

4. Associated cofinite systems.

With every inverse system $X=(X_\lambda, p_{\lambda\lambda'}, \Lambda)$ we will now associate a new system $X^*=(X^*_\alpha, p^*_{\alpha\alpha'}, \Lambda^*)$, defined as follows (see [9], the proof I, 1, Theorem 2). The set Λ^* consists of all finite subsets $\alpha\subseteqq\Lambda$ having a maximum. Since we have assumed that Λ is antisymmetric, the maximum of α is unique and we denote it shortly by $\bar{\alpha}$. We order Λ^* by inclusion $\subseteqq$. Clearly, Λ^* is directed, antisymmetric and cofinite. We now put

$$(1)\qquad X^*_\alpha=X_{\bar{\alpha}},$$

$$(2)\qquad p^*_{\alpha\alpha'}=p_{\bar{\alpha}\bar{\alpha}'}.$$

We refer to X^* as to the *cofinite system associated with* X.

We now define a map of systems $u=u_X: X\to X^*$ (in the sense of [4]). It is given by the increasing function $\alpha\mapsto\bar{\alpha}$, $\alpha\in\Lambda^*$, and by the maps $u_\alpha=id: X_{\bar{\alpha}}\to X^*_\alpha=X_{\bar{\alpha}}$. It generates a chain mapping $u_\#=u_{X\#}: C_\#(X)\to C_\#(X^*)$ (§ 3. of [4]), given by

$$(u_\# x)_{\alpha_0\cdots\alpha_n}=x_{\bar{\alpha}_0\cdots\bar{\alpha}_n},\quad (\alpha_0, \cdots, \alpha_n)\in\Lambda^{*n}. \tag{3}$$

If $x\in C_p(X)$, then $x_{\bar{\alpha}_0\cdots\bar{\alpha}_n}\in C_{p+n}(X_{\bar{\alpha}_0})=C_{p+n}(X^*_{\alpha_0})$ as desired.

REMARK 1. If one first associates with u a special coherent map u' as in (2.8) of [5], then the induced chain mapping $u'_\#$ is chain homotopic to $u_\#$ and is given by a formula more complicated than (3).

Let $Y^*=(Y^*_\beta, q^*_{\beta\beta'}, M^*)$ be the cofinite system associated with $Y=(Y_\mu, q_{\mu\mu'}, M)$. With every coherent map $f: X\to Y$ we now associate a coherent map $f^*: X^*\to Y^*$. If f is given by φ and $f_{\beta_0\cdots\beta_n}$, we define f^* by φ^* and $f^*_{\beta_0\cdots\beta_n}: \Delta^n\times X^*_{\varphi^*(\beta_0,\cdots,\beta_n)}\to Y^*_{\beta_0}$, where

$$\varphi^*(\beta_0, \cdots, \beta_n)=\{\varphi(\bar{\beta}_{i_0}, \cdots, \bar{\beta}_{i_k}): 0\leqq i_0< \cdots <i_k\leqq n,\ 0\leqq k\leqq n\}, \tag{4}$$

$$f^*_{\beta_0\cdots\beta_n}=f_{\bar{\beta}_0\cdots\bar{\beta}_n}. \tag{5}$$

Note that $\beta_0\subseteq\cdots\subseteq\beta_n$ implies $\bar{\beta}_{i_0}\leqq\cdots\leqq\bar{\beta}_{i_k}$ so that $\varphi^*(\beta_0, \cdots, \beta_n)$ is a well-defined finite subset of Λ. It belongs to Λ^* because (2.1) implies that $\varphi(\bar{\beta}_0, \cdots, \bar{\beta}_n)$ is the maximum of $\varphi^*(\beta_0, \cdots, \beta_n)$, i.e.,

$$\overline{\varphi^*(\beta_0, \cdots, \beta_n)}=\varphi(\bar{\beta}_0, \cdots, \bar{\beta}_n) \tag{6}$$

That φ^* and $f^*_{\beta_0\cdots\beta_n}$ satisfy 2.(2)-2.(3) is immediate. We refer to f^* as to the *coherent map associated with f*.

If $f: X\to Y$ is a special coherent map, given by an increasing function $\varphi: M\to\Lambda$ and by maps $f_m: \Delta^n\times X_{\varphi(\mu_n)}\to Y_{\mu_0}$, then one can associate with f a special coherent map $f^+: X^*\to Y^*$, given by $\varphi^+: M^*\to\Lambda^*$ and $f^+_{\beta_0\cdots\beta_n}: \Delta^n\times X^*_{\varphi^+(\beta_n)}\to Y^*_{\beta_0}$, defined as follows.

$$\varphi^+(\beta)=\varphi(\beta),\quad \beta\in M^*, \tag{7}$$

$$f^+_{\beta_0\cdots\beta_n}=f_{\bar{\beta}_0\cdots\bar{\beta}_n}. \tag{8}$$

Note that $\varphi(\beta)$ is a finite subset of Λ, because β is a finite subset of M. Moreover, $\varphi(\bar{\beta})$ is the maximum of $\varphi(\beta)$, i.e.,

$$\varphi(\bar{\beta})=\overline{\varphi^+(\beta)},\quad \beta\in M^*, \tag{9}$$

which shows that indeed $\varphi^+(\beta)\in\Lambda^*$. Furthermore, $\beta_0\subseteq\beta_1$ implies $\varphi(\beta_0)\subseteq\varphi(\beta_1)$, which shows that φ^+ is an increasing function.

Also note that (9) implies $X^*_{\varphi^+(\beta_n)}=X_{\overline{\varphi^+(\beta_n)}}=X_{\varphi(\bar\beta_n)}$, $Y^*_{\beta_0}=Y_{\bar\beta_0}$, so that (8) defines a map $f^+_{\beta_0\cdots\beta_n}:\Delta^n\times X^*_{\varphi^+(\beta_n)}\to Y^*_{\beta_0}$. A straightforward verification shows that f^+ satisfies (2.2) and (2.3) (put $\varphi^+(\beta_0,\cdots,\beta_n)=\varphi^+(\beta_n)$).

REMARK 2. If f is a special coherent map, then the coherent maps f^*, f^+: $\boldsymbol{X}^*\to\boldsymbol{Y}^*$ are coherently homotopic. Indeed, if we put $\Phi(\beta_0,\cdots,\beta_n)=\varphi(\beta_n)$, then

$$\varphi^*(\beta_0,\cdots,\beta_n)\subseteqq\Phi(\beta_0,\cdots,\beta_n). \tag{10}$$

Indeed, when f is a special coherent map, $\varphi(\bar\beta_{i_0},\cdots,\bar\beta_{i_k})=\varphi(\bar\beta_{i_k})$ and therefore (4) becomes $\varphi^*(\beta_0,\cdots,\beta_n)=\{\varphi(\bar\beta_0),\cdots,\varphi(\bar\beta_n)\}$. However, $\bar\beta_i\in\beta_i\subseteqq\beta_n$ and so $\varphi(\bar\beta_i)\in\varphi(\beta_n)=\Phi(\beta_0,\cdots,\beta_n)$, $0\leqq i\leqq n$. Also, $\varphi^+(\beta_0,\cdots,\beta_n)=\varphi(\beta_n)=\Phi(\beta_0,\cdots,\beta_n)$. Furthermore, note that

$$\begin{aligned} &f^*_{\beta_0\cdots\beta_n}(p^*_{\varphi^*(\beta_0,\cdots,\beta_n)\Phi(\beta_0,\cdots,\beta_n)}(x),t)\\ &\quad=f_{\bar\beta_0\cdots\bar\beta_n}(x,t)=f^+_{\beta_0\cdots\beta_n}(p^*_{\varphi^+(\beta_0,\cdots,\beta_n)\Phi(\beta_0,\cdots,\beta_n)}(x),t). \end{aligned} \tag{11}$$

Indeed, since f is special (φ increases), (6) yields $\overline{\varphi^*(\beta_0,\cdots,\beta_n)}=\varphi(\bar\beta_0,\cdots,\bar\beta_n)$ $=\varphi(\bar\beta_n)=\overline{\varphi(\beta_n)}$. Since f^+ is special, formula (7) yields $\overline{\varphi^+(\beta_0,\cdots,\beta_n)}=\overline{\varphi^+(\beta_n)}=\overline{\varphi(\beta_n)}$. By definition, $\overline{\Phi(\beta_0,\cdots\beta_n)}=\overline{\varphi(\beta_n)}$. All this shows that the constant homotopies $F_{\beta_0\cdots\beta_n}$ yield a coherent homotopy between f^* and f^+.

REMARK 3. If f is a special coherent map, then the induced chain mappings $f_\#$ and $f^+_\#$ satisfy

$$f^+_\# u_{X\#}=u_{Y\#}f_\#, \tag{12}$$

i.e. the following diagram of chain mappings commutes

$$\begin{array}{ccc} C_\#(\boldsymbol{X}^*) & \xleftarrow{\;u_{X\#}\;} & C_\#(\boldsymbol{X}) \\ {\scriptstyle f^+_\#}\big\downarrow & & \big\downarrow{\scriptstyle f_\#} \\ C_\#(\boldsymbol{Y}^*) & \xleftarrow[\;u_{Y\#}\;]{} & C_\#(\boldsymbol{Y})\,. \end{array} \tag{13}$$

Indeed,

$$\begin{aligned} &(f^+_\# u_{X\#}x)_{\beta_0\cdots\beta_n}\\ &\quad=\sum_{i=0}^n f^+_{\beta_0\cdots\beta_i\#}(\Delta^i\times(u_{X\#}x)_{\varphi^+(\beta_i)\cdots\varphi^+(\beta_n)})\\ &\quad=\sum_{i=0}^n f_{\bar\beta_0\cdots\bar\beta_i\#}(\Delta^i\times x_{\varphi(\bar\beta_i)\cdots\varphi(\bar\beta_n)})\\ &\quad=(f_\# x)_{\bar\beta_0\cdots\bar\beta_n}=(u_{Y\#}f_\# x)_{\beta_0\cdots\beta_n}. \end{aligned} \tag{14}$$

Passing to homology and the induced homomorphisms, we conclude that the following diagrams commute for $p\geqq 0$:

(15)
$$\begin{array}{ccc} H_p(X^*) & \xleftarrow{\;u_{X*}\;} & H_p(X) \\ f^+_* \big\downarrow & & \big\downarrow f_* \\ H_p(Y^*) & \xleftarrow[\;u_{Y*}\;]{} & H_p(Y) . \end{array}$$

The main result of this paper is the following theorem proved in 5-7.

THEOREM 1. *$u_{X\#}:C_\#(X)\rightarrow C_\#(X^*)$ is a chain equivalence and theorefore, $u_{X*}:H_p(X)\rightarrow H_p(X^*)$ is an isomorphism.*

5. The homotopy inverse v of $u_\#$.

For $u_\#=u_{X\#}:C_\#(X)\rightarrow C_\#(X^*)$ we will now define a chain homotopy inverse $v=v_X:C^*_\#(X)\rightarrow C_\#(X)$.

We first introduce some notation. $P(n)$, $n\geqq 0$, will denote the group of all permutations of the set $\{0, 1, \cdots, n\}$. If $\boldsymbol{l}=(\lambda_0, \cdots, \lambda_n)\in\Lambda^n$ and $\pi\in P(n)$, we put

$$(1)\qquad \boldsymbol{l}\pi=(\lambda_{\pi(0)}, \cdots, \lambda_{\pi(n)}),$$

$$(2)\qquad [\boldsymbol{l}\pi]=(\{\lambda_{\pi(0)}\}, \{\lambda_{\pi(0)}, \lambda_{\pi(1)}\}, \cdots, \{\lambda_{\pi(0)}, \cdots, \lambda_{\pi(n)}\}).$$

$\boldsymbol{l}\pi$ is a sequence in Λ of length n (which need not be increasing).

Each $\{\lambda_{\pi(0)}, \cdots, \lambda_{\pi(i)}\}\subseteqq\{\lambda_0, \cdots, \lambda_n\}$, $0\leqq i\leqq n$, is an element of Λ^*, because it is a finite totally ordered subset of Λ. Moreover, $[\boldsymbol{l}\pi]$ is an increasing sequence in Λ^* of length n, so that $[\boldsymbol{l}\pi]\in(\Lambda^*)^n$.

For $y\in C_p(X^*)$, we now define $vy\in C_p(X)$ as follows. If $n\geqq 0$ and $\boldsymbol{l}=(\lambda_0, \cdots, \lambda_n)\in\Lambda^n$, we put

$$(3)\qquad (vy)_{\boldsymbol{l}}=\sum_{\pi\in P(n)}\operatorname{sgn}\pi p_\pi y_{[\boldsymbol{l}\pi]},$$

where

$$(4)\qquad p_\pi=p_{\lambda_0\lambda_{\pi(0)}\#}.$$

Clearly, $y_{[\boldsymbol{l}\pi]}\in C_{p+n})X^*_{\{\lambda_{\pi(0)}\}})=C_{p+n}(X_{\lambda_{\pi(0)}})$ so that $(vy)_{\boldsymbol{l}}\in C_{p+n}(X_{\lambda_0})$ as desired.

$v:C_p(X^*)\rightarrow C_p(X)$ is a homomorphism for each $p\geqq 0$. Moreover, we have the following assertion.

LEMMA 1. *$v:C_\#(X^*)\rightarrow C_\#(X)$ is a chain mapping, i.e.*

$$(5)\qquad (vdy)_{\boldsymbol{l}}=(dvy)_{\boldsymbol{l}},\quad \boldsymbol{l}\in\Lambda^n.$$

PROOF. Let $\boldsymbol{l}=(\lambda_0, \cdots, \lambda_n)$. If $n=0$, then

$$(dvy)_{\lambda_0}=\partial(vy)_{\lambda_0}=\partial y_{\{\lambda_0\}}=(dy)_{\{\lambda_0\}}=(vdy)_{\lambda_0}.$$

We will therefore assume that $n\geqq 1$.

By (3) and 3.(2), we see that

$$(6)\qquad (vdy)_{\boldsymbol{l}}=(-1)^n \sum_{\pi\in P(n)} \operatorname{sgn}\pi p_\pi \partial(y_{[\boldsymbol{l}\pi]}) +(-1)^{n-1}\sum_{\pi\in P(n)} \operatorname{sgn}\pi p_\pi p_\pi^* y_{[\boldsymbol{l}\pi]_0} +(-1)^{n-1}\sum_{j=1}^{n-1}\sum_{\pi\in P(n)}(-1)^j \operatorname{sgn}\pi p_\pi y_{[\boldsymbol{l}\pi]_j} -\sum_{\pi\in P(n)} \operatorname{sgn}\pi p_\pi y_{[\boldsymbol{l}\pi]},$$

where we have put

$$(7)\qquad p_\pi^*=p^*_{\nu_0(\pi)\nu_1(\pi)\#}$$

$$(8)\qquad \nu_0(\pi)=\{\lambda_{\pi(0)}\},\quad \nu_1(\pi)=\{\lambda_{\pi(0)}, \lambda_{\pi(1)}\}.$$

We will now show that

$$(9)\qquad \sum_{\pi\in P(n)} \operatorname{sgn} p_\pi p_\pi^* y_{[\boldsymbol{l}\pi]_0}=0,$$

$$(10)\qquad \sum_{\pi\in P(n)} \operatorname{sgn}\pi p_\pi y_{[\boldsymbol{l}\pi]_j}=0,\quad 1\leqq j\leqq n-1,$$

so that (6) reduces to the first and the last sum.

Indeed, put

$$(11)\qquad P_j(n)=\{\pi\in P(n):\ \pi(j)<\pi(j+1)\},\quad 0\leqq j\leqq n-1,$$

For any $\pi\in P_j(n)$, $0\leqq j\leqq n-1$, define $\pi'\in P(n)\backslash P_j(n)$ by

$$(12)\qquad \pi'(i)=\begin{cases}\pi(i), & i\neq j,\ j+1,\\ \pi(j+1), & i=j,\\ \pi(j), & i=j+1.\end{cases}$$

Clearly, in $[\boldsymbol{l}\pi]$ and $[\boldsymbol{l}\pi']$ only the j^{th} terms differ so that

$$(13)\qquad [\boldsymbol{l}\pi]_j=[\boldsymbol{l}\pi']_j,\quad \pi\in P_j(n),\quad 0\leqq j\leqq n-1.$$

Furthermore, for $1\leqq j\leqq n-1$, $\pi'(0)=\pi(0)$ so that

$$(14)\qquad p_\pi=p_{\pi'},\quad \pi\in P_j(n),\quad 1\leqq j\leqq n-1.$$

By (4) and (7), for any $\pi\in P(n)$, we have $p_\pi p_\pi^*=p_{\lambda_0\overline{\nu_1(\pi)}\#}$ because $\overline{\nu_0(\pi)}=\lambda_{\pi(0)}$. However, for $\pi\in P_0(n)$, $\nu_1(\pi)=\{\lambda_{\pi(0)}, \lambda_{\pi(1)}\}=\nu_1(\pi')$, so that

(15) $$p_\pi p_\pi^* = p_{\pi'} p_{\pi'}^*, \quad \pi \in P_0(n).$$

Since $\operatorname{sgn}\pi = -\operatorname{sgn}\pi'$ and the permutations of $P(n)$ come in pairs π, π', where $\pi \in P_j$, we obtain (9) and (10).

We now consider $(dvy)_{\boldsymbol{l}}$. By 3. (2), we have

(16) $$(dvy)_{\boldsymbol{l}} = (-1)^n \partial(vy)_{\boldsymbol{l}} + (-1)^{n-1} p_{\lambda_0\lambda_1\#}(vy)_{\boldsymbol{l}_0} + \sum_{j=1}^{n} (-1)^{n+j-1} (vy)_{\boldsymbol{l}_j}.$$

We will now examine $(vy)_{\boldsymbol{l}_j}$ for $0 \leqq j \leqq n$. We first define $\lambda'_0, \cdots, \lambda'_{n-1}$ by

(17) $$\lambda'_i = \begin{cases} \lambda_i, & i \leqq j-1 \\ \lambda_{i+1}, & i \geqq j. \end{cases}$$

Clearly, $\boldsymbol{l}_j = (\lambda'_0, \cdots, \lambda'_{n-1})$. With every $\pi \in P(n-1)$ and $0 \leqq j \leqq n$, we associate a permutation $\pi'_j \in P(n)$ by putting

(18) $$\pi'_j(i) = \begin{cases} \pi(i), & \text{if } 0 \leqq i \leqq n-1 \text{ and } \pi(i) \leqq j-1, \\ \pi(i)+1, & \text{if } 0 \leqq i \leqq n-1 \text{ and } \pi(i) \geqq j, \\ j, & \text{if } i = n. \end{cases}$$

Note that π'_i belongs to

(19) $$Q_j(n) = \{\pi \in P(n) : \pi(n) = j\}, \quad 0 \leqq j \leqq n,$$

(20) $$\operatorname{sgn}\pi'_j = (-1)^{n-j} \operatorname{sgn}\pi, \quad 0 \leqq j \leqq n,$$

(21) $$\lambda'_{\pi(i)} = \lambda_{\pi'_j}(i), \quad 0 \leqq i \leqq n-1,\ 0 \leqq j \leqq n.$$

This shows that the sequence $\boldsymbol{l}_j\pi = (\lambda'_{\pi(0)}, \cdots, \lambda'_{\pi(n-1)})$ (of length $n-1$) is obuained from the sequence $\boldsymbol{l}\pi'_j = (\lambda_{\pi'_j(0)}, \cdots, \lambda_{\pi'_j(n)})$ (of length n) by omitting the last term. Therefore,

(22) $$[\boldsymbol{l}_j\pi] = [\boldsymbol{l}\pi'_j]_n, \quad 0 \leqq j \leqq n.$$

Also note that

(23) $$p_{\pi'_j} = \begin{cases} p_{\lambda_0\lambda_1} p_\pi, & j = 0, \\ p_\pi, & 1 \leqq j \leqq n. \end{cases}$$

Indeed, by (21), for $0 \leqq j \leqq n$, $p_{\pi'_j} = p_{\lambda_0\lambda_{\pi'_j(0)}\#} = p_{\lambda_0\lambda'_{\pi(0)}\#}$. If $j \geqq 1$ the first term of $\boldsymbol{l}_j$ is λ_0 and the first term of $\boldsymbol{l}_j\pi$ is $\lambda'_{\pi(0)}$. Therefore, $p_\pi = p_{\lambda_0\lambda'_{\pi(0)}\#}$ and we have $p_{\pi'_j} = p_\pi$. If $j = 0$, then the first term of $\boldsymbol{l}_0$ is λ_1 and the first term of $\boldsymbol{l}_0\pi$ is $\lambda'_{\pi(0)}$ so that $p_\pi = p_{\lambda_1\lambda'_{\pi(0)}\#}$ and $p_{\lambda_0\lambda_1} p_\pi = p_{\lambda_0\lambda'_{\pi(0)}\#} = p_{\pi'_0}$.

We therefore have, by (3),

(24) $$(-1)^{n-1} p_{\lambda_0\lambda_1}(vy)_{\boldsymbol{l}_0} = -\sum_{\pi' \in Q_0(n)} \operatorname{sgn}\pi' p_{\pi'} y_{[\boldsymbol{l}\pi']_n},$$

$$(25)\qquad (-1)^{n+j-1}(vy)_{l_j}=-\sum_{\pi'\in Q_j(n)}\operatorname{sgn}\pi' p_{\pi'}y_{[l\pi']_n},\quad 1\leqq j\leqq n.$$

Now notice that $P(n)=\bigcup_{j=0}^{n}Q_j(n)$ is a decomposition of $P(n)$ in disjoint sets. Therefore, by (16),

$$(26)\qquad (dvy_l)=(-1)^n\partial(vy)_l-\sum_{\pi\in P(n)}\operatorname{sgn}\pi p_\pi y_{[l\pi]_n}.$$

Since, by (3), $(-1)^n\partial(vy)_l$ equals the first sum in (6), we obtain the desired conclusion (5).

6. The homotopy $u_\# v\cong 1$.

In this section we will define a chain homotopy D on $C_\#(\boldsymbol{X}^*)$ such that

$$(1)\qquad (dDx)_{\boldsymbol{a}}+(Ddx)_{\boldsymbol{a}}=u_{\boldsymbol{a}}-(u_\# vx)_{\boldsymbol{a}}$$

for every $\boldsymbol{a}=(\alpha_0,\cdots,\alpha_n)\in\Lambda^{*n}$ and $x\in C_p(\boldsymbol{X}^*)$.

Note that

$$(2)\qquad (u_\# vx)_{\boldsymbol{a}}=(vx)_{\boldsymbol{a}}=\sum_{\pi\in P(n)}\operatorname{sgn}\pi\bar{p}_\pi x_{[\bar{\boldsymbol{a}}\pi]},$$

where $\bar{\boldsymbol{a}}=(\bar{\alpha}_0,\cdots,\bar{\alpha}_n)$,

$$(3)\qquad \bar{p}_\pi=p_{\alpha_0\overline{\alpha_{\pi(0)}}\#}.$$

In order to define D we need more notation. Let $n\geqq 0$, $\boldsymbol{a}=(\alpha_0,\cdots,\alpha_n)\in\Lambda^{*n}$, $0\leqq k\leqq n$, $\pi\in P(k)$. Then we put

$$(4)\qquad \bar{\boldsymbol{a}}\pi=(\overline{\alpha_{\pi(0)}},\cdots,\overline{\alpha_{\pi(k)}}),$$

$$(5)\qquad [\bar{\boldsymbol{a}}\pi]=(\{\overline{\alpha_{\pi(0)}}\},\cdots,\{\overline{\alpha_{\pi(0)}},\cdots,\overline{\alpha_{\pi(k)}}\}).$$

$$(6)\qquad \boldsymbol{a}(k)=(\alpha_k,\cdots,\alpha_n).$$

Since $\alpha_0\subseteqq\cdots\subseteqq\alpha_n$, we have $\bar{\alpha}_0\leqq\cdots\leqq\bar{\alpha}_n$ and therefore $\{\overline{\alpha_{\pi(0)}},\cdots,\overline{\alpha_{\pi(i)}}\}$, $0\leqq i\leqq k$, is a totally ordered finite set, hence, an element of Λ^*. Consequently, $[\bar{\boldsymbol{a}}\pi]\in\Lambda^{*k}$. Moreover, $\boldsymbol{a}(k)\in\Lambda^{*n-k}$ and $[\bar{\boldsymbol{a}}\pi]\boldsymbol{a}(k)\in\Lambda^{*n+1}$, because $\{\overline{\alpha_{\pi(0)}},\cdots,\overline{\alpha_{\pi(k)}}\}=\{\bar{\alpha}_0,\cdots,\bar{\alpha}_k\}\subseteqq\alpha_k$ E.g., if $\boldsymbol{a}=(\alpha_0,\alpha_1,\alpha_2)$, $k=1$ and π permutes 0 and 1, then $[\bar{\boldsymbol{a}}\pi]a(k)=(\{\bar{\alpha}_1\},\{\bar{\alpha}_0,\bar{\alpha}_1\},\alpha_1,\alpha_2)$. Note that $\bar{\boldsymbol{a}}\pi$ and $[\bar{\boldsymbol{a}}\pi]$ can be interpreted as 5.(1) and 5.(2) for $\boldsymbol{l}=(\bar{\alpha}_0,\cdots,\bar{\alpha}_k)$. For $x\in C_p(\boldsymbol{X}^*)$ we now put

$$(7)\qquad (Dx)_{\boldsymbol{a}}=(-1)^n\sum_{k=0}^{n}\sum_{\pi\in P(k)}(-1)^k\operatorname{sgn}\pi\bar{p}_\pi x_{[\bar{\boldsymbol{a}}\pi]\boldsymbol{a}(k)}.$$

LEMMA 2. *D is a chain homotopy connecting identity with uv, i.e., D satisfies* (1).

Note that $x_{[a\pi]a(k)} \in C_{p+n+1}(X^*_{\{\overline{\alpha_{\pi(0)}}\}}) = C_{p+n+1}(X_{\overline{\alpha_{\pi(0)}}})$ so that $\bar{p}_\pi x_{[a\pi]a(k)} \in C_{p+n+1}(X_{\bar{\alpha}_0}) = C_{p+n+1}(X^*_{\alpha_0})$ as desired.

In the verifiecation of formula (1) we omit the easier cases $n=0$ and $n=1$ and concentrate on $n \geqq 2$. By 3.(2) and (7), we have

$$(Dd x)_a = S_1 + S_2 + S_3, \tag{8}$$

where

$$S_1 = -\sum_{k=0}^{n} \sum_{\pi \in P(k)} (-1)^k \operatorname{sgn} \pi \bar{p}_\pi \partial(x_{[a\pi]a(k)}), \tag{9}$$

$$S_2 = \sum_{k=0}^{n} \sum_{\pi \in P(k)} (-1)^k \operatorname{sgn} \pi \bar{p}_\pi \bar{p}^*_\pi x_{([a\pi]a(k))_0}, \tag{10}$$

$$S_3 = \sum_{k=0}^{n} \sum_{j=1}^{n+1} \sum_{\pi \in P(k)} (-1)^{k+j} \operatorname{sgn} \pi \bar{p}_\pi x_{([a\pi]a(k))_j}, \tag{11}$$

$$\bar{p}^*_\pi = p^*_{\nu_0(\pi)\nu_1(\pi)\#}, \tag{12}$$

$$\nu_0(\pi) = \{\overline{\alpha_{\pi(0)}}\}, \quad \nu_1(\pi) = \begin{cases} \{\overline{\alpha_{\pi(0)}}, \overline{\alpha_{\pi(1)}}\}, & 1 \leqq k \leqq n, \\ \alpha_0, & k=0 \end{cases} \tag{13}$$

(cf. with 5.(7) and 5.(8)).

For $k=0$ the only permutation of $\{0\}$ is the identity so that $[\bar{a}\pi] = [\bar{a}] = \{\bar{\alpha}_0\}$ and thus $([\bar{a}\pi]a(0))_0 = a(0) = a$. Moreover, $\bar{p}_\pi = id$, $\bar{p}^*_\pi = id$. Therefore, the first term in S_2 equals x_a. The sum of all the remainig terms of S_2 equals 0, because we will see that

$$\sum_{\pi \in P(k)} \operatorname{sgn} \pi \bar{p}_\pi \bar{p}^*_\pi x_{([a\pi]a(k))_0} = 0, \quad 1 \leqq k \leqq n. \tag{14}$$

This will prove that

$$S_2 = x_a. \tag{15}$$

Similarly, we will show that a part of the triple sum S_3, vanishes, because

$$\sum_{\pi \in P(k)} \operatorname{sgn} \pi \bar{p}_\pi x_{([a\pi]a(k))_j} = 0, \quad 1 \leqq j \leqq k-1,\ 2 \leqq k \leqq n. \tag{16}$$

In order to prove (14) and (16), we use some arguments from 5. In particular, since

$$([\bar{a}\pi]a(k))_j = [\bar{a}\pi]_j a(k), \quad 0 \leqq j \leqq k-1,\ \pi \in P(k), \tag{17}$$

5.(13) for $l = (\bar{\alpha}_0, \cdots, \bar{\alpha}_k)$ implies

$$([\bar{a}\pi]a(k))_j = ([\bar{a}\pi']a(k))_j, \quad \pi \in P_j(k),\ 0 \leqq j \leqq k-1. \tag{18}$$

Furthermore, 5.(14) and 5.(15) imply

$$\bar{p}_\pi = \bar{p}_{\pi'}, \quad \pi \in P_j(k), \quad 1 \leqq j \leqq k-1, \tag{19}$$

(20) $$\bar{p}_\pi \bar{p}_\pi^* = \bar{p}_{\pi'} \bar{p}_{\pi'}^*, \quad \pi \in P_0(k).$$

Since $\operatorname{sgn} \pi' = -\operatorname{sgn} \pi$ and the permutations of $P(k)$ come in pairs π, π', where $\pi \in P_j(k)$, we conclude that (14) and (16) hold indeed.

The summation in S_3 is over the set $\{(k, j): 0 \leqq k \leqq n, 1 \leqq j \leqq n+1\}$, which decomposes in the following subsets of $\boldsymbol{Z} \times \boldsymbol{Z}$:

$$U_1 = \{(k, j): 2 \leqq k \leqq n, 1 \leqq j \leqq k-1\},$$

$$U_2 = \{(k, j): 1 \leqq k \leqq n, j = k\},$$

$$U_3 = \{(k, j): 0 \leqq k \leqq n-1, j = k+1\},$$

$$U_4 = \{(n, n+1)\},$$

$$U_5 = \{(k, j): 0 \leqq k \leqq n-1, k+2 \leqq j \leqq n+1\},$$

Denote the part of S_3 corresponding to U_i by S_3^i. Then

(21) $$S_3 = \sum_{j=1}^{5} S_3^i.$$

Now (16) implies

(22) $$S_3^1 = 0.$$

The terms of S_3^4 equal

(23) $$-\operatorname{sgn} \pi \bar{p}_\pi x_{[\boldsymbol{a}\pi]}, \quad \pi \in P(n),$$

because

(24) $$([\bar{\boldsymbol{a}}\pi] a(n))_{n+1} = ([\bar{\boldsymbol{a}}\pi] \alpha_n)_{n+1} = [\bar{\boldsymbol{a}}\pi].$$

Hence, by (2),

(25) $$S_3^4 = -(u_{\#} v x)_{\boldsymbol{a}}.$$

We will now consider $S_3^2 + S_3^3$. If we replace in the expression for S_3^2 the summation index k by $k+1$, we obtain

(26) $$S_3^2 = \sum_{k=0}^{n-1} \sum_{\pi \in P(k+1)} \operatorname{sgn} \pi \bar{p}_\pi x_{([\boldsymbol{a}\pi]\boldsymbol{a}(k+1))_{k+1}}.$$

On the other hand,

(27) $$S_3^3 = -\sum_{k=0}^{n-1} \sum_{\pi \in P(k)} \operatorname{sgn} \pi \bar{p}_\pi x_{([\boldsymbol{a}\pi]\boldsymbol{a}(k))_{k+1}}.$$

With every permutation $\pi \in P(k)$ we now associate a permutation $\pi' \in Q_{k+1}(k+1)$ such that $\pi'(i) = \pi(i)$ for $0 \leqq i \leqq k$ and $\pi'(k+1) = k+1$ (see 5. (18)). Note that

(28) $$[\bar{\boldsymbol{a}}\pi'] \boldsymbol{a}(k+1) = [\bar{\boldsymbol{a}}\pi], \{\overline{\alpha_{\pi(0)}}, \cdots, \overline{\alpha_{\pi(k)}}, \overline{\alpha_{k+1}}\}, \alpha_{k+1}, \cdots, \alpha_n,$$

(29) $$[\bar{\boldsymbol{a}}\pi]\boldsymbol{a}(k)=[\bar{\boldsymbol{a}}\alpha],\ \alpha_k,\ \alpha_{k+1},\ \cdots,\ \alpha_n,$$

so that

(30) $$([\bar{\boldsymbol{a}}\pi']\boldsymbol{a}(k+1))_{k+1}=([\bar{\boldsymbol{a}}\pi]\boldsymbol{a}(k))_{k+1}.$$

We also have $\operatorname{sgn}\pi'=\operatorname{sgn}\pi$ and $\bar{p}_{\pi'}=\bar{p}_\pi$, because $\pi'(0)=\pi(0)$, Since $\pi\mapsto\pi'$ is a bijection $P(k)\to Q_{k+1}(k+1)\subseteq P(k+1)$ and $P(k+1)=\bigcup_{j=0}^{k+1}Q_j(k+1)$, we conclude that

(31) $$S_3^2+S_3^3=\sum_{k=0}^{n-1}\sum_{j=0}^{k}\sum_{\pi\in Q_j(k+1)}\operatorname{sgn}\pi\,\bar{p}_\pi x_{([\bar{\boldsymbol{a}}\pi]\boldsymbol{a}(k+1))_{k+1}}.$$

We will now show that

(32) $$S_3^5=-\sum_{k=0}^{n-1}\sum_{j=k+1}^{n}\sum_{\pi\in P(k)}(-1)^{k+j}\operatorname{sgn}\pi\,\bar{p}_\pi x_{([\bar{\boldsymbol{a}}_j\pi]\boldsymbol{a}_j(k))}.$$

Let $0\leqq k\leqq n$, $\pi\in P(k)$, and let $k+1\leqq j\leqq n+1$. Then

(33) $$([\bar{\boldsymbol{a}}\pi]\boldsymbol{a}(k))_j=[\bar{\boldsymbol{a}}\pi](\boldsymbol{a}(k))_{j-k-1}.$$

On the other hand, if $0\leqq k\leqq n-1$, $\pi\in P(k)$ and $k+1\leqq j\leqq n$, then

(34) $$[\bar{\boldsymbol{a}}_j\pi]\boldsymbol{a}_j(k)=[\bar{\boldsymbol{a}}\pi](\boldsymbol{a}(k))_{j-k},$$

because $[\bar{\boldsymbol{a}}_j\pi]=[\bar{\boldsymbol{a}}\pi]$. Consequently, (33) and (34) imply

(35) $$([\bar{\boldsymbol{a}}\pi]\boldsymbol{a}(k))_j=[\bar{\boldsymbol{a}}_{j-1}\pi]\boldsymbol{a}_{j-1}(k),\quad 0\leqq k\leqq n-1,\ k+2\leqq j\leqq n+1,$$

and (32) follows.

We will now compute $(dDx)_{\boldsymbol{a}}$. By 3.(2) and (7) we see that

(36) $$(dDx)_{\boldsymbol{a}}=T_1+T_2+T_3,$$

where

(37) $$T_1=\sum_{k=0}^{n}\sum_{\pi\in P(k)}(-1)^k\operatorname{sgn}\pi\,\bar{p}_\pi\partial(x_{[\bar{\boldsymbol{a}}\pi]\boldsymbol{a}(k)}),$$

(38) $$T_2=\sum_{k=1}^{n-1}\sum_{\pi\in P(k)}(-1)^k\operatorname{sgn}\pi\,p_{\bar{a}_0\bar{a}_1\#}\bar{p}_\pi x_{[\bar{\boldsymbol{a}}_0\pi]\boldsymbol{a}_0(k)},$$

(39) $$T_3=\sum_{k=0}^{n-1}\sum_{j=1}^{n}\sum_{\pi\in P(k)}(-1)^{k+j}\operatorname{sgn}\pi\,\bar{p}_\pi x_{[\bar{\boldsymbol{a}}_j\pi]\boldsymbol{a}_j(k)}.$$

We see, by (9) and (37), that

(40) $$S_1+T_1=0.$$

We now decompose T_3 in two summands T_3^1, T_3^2 corresponding to the decomposition of the set $V=\{(k,j):\ 0\leqq k\leqq n-1,\ 1\leqq j\leqq n\}$ in sets

$$V_1=\{(k,j):\ 1\leqq k\leqq n-1,\ 1\leqq j\leqq k\},$$

$$V_2=\{(k, j): 0 \leqq k \leqq n-1,\ k+1 \leqq j \leqq n\}.$$

It follows from (32) that

(41) $$S_3^5+T_3^2=0.$$

Taking into account (15), (22), (25), (40) and (41), in order to prove (1), it remains to show that

(42) $$S_3^2+S_3^3=-(T_2+T_3^1).$$

We now analyze T_2 and T_3^1. Let $0 \leqq k \leqq n-1$. With every $\pi \in P(k)$ and $0 \leqq j \leqq k$ we associate a permutation $\pi_j' \in Q_j(k+1)$, defined by 5.(18) (with $k+1$ in place of n). Then (see 5.(20), 5.(22))

(43) $$\operatorname{sgn} \pi_j'=(-1)^{k+1-j} \operatorname{sgn} \pi, \quad 0 \leqq j \leqq k,$$

(44) $$[\bar{\boldsymbol{a}}_j \pi]=[\bar{\boldsymbol{a}} \pi_j']_{k+1}, \quad 0 \leqq j \leqq k.$$

Moreover, $\boldsymbol{a}_j(k)=\boldsymbol{a}(k+1)$ for $j \leqq k$ and (since $[\bar{\boldsymbol{a}} \pi_j']$ is of length $k+1$)

$$[\bar{\boldsymbol{a}} \pi_j']_{k+1} \boldsymbol{a}(k+1)=([\bar{\boldsymbol{a}} \pi_j'] \boldsymbol{a}(k+1))_{k+1}.$$

We thus obtain

(45) $$[\bar{\boldsymbol{a}}_j \pi] \boldsymbol{a}_j(k)=([\bar{\boldsymbol{a}} \pi_j'] \boldsymbol{a}(k+1))_{k+1}, \quad 0 \leqq j \leqq k.$$

Also note (see 5.(23)) that

(46) $$\bar{p}_{\pi_j'}=\begin{cases} p_{\bar{a}_0 \bar{a}_1 \#} \bar{p}_\pi, & j=0, \\ \bar{p}_\pi, & 1 \leqq j \leqq k. \end{cases}$$

Finally, observe that $\pi \mapsto \pi_j'$ is a bijection $P(k) \rightarrow Q_j(k+1)$. Therefore,

(47) $$T_2=-\sum_{k=0}^{n-1} \sum_{\pi' \in Q_0(k+1)} \operatorname{sgn} \pi' \bar{p}_{\pi'} x_{([\bar{\boldsymbol{a}} \pi'] \boldsymbol{a}(k+1))_{k+1}},$$

(48) $$T_3^1=-\sum_{k=1}^{n-1} \sum_{j=1}^{k} \sum_{\pi' \in Q_j(k+1)} \operatorname{sgn} \pi' \bar{p}_{\pi'} x_{[\bar{\boldsymbol{a}} \pi'] \boldsymbol{a}_j(k)}.$$

The summation in (31) is over the set $\{(k, j): 0 \leqq k \leqq n-1,\ 0 \leqq j \leqq k\}$, which decomposes in $\{(k, j): 0 \leqq k \leqq n-1,\ j=0\}$ and $\{(k, j): 1 \leqq k \leqq n-1,\ 1 \leqq j \leqq k\}$. Therefore, (31), (47) and (48) show that indeed (42) holds. This completes the proof of Lemma 2.

7. The homotopy $vu_\# \simeq 1$.

In §7 of [4] with every inverse system $\boldsymbol{X}$ a reduced chain complex $\hat{C}_\#(\boldsymbol{X})$ was defined. It was the restriction of $C_\#(\boldsymbol{X})$ to non-degenerate sequences $\boldsymbol{l}=(\lambda_0, \cdots, \lambda_n) \in \Lambda^n$. These are sequences such that $\lambda_i \leqq \lambda_{i+1}$ and $\lambda_i \neq \lambda_{i+1}$, $i=0, \cdots, n-1$.

There is a chain mapping $i: C_{\#}(\boldsymbol{X}) \to \hat{C}_{\#}(\boldsymbol{X})$ defined by

(1) $$(i(x))_{\boldsymbol{l}} = x_{\boldsymbol{l}}, \quad \boldsymbol{l} \in \hat{\Lambda}^n.$$

Also in §7 of [4], a chain mapping $r: \hat{C}_{\#}(\boldsymbol{X}) \to C_{\#}(\boldsymbol{X})$ was defined by

(2) $$(r(y))_{\boldsymbol{l}} = \begin{cases} y_{\boldsymbol{l}}, & \boldsymbol{l} \in \hat{\Lambda}^n, \\ 0, & \boldsymbol{l} \in \Lambda^n \setminus \hat{\Lambda}^n, \end{cases}$$

and it was shown that $ir=1$, $ri \cong 1$. The latter relation means that there exists a chain homotopy E such that

(3) $$dEx + Edx = x - rix, \quad x \in C_{\#}(X).$$

LEMMA 3. *The chain mapping $vu_{\#}: C_{\#}(\boldsymbol{X}) \to C_{\#}(\boldsymbol{X})$ satisfies*

(4) $$vu_{\#} ry = ry, \quad y \in \hat{C}_{\#}(\boldsymbol{X}).$$

PROOF. It suffices to show that for and $x \in C_p(\boldsymbol{X})$ and any $\boldsymbol{l} = (\lambda_0, \cdots, \lambda_n) \in \Lambda^n$ the chain $(vu_{\#}x - x)_{\boldsymbol{l}}$ is a finite sum of terms of the form $\pm p_{\lambda_0 \lambda'_0 \#} x_{\boldsymbol{l}'}$, where $\lambda_0 \leqq \lambda'_0$, $\boldsymbol{l}' = (\lambda'_0, \cdots, \lambda'_n) \in \Lambda^n \setminus \hat{\Lambda}^n$. Indeed, if this is the case, then for $x = ry$, $y \in \hat{C}_{\#}(\boldsymbol{X})$, one can express $(vu_{\#}ry - ry)_{\boldsymbol{l}}$, $\boldsymbol{l} \in \Lambda^n$, as a finite sum of terms of the form $\pm p_{\lambda_0 \lambda'_0 \#}(ry)_{\boldsymbol{l}'}$. Since, by (2), each of these terms vanishes, we conclude that (4) holds.

By 5. (3), we have

(5) $$(vu_{\#}x)_{\boldsymbol{l}} = \sum_{\pi \in P(n)} \operatorname{sgn} \pi p_{\pi} (u_{\#}x)_{[\boldsymbol{l}\pi]}, \quad \boldsymbol{l} \in \Lambda^n.$$

If π is the identity map, then $\boldsymbol{l}\pi = \boldsymbol{l}$, so that

(6) $$[\boldsymbol{l}\pi] = (\{\lambda_0\}, \cdots, \{\lambda_0, \cdots, \lambda_n\}), \quad \pi = id.$$

Since $\max\{\lambda_0, \cdots, \lambda_i\} = \lambda_i$, we conclude that

(7) $$\operatorname{sgn} \pi p_{\pi} (u_{\#}x)_{[\boldsymbol{l}\pi]} = x_{\boldsymbol{l}}, \quad \pi = id \in P(n).$$

Therefore, $(vu_{\#}x - x)_{\boldsymbol{l}}$ is a finite sum of terms of $\pm p_{\lambda_0 \lambda_{\pi(0)} \#}(ux)_{[\boldsymbol{l}\pi]}$, where $\pi \in P(n)$ and $\pi \neq id$. For any such π there exist indexes j, $0 \leqq j < n$, such that $\pi(j+1) < \pi(j)$. For the smallest such j we have $\pi(0) < \cdots < \pi(j)$, $\pi(j+1) < \pi(j)$ so that

(8) $$\max\{\lambda_{\pi(0)}, \cdots, \lambda_{\pi(j)}\} = \lambda_{\pi(j)} = \max\{\lambda_{\pi(0)}, \cdots \lambda_{\pi(j)}, \lambda_{\pi(j+1)}\}.$$

Since $[\boldsymbol{l}\pi]$ is of the form $(\nu_0, \cdots, \nu_n)$ with $\nu_i = \{\lambda_{\pi(0)}, \cdots, \lambda_{\pi(i)}\}$, we see that $\bar{\nu}_j = \bar{\nu}_{j+1}$, so that $\boldsymbol{l}' = (\bar{\nu}_0, \cdots, \bar{\nu}_n) \in \Lambda^n \setminus \hat{\Lambda}^n$. However, $(u_{\#}x)_{[\boldsymbol{l}\pi]} = (u_{\#}x)_{\nu_0 \cdots \nu_n} = x_{\bar{\nu}_0 \cdots \bar{\nu}} = x_{\boldsymbol{l}'}$ as desired.

LEMMA 4. *Let E be a homotopy satisfying* (3). *Then*

(9) $$D=vuE-E$$

is a chain homotopy satisgying

(10) $$dDx+Ddx=vux-x, \quad x\in C_p(\boldsymbol{X}).$$

PROOF. By (3) and (4), we have

$$dDx+Ddx=dvuEx+vuEdx$$
$$-dEx-Edx=vu(dEx+Edx)-(dEx+Edx)$$
$$=vu(x-rix)-(x-rix)$$
$$=(vux-eix)-(x-rix)=vux-x.$$

Lemmas 1, 2 and 4 complete the proof of Theorem 1.

8. Homomorphisms induced by arbitrary coherent maps.

Let $f:\boldsymbol{X}\to\boldsymbol{Y}$ be an arbitrary coherent map. In order to define the induced homomorphisms $f_*:H_p(\boldsymbol{X})\to H_p(\boldsymbol{Y})$ one proceeds as follows. Consider the cofinite systems $\boldsymbol{X}^*$ and $\boldsymbol{Y}^*$ associated with $\boldsymbol{X}$ and $\boldsymbol{Y}$ respectively and consider the chain mappings $u_{\boldsymbol{X}\#}:C_\#(\boldsymbol{X})\to C_\#(\boldsymbol{X}^*)$ and $u_{\boldsymbol{Y}\#}:C_\#(\boldsymbol{Y})\to C_\#(\boldsymbol{Y}^*)$. Let $v_{\boldsymbol{Y}\#}:C_\#(\boldsymbol{Y}^*)\to C_\#(\boldsymbol{X}^*)$ be the homotopy inverse of $u_{\boldsymbol{Y}\#}$. Let $f^*:\boldsymbol{X}^*\to\boldsymbol{Y}^*$ be the coherent mapping associated with f. Since $\boldsymbol{Y}^*$ is cofinite, there exists a special coherent map $f^+:\boldsymbol{X}^*\to\boldsymbol{Y}^*$, which is coherently homotopic to f^* (see Lemma 6.5 of [5]). We now take for f_* the homomorphism induced by the chain mapping

(1) $$f_\#=v_{\boldsymbol{Y}\#}f^+_\#u_{\boldsymbol{X}\#}.$$

The homomorphism f_* is independent of the choice of f^+. Indeed, for another choice f^+_1, one has $f^+\cong f\cong f^+_1$, so that $f_\#\cong v_{\boldsymbol{Y}\#}f^+_{1\#}u_{\boldsymbol{X}\#}$.

REMARK 4. If $f:\boldsymbol{X}\to\boldsymbol{Y}$ is a special coherent map, one can choose $f^+\cong f^*$ as in Remark 2. Then, by Remark 3, $f^+_\#u_{\boldsymbol{X}\#}=u_{\boldsymbol{Y}\#}f_\#$, and therefore

(2) $$v_{\boldsymbol{Y}\#}f^+_\#u_{\boldsymbol{X}\#}\cong f_\#.$$

This shows that for special f the new definition of f_* agrees with the previous one, given in 3.

REMARK 5. If $f, g:\boldsymbol{X}\to\boldsymbol{Y}$ are coherently homotopic coherent maps, then $f_\#, g_\#:C_\#(\boldsymbol{X})\to C_\#(\boldsymbol{Y})$ are chain homotopic chain mappings and therefore $f_*=g_*: H_p(\boldsymbol{X})\to H_p(\boldsymbol{Y})$.

In order to establish this assertion, it suffices to show that the associated coherent mappings $f^*, g^*: \boldsymbol{X}^* \to \boldsymbol{Y}^*$ (defined in § 4) are coherently homotopic. Let g be given by ψ and g_m and let $F: I\times\boldsymbol{X}\to\boldsymbol{Y}$ be a coherent homotopy from f to g, given by Φ and F_m. We associate with F the coherent map $F^*: I\times\boldsymbol{X}^* \to\boldsymbol{Y}^*$ as in § 4. It is given by Φ^* and $F^*_{\beta_0\cdots\beta_n}$. It is then straightforward to verify that F^* is a coherent homotopy from f^* to g^*.

REMARK 6. Let $f:\boldsymbol{X}\to\boldsymbol{Y}$ be a coherent map. If $\boldsymbol{Y}$ is cofinite, there exists a special coherent map $f_1:\boldsymbol{X}\to\boldsymbol{Y}$ such that $f\cong f_1$. It is a consequence of Remark 5 that $f_*=f_{1*}$. However, by Remark 4, f_{1*} can be obtained directly using the induced chain mapping $f_{1\#}$ of the special coherent map f_1.

REMARK 7. The definition of f_* for an arbitrary coherent map $f:\boldsymbol{X}\to\boldsymbol{Y}$ shows that also in this case the diagram 4. (15) commutes. Moreover, by Remark 6, one can replace in this diagram $f^{\ddagger}_*$ by f^*_*. This shows the naturality of the isomorphisms u_{X*}.

9. Homology of spaces using arbitrary ANR-resolutions.

Let $\boldsymbol{X}=(X_\lambda, p_{\lambda\lambda'}, \Lambda)$ be an inverse system and let $p: X\to\boldsymbol{X}$ be a morphism of pro-Top, i.e. a collection of maps $p_\lambda: X\to X_\lambda$ such that $p_{\lambda\lambda'}p_{\lambda'}=p_\lambda$ for $\lambda\leqq\lambda'$. We say that p is a resolution of X (see [7], (8) and [9]) provided the following two conditions are satisfied:

(R1) Let P be an ANR (for metric spaces), let $\mathcal{V}$ be an open covering of P and let $f: X\to P$ be a map. Then there exist a $\lambda\in\Lambda$ and a map $g: X_\lambda\to P$ such that gp_λ and f are $\mathcal{V}$-near maps.

(R2) Let P be an ANR and $\mathcal{V}$ an open covering of P. Then there exists an open covering $\mathcal{V}'$ of P such that whenever $\lambda\in\Lambda$ and $g, g': X_\lambda\to P$ are maps such that gp_λ and $g'p_\lambda$ are $\mathcal{V}'$-near maps, then there exists a $\lambda'\geqq\lambda$ such that $gp_{\lambda\lambda'}$ and $g'p_{\lambda\lambda'}$ are $\mathcal{V}$-near maps.

If all X_λ are ANR's we say that $p: X\to\boldsymbol{X}$ is an ANR-resolution.

Let $\boldsymbol{X}^*=(X^*_\alpha, p^*_{\alpha\alpha'}, \Lambda^*)$ be the cofinite system associated with $\boldsymbol{X}$ described in 4. We define $p^*_\alpha: X\to X^*_\alpha$, $\alpha\in\Lambda^*$, by $p^*_\alpha=p_{\bar\alpha}: X\to X_{\bar\alpha}=X^*_\alpha$. Note that the maps p^*_α, $\alpha\in\Lambda^*$, define a morphism $p^*: X\to\boldsymbol{X}^*$ of pro-Top, because

$$p^*_{\alpha\alpha'}p^*_{\alpha'}=p^*_\alpha \quad \text{for } \alpha\leqq\alpha'.$$

THEOREM 2. *If $p: X\to\boldsymbol{X}$ is an arbitrary resolution (ANR-resolution) of the space X, then $p^*: X\to\boldsymbol{X}^*$ is also a resolution (ANR-resolution) of X and*

$$u_X p=p^*. \tag{3}$$

PROOF. (1) is an immediate consequence of the definitions.

In order to verify (R1), consider an ANR P, an open covering $\mathcal{V}$ of P and a map $f: X \to P$. Choose $\lambda \in \Lambda$ and $g: X_\lambda \to P$ as in (R1) for p. If we put $\alpha = \{\lambda\} \in \Lambda^*$, then $X^*_\alpha = X_\lambda$, $p^*_\alpha = p_\lambda$ and $gp^*_\alpha = gp_\lambda$ is $\mathcal{V}$-near f.

In order to verify (R2), consider $P \in$ ANR and an open covering $\mathcal{V}$ of P. Choose $\mathcal{V}'$ as in (R2) for p. We claim that $\mathcal{V}'$ also satisfies (R2) for p^*.

Indeed, let $\alpha \in \Lambda^*$ and let $g, g': X^*_\alpha \to P$ be maps such that gp^*_α and $g'p^*_\alpha$ are $\mathcal{V}'$-near. Since $gp^*_\alpha = gp_{\bar{\alpha}}$, $g'p^*_\alpha = g'p_{\bar{\alpha}}$, we conclude that there is a $\lambda' \in \Lambda$, $\lambda' \geqq \bar{\alpha}$, such that $gp_{\bar{\alpha}\lambda'}$ and $g'p_{\bar{\alpha}\lambda'}$ are $\mathcal{V}$-near maps. Put $\alpha' = \alpha \cup \{\lambda'\}$. Clearly, $\alpha' \in \Lambda^*$ and $\bar{\alpha}' = \lambda'$. Since $p^*_{\alpha\alpha'} = p_{\bar{\alpha}\bar{\alpha}'} = p_{\bar{\alpha}\lambda'}$, we conclude that $gp^*_{\alpha\alpha'}$ and $g'p^*_{\alpha\alpha'}$ are $\mathcal{V}$-near maps.

REMARK 8. Let X be an arbitrary space and let $p: X \to \boldsymbol{X}$ be an ANR-resolution of $\boldsymbol{X}$. By definition [1], [6], the homology group $H^S_p(X)$ of the space X can be identified with the homology group $H_p(\boldsymbol{X}^*)$ of the cofinite ANR-resolution $\boldsymbol{X}^*$. However, by Theorem 1, $u_{\boldsymbol{X}^*}$ establishes a natural isomorphism $H_p(\boldsymbol{X}) \to H_p(\boldsymbol{X}^*)$. Therefore, $H^S_p(X)$ can also be identified with the homology group $H_p(\boldsymbol{X})$, where $p: X \to \boldsymbol{X}$ is an arbitrary (non-cofinite) ANR-resolution of the space X.

10. Eliminating the assumption of anti-symmetry.

In this section we assume that Λ is a directed set, which need not be anti-symmetric. If $\lambda_0 \leqq \lambda_1$ and $\lambda_1 \leqq \lambda_0$, we put $\lambda_0 \sim \lambda_1$. Clearly, $\sim$ is an equivalence relation. Let $\Lambda' \subseteqq \Lambda$ be a subset of Λ which contains precisely one element from every equivalence class of Λ with respect to $\sim$. The set Λ' is directed and antisymmetric

With every system $\boldsymbol{X} = (X_\lambda, p_{\lambda_0\lambda_1}, \Lambda)$ we now associate its restriction $\boldsymbol{X}' = (X_\lambda, p_{\lambda_0\lambda_1}, \Lambda')$ to $\Lambda' \subseteqq \Lambda$. We then define a map of systems $s = s_{\boldsymbol{X}}: \boldsymbol{X} \to \boldsymbol{X}'$ by the inclusion map $\Lambda' \to \Lambda$ and by the identity maps $s_\lambda = id: X_\lambda \to X_\lambda$. The induced chain mappings $s_\#: C_\#(\boldsymbol{X}) \to C_\#(\boldsymbol{X}')$ is given by

$$(1) \qquad (s_\# x)_{\lambda_0 \cdots \lambda_n} = x_{\lambda_0 \cdots \lambda_n}, \quad (\lambda_0, \cdots, \lambda_n) \in \Lambda'^n.$$

With every coherent map $f: \boldsymbol{X} \to \boldsymbol{Y}$ we associate a coherent map $f': \boldsymbol{X}' \to \boldsymbol{Y}'$. If f is given by φ and $f_{\mu_0 \cdots \mu_n}$, then f' is given by φ' and $f'_{\mu_0 \cdots \mu_n}$ defined as follows: $\varphi'(\boldsymbol{m})$, $\boldsymbol{m} = (\mu_0, \cdots, \mu_n) \in M^n$, is the only element of Λ' such that $\varphi'(\boldsymbol{m}) \sim \varphi(\boldsymbol{m})$. The mapping $f'_{\boldsymbol{m}}: \Delta^n \times X_{\varphi'(\boldsymbol{m})} \to Y_{\mu_0}$ is given by

$$(2) \qquad f'_{\boldsymbol{m}}(t, x) = f_{\boldsymbol{m}}(t, p_{\varphi(\boldsymbol{m})\varphi'(\boldsymbol{m})}(x)).$$

REMARK 9. If $f: X \to Y$ is special, then also $f': X' \to Y'$ is special and the induced chain mappings $f_\#, f'_\#$ satisfy the naturality condition

$$f'_\# s_{X\#} \cong s_{Y\#} f_\# . \tag{3}$$

To see this first notice that $\varphi'(m) \sim \varphi(m) = \varphi(\mu_n) \sim \varphi'(\mu_n)$, so that $\varphi'(m) = \varphi'(\mu_n)$. Furthermore, if $\mu_0 \leqq \mu_1$, then $\varphi'(\mu_0) \sim \varphi(\mu_0) \leqq \varphi(\mu_1) \sim \varphi'(\mu_1)$, so that φ' increases. This proves that f' is also a special coherent map.

To verify (3) we first consider the special coherent map $f_1: X \to Y'$, given by φ and f_m, $m \in M'^n$. The induced chain map $f_{1\#}: C_\#(X) \to C_\#(Y')$ satisfies

$$f_{1\#} = s_{Y\#} f_\# . \tag{4}$$

We then consider the special coherent map $f_2: X \to Y'$, given by φ' and (2), and observe that

$$f_{2\#} = f'_\# s_{X\#} . \tag{5}$$

Finally, since $\varphi \leqq \varphi'$, f_1 and f_2 are coherently homotopic (even congruent in the sense of § 5 of [5]) so that (see § 3).

$$f_{1\#} \cong f_{2\#} . \tag{6}$$

THEREM 3. *$s_{X\#}: C_\#(X) \to C_\#(X')$ is a chain equivalence and, therefore, $s_{X*}: H_p(X) \to H_p(X')$ is an isomorphism.*

PROOF. We define the inverse chain mapping $w = w_X: C_\#(X') \to C_\#(X)$ by

$$(wy)_{\lambda_0 \cdots \lambda_n} = p_{\lambda_0 \lambda'_0 \#} y_{\lambda'_0 \cdots \lambda'_n} , \tag{7}$$

where λ' is the only element of Λ' such that $\lambda \sim \lambda'$. Clearly,

$$(s_\# wy)_{\lambda_0 \cdots \lambda_n} = (wy)_{\lambda_0 \cdots \lambda_n} = y_{\lambda_0 \cdots \lambda_n} , \tag{8}$$

when $\lambda_0, \cdots, \lambda_n \in \Lambda'$. Therefore, $s_\# w = 1$.

On the other hand,

$$(ws_\# x)_{\lambda_0 \cdots \lambda_n} = p_{\lambda_0 \lambda'_0 \#} x_{\lambda'_0 \cdots \lambda'_n} , \tag{9}$$

so that $sw_\#$ can differ from 1. However,

$$ws_\# \cong 1 . \tag{10}$$

To establish (10), we put

$$(-1)^n (Dx)_{\lambda_0 \cdots \lambda_n} = \sum_{k=0}^{n} (-1)^k x_{\lambda_0 \cdots \lambda_k \lambda'_k \cdots \lambda'_n} . \tag{11}$$

We will now verify

$$(dDx + Ddx)_{\lambda_0 \cdots \lambda_n} = (ws_\# x - x)_{\lambda_0 \cdots \lambda_n} . \tag{12}$$

We concentrate on the case $n \geqq 2$. If $\boldsymbol{l}=(\lambda_0, \cdots, \lambda_n)$, $n \geqq 2$, and $0 \leqq k \leqq n$, we have

(13) $$(Ddx)_{\lambda_0 \cdots \lambda_n} = \sum_{i=0}^{9} A_i,$$

where

$$A_1 = -\sum_{k=0}^{n} (-1)^k \partial(x_{\lambda_0 \cdots \lambda_k \lambda'_k \cdots \lambda'_n}),$$

(14) $$A_2 = p_{\lambda_0 \lambda'_0 \#} x_{\lambda'_0 \cdots \lambda'_n} = (wsx)_{\lambda_0 \cdots \lambda_n},$$

$$A_3 = \sum_{k=1}^{n} (-1)^k p_{\lambda_0 \lambda_1 \#} x_{\lambda_1 \cdots \lambda_k \lambda'_k \cdots \lambda'_n},$$

$$A_4 = \sum_{k=2}^{n} \sum_{j=1}^{k-1} (-1)^{k+j} x_{\lambda_0 \cdots \hat{\lambda}_j \cdots \lambda_k \lambda'_k \cdots \lambda'_n},$$

$$A_5 = \sum_{k=1}^{n} x_{\lambda_0 \cdots \lambda_{k-1} \lambda'_k \cdots \lambda'_n},$$

$$A_6 = -\sum_{k=0}^{n-1} x_{\lambda_0 \cdots \lambda_k \lambda'_{k+1} \cdots \lambda'_n},$$

$$A_7 = \sum_{k=0}^{n-1} \sum_{j=k+1}^{n-1} (-1)^{k+j+1} x_{\lambda_0 \cdots \lambda_k \lambda'_k \cdots \hat{\lambda}'_j \cdots \lambda'_n},$$

$$A_8 = \sum_{k=0}^{n-1} (-1)^{k+n+1} x_{\lambda_0 \cdots \lambda_k \lambda'_k \cdots \lambda'_{n-1}},$$

(15) $$A_9 = -x_{\lambda_0 \cdots \lambda_n}.$$

We see immediately that

(16) $$A_5 + A_6 = 0.$$

On the other hand, we have

(17) $$(dDx)_{\lambda_0 \cdots \lambda_n} = \sum_{i=1}^{5} B_i,$$

where

$$B_1 = \sum_{k=0}^{n} (-1)^k \partial(x_{\lambda_0 \cdots \lambda_k \lambda'_k \cdots \lambda'_n}),$$

$$B_2 = p_{\lambda_0 \lambda_1 \#} \sum_{k=1}^{n} (-1)^{k+1} x_{\lambda_1 \cdots \lambda_k \lambda'_k \cdots \lambda'_n},$$

$$B_3 = \sum_{k=1}^{n-1} \sum_{j=1}^{k} (-1)^{k+j} x_{\lambda_0 \cdots \hat{\lambda}_j \cdots \lambda_{k+1} \lambda'_{k+1} \cdots \lambda'_n},$$

$$B_4 = \sum_{k=0}^{n-1} \sum_{j=k+1}^{n-1} (-1)^{k+j} x_{\lambda_0 \cdots \lambda_k \lambda'_k \cdots \hat{\lambda}'_j \cdots \lambda'_n},$$

$$B_5=\sum_{k=0}^{n-1}(-1)^{k+n}x_{\lambda_0\cdots\lambda_k\lambda'_k\cdots\lambda'_{n-1}}.$$

We see that

$$A_1+B_1=0,\quad A_3+B_2=0,\quad A_4+B_3=0,\quad A_7+B_4=0,\quad A_8+B_5=0.$$

Therefore, (12) follows from (13)-(17).

REMARK 10. We can now define the induced homomorphism $f_*: H_p(\boldsymbol{X})\rightarrow H_p(\boldsymbol{Y})$ of an arbitrary coherent map $f:\boldsymbol{X}\rightarrow\boldsymbol{Y}$ between systems, which need not be antisymmetric. By definition, f_* is the homomorphism induced by the chain mapping $f_\#=w_{Y\#}f'_\# s_{X\#}: C_\#(\boldsymbol{X})\rightarrow C_\#(\boldsymbol{Y})$. Clearly, f_* satisfies the naturality condition

$$s_{Y*}f_*=f'_*s_{X*}. \tag{18}$$

Moreover, $f\cong f_1$ implies $f_*=f_{1*}$. Also note that whenever f is special, f_* can be obtained directly from the chain mapping induced by f as in 3. This is a consequence of Remark 9.

REMARK 11. If $p: X\rightarrow\boldsymbol{X}$ is a resolution (ANR-resolution) of the space X, then we define a morphism $p': X\rightarrow\boldsymbol{X}'$ of pro-Top by putting $p'_\lambda=p_\lambda$ for $\lambda\in\Lambda'$. That p' is also a resolution (ANR-resolution) of X is obvious. Moreover, $s_X p=p'$. Since s_{X*} is a natural isomorphism, one can identify $H_p(\boldsymbol{X}')$ with $H_p(\boldsymbol{X})$. This and Remark 8 show that the homology group $H_p^S(X)$ of the space X can be identified with $H_p(\boldsymbol{X})$, where $p: X\rightarrow\boldsymbol{X}$ is an arbitrary ANR-resolution of X (Λ need not even be anti-symmetric).

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