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ON COMPACTA WHICH ARE l-EQUIVALENT TO I^n

By

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1. Introduction.

All spaces considered in this paper are assumed to be *metrizable*. A compactum is a compact space. A continuum is a connected compactum, and a mapping is a continuous function. For a space X we denote by $C(X)$ the space of all real-valued mappings on X with the topology of *uniform convergence*. Then by Milutin's interesting work [8], we have known that for each pair of uncountable compacta X and Y, $C(X)$ is linearly isomorphic to $C(Y)$ (see [12] for the details and generalizations). On the other hand, for space X we denote by $C_p(X)$ the space of all real-valued mappings on X with the topology of *pointwise convergence*. Spaces X and Y are said to be *l-equivalent* [1] provided that $C_p(X)$ is linearly isomorphic to $C_p(Y)$, written $C_p(X)\cong C_p(Y)$. Recently, Pavlovskiĭ [11] showed the following.

1.1. THEOREM. (1) *If locally compact spaces X and Y are l-equivalent, then for each non-empty open or closed set $\tilde{X}$ of X, there exists a non-empty open set in $\tilde{X}$ which can be embedded in Y. Therefore,* $\dim X=\dim Y$ (*see also* [4] *and* [13]).

(2) *Non-zero-dimensional compact polyhedra P and Q are l-equivalent if and only if* $\dim P=\dim Q$.

(3) *Let B be the Pontryagin's 2-dimensional continum with the property* $\dim(B\times B)=3$. *Then B is not l-equivalent to I^2, where I is the unit interval* $[0, 1]$.

Being motivated by Theorem 1.1 (2), readers may consider that for $n\geqq 1$, all n-dimensional compact ANR's are l-equivalent to I^n. However, by Theorem 1.1 (1) and [3, Theorem VI. (6.1)], we can easily see that *for each $n\geqq 1$, there exists a collection of $2^{\aleph_0}$ n-dimensional compact AR's in R^{n+1} which are not l-equivalent to each other.* On the other hand, let X be a compactification of the half-open interval $[0, 1)$ whose remainder is I^n. Then X is l-equivalent to I^n, although X is not even locally connected. Therefore it seems to be difficult to

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control n-dimensional compacta which are l-equivalent to I^n.

In this paper we will show a criterion of an n-dimensional locally compact space which is l-equivalent to an n-manifold. Concerning 1-dimensional compacta, Lelek [7] introduced the class of finitely Suslinian compacta, which contains all hereditarily locally connected continua, and therefore all 1-dimensional comapct ANR's. We will also show a simple criterion of a curve (=1-dimensional continuum) which is l-equivalent to a finitely Suslinian compactum. Hence we can easily see that neither the Cantor fan nor the Knaster indecomposable curve are l-equivalent to any finitely Suslinian compacta. Moreover, we will investigate a class of curves which are l-equivalent to I. So we have a desired class of special comapct ANR's which contains all graphs, and show that every continuum which is a one-to-one continuous image of $[0, \infty)$ is l-equivalent to I.

Most of our results can be applied to the theory of free topological groups in the sense of Graev [5]. So we may have interesting examples concerning free topological groups in the sense of Graev.

We denote by $\dim X$ the *covering dimension* of a space X. Let A be a subset of a space X. We denote its *interior* and *closure* in X by *int* A and *cl* A, respectively. The symbol ANR is used to specify an *absolute neighborhood retract* for the class of all metric spaces. Undefined terms and notations in continuum theory may be found in [6] and [14].

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2. Criterions for being l-equivalent to special spaces.

First, we will discuss a compactum which is l-equivalent to I^n. A space X is *locally contractible at a point* x of X if for every open neighborhood U of x in X, there exists an open neighborhood V of x in X such that $V \subset U$ and V is contractible in U. We denote the set of all points of X at which X are locally contractible by $L_c(X)$. Now we have

2.1. THEOREM. *Let X be an n-dimensional locally compact space and $\tilde{X}$ be the closure of the set of all points of X whose local dimensions are exactly n. If X is l-equivalent to an n-manifold, then $L_c(\tilde{X})$ is dense in $\tilde{X}$.*

PROOF. Note that $\dim A = n$ for any non-empty open subset A of $\tilde{X}$. Suppose that X is l-equivalent to an n-manifold M. First, we show that for an arbitrary open subset U of $\tilde{X}$, there is an open subset of U which is contractible in U. By Theorem 1.1 (1), there exists a non-empty open subset V of

U and there exist maps $f: V \to M$ and $g: f(V) \to V$ such that $gf = 1_V$. Since $f(V)$ is the n-dimensional subset of M, $int\, f(V) \neq \emptyset$. Hence there is a point x_0 of V and there is an open subset W of M such that $f(x_0) \in W \subset cl\, W \subset int\, f(V)$ and $cl\, W$ is homeomorphic to I^n. Particularly, W is contractible in $f(V)$, and therefore there is a homotopy $G: W \times I \to f(V)$ such that $g(y, 0) = y$ and $G(y, 1) = f(x_0)$ for all $y \in W$. Take an open subset V_0 in V such that $x_0 \in V_0$ and $f(V_0) \subset W$ and define a homotopy $H: V_0 \times I \to U$ by $H(x, t) = gG(f(x), t)$ for $(x, t) \in V_0 \times I$. Then $H(x, 0) = x$ and $H(x, 1) = x_0$ for all $(x, t) \in V_0 \times I$. Hence V_0 is contractible in U.

Next, we show that $L_c(\tilde{X})$ is dense in $\tilde{X}$. Let U an arbitrary non-empty open subset of $\tilde{X}$. By the first part of the proof, we have a sequence $\{U_n\}_{n \geq 0}$ of non-empty open subsets of $\tilde{X}$ such that for every $n = 0, 1, 2, \cdots$,

(1) $cl\, U_{n+1} \subset U_n$, where $U_0 = U$

(2) $\operatorname{diam}[U_n] < \dfrac{1}{n}$, and

(3) U_{n+1} is contractible in U_n.

Then by (1) and (2), we have a point $x_* \in \bigcap_{n \geq 0} U_n \subset U$, and by (2) and (3), we can see that $x_* \in L_c(\tilde{X})$. Therefore $L_c(\tilde{X})$ is dense in $\tilde{X}$.

2.2. COROLLARY. *Let X be an n-dimensional compactum and $\tilde{X}$ be the closure of the set of all points of X whose local dimensions are exactly n. Then if X is l-equivalent to I^n, $L_c(\tilde{X})$ is dense in $\tilde{X}$.*

Next, we will consider the case of curves. A compactum X is *finitely Suslinian* [7] if for every $\varepsilon > 0$, each collection of pairwise disjoint subcontinua of X having diameters greater than ε is finite. We note that every finitely Suslinian continuum is at most 1-dimensional, and that every hereditarily locally connected continuum is finitely Suslinian. Hence every 1-dimensional compact ANR is finitely Suslinian, and there exist finitely Suslinian compacta which are not ANR's. In order to show a criterion of a curve which is l-equivalent to I, we introduce a notation as follows. A space X is *locally connected at a point x* of X if for every open neighborhood U of x in X, there exists a connected open neighborhood V of x in U. By $L(X)$, we denote the set of all points of X at which X is locally connected. Clearly a space X is locally connected if and only if $L(X) = X$. Then we have

2.3. THEOREM. *If a curve X is l-equivalent to a finitely Suslinian compactum, then the following conditions are satisfied:*

(i) *$L(X)$ is dense in X, and*
(ii) *$L(X)$ has non-empty interior in X.*

PROOF. Suppose that X is l-equivalent to a finitely Suslinian compactum Y but $L(X)$ is not dense in X. Then there is a non-empty open subset U of X such that $U\cap L(X)=\emptyset$. By Theorem 1.1 (1), there is a non-empty open subset V of U such that $cl\,V\subset U$ and there exists an embedding $f: cl\,V\rightarrow Y$. Since $V\cap L(X)=\emptyset$, by [14, Theorem I.12.1], there exist a positive number $\varepsilon>0$ and a sequence $K_0, K_1, K_2, \cdots$ of pairwise disjoint subcontinua of $cl\,V$ such that

$$\operatorname{diam}[K_i]>\varepsilon \quad \text{for all } i\geqq 0, \text{ and } K_0=\operatorname{Lim}_i K_i.$$

Then the sequence $f(K_0), f(K_1), f(K_2), \cdots$ consists of pairwise disjoint subcontinua in Y and satisfies the following properies:

$$f(K_0)=\operatorname{Lim}_i f(K_i), \text{ and } \operatorname{diam}[f(K_0)]>0.$$

But this contradicts to the assumption that Y is finitely Suslinian, because $\operatorname{diam}[f(K_i)]\geqq 1/2\operatorname{diam}[f(K_0)]$ for almost all $i\geqq 1$. Namely, the curve X satisfies the condition (i).

If $int\,L(X)=\emptyset$, then $X-L(X)$ is dense in X. Hence we can similarly prove that the condition (ii) is satisfied.

2.4. COROLLARY. *Neither the Cantor fan nor the Knaster indecomposable curve* (see [6, Example 1, p. 204]) *are l-equivalent to any finitely Suslinian compactum.*

A space X has a *decomposable local system* if every non-empty open subset of X contains a non-degenarate decomposable continuum. For example, n-manifolds, polyhedra, hereditarily decomposable continua, the Knaster indecomposable curve, the dyadic solenoid have decomposable local system. By Theorem 1.1 (1), we can easily show the following.

2.5. LEMMA. *No compactum which has a decomposable local system is l-equivalent to any hereditarily indecomposable continuum.*

Considering the arc, the Knaster indecomposable curve and the pseudo-arc [2], by Corollary 2.4 and Lemma 2.5, we have.

2.6. COROLLARY. *There exist three arc-like continua which are not l-equivalent to each other.*

Finally, we will construct a finitely Suslinian continuum which is not locally contractible at any point. Namely, for a curve X, the density of $L(X)$ is a criterion for being l-equivalent to a finitely Suslinian compactum but is not one for being l-equivalent to I.

2.7. EXAMPLE. Let S_0 be the unit circle in the plane R^2. Let $\{a_i | i \geqq 1\}$ be a countable dense subset of S_0. Then we can take a sequence $\{S_{1,i}\}_{i\geqq 1}$ of pairwise disjoint circles inside of S_0 satisfying the conditions;

(1) $S_0 \cap S_{1,i} = \{a_i\}$ for every $i \geqq 1$, and

(2) $\operatorname{diam}[S_{1,i}] \leqq \dfrac{1}{2^i}$ for every $i \geqq 1$.

Define

$$X_1 = S_0 \cup (\bigcup_{i\geqq 1} S_{1,i}).$$

For $n \geqq 1$, assume that we have constructed a sequence $\{S_{n,i}\}_{i\geqq 1}$ of pairwise disjoint circles and a continuum X_n of the form $X_{n-1} \cup (\bigcup_{i\geqq 1} S_{n,i})$, where $X_0 = S_0$, such that for every $i \geqq 1$,

(3) $X_{n-1} \cap S_{n,i} = \{a_{n,i}\}$, $X_{n-2} \cap S_{n,i} = \emptyset$,

(4) $\operatorname{diam}[S_{n,i}] \leqq \dfrac{1}{n \cdot 2^i}$,

(5) $\{a_{n,i} | i \geqq 1\}$ is dense in X_{n-1}.

Then for every $i \geqq 1$, take a countable subset $\{b_{i,j} | j \geqq 1\}$ of $S_{n,i} - X_{n-1}$ which is dense in $S_{n,i}$. Further let us take a sequence $\{S_{n,i,j}\}_{j\geqq 1}$ of pairwise disjoint circles inside of $S_{n,i}$ such that for every $i \geqq 1$,

(6) $X_n \cap S_{n,i,j} = \{b_{i,j}\}$, and

(7) $\operatorname{diam}[S_{n,i,j}] \leqq \dfrac{1}{(n+1) \cdot 2^{i^2+j}}$.

Then define

$$X_{n+1} = X_n \cup [\bigcup_{i\geqq 1} (\bigcup_{j\geqq 1} S_{n,i,j})].$$

It is easily seen that X_{n+1} can be represented in the form which satisfies the inductive assumptions (3)-(5) in replacement of X_n by X_{n+1}. So we define a curve

$$X = \bigcup_{n\geqq 1} X_n.$$

Now we can rewrite X as follows;

$$Y_i = S_{1,i} \cup (\bigcup_{j\geqq 1} S_{1,i,j}) \cup (\bigcup_{j\geqq 1} \bigcup_{k\geqq 1} S_{1,i,j,k}) \cup \cdots \quad \text{for } i \geqq 1, \text{ and } X = \bigcup_{i\geqq 1} Y_i.$$

By the construction, every subcontinuum of X having diameter greater than $1/2^i$, which intersects Y_i, must contain a_i. Hence it is easily seen that X is finitely Suslinian. By the conditions (3)-(7), every non-empty open subset of X contains simple closed curves. Hence $L_c(X)=\emptyset$. Therefore the curve X is the required one.

3. Curves which are l-equivalent to I.

In this section we will show that certain curves are l-equivalent to I. We need the following lemma as elementary and key tools for calculations.

3.1. LEMMA (Pavlovskii [8]). (1) *For a closed subset S of I, $C_p(I)\cong C_p(S)\times C_p(I;S)$, where for a subset A of a space X, we define $C_p(X;A)=\{f\in C_p(X)\mid f(A)=0\}$, and if $A=\{a\}$, we write $C_p(X;A)=C_p(X;a)$.*

(2) *Let A be a closed subset of a space X, which is a neighborhood retract of X. Then $C_p(X)\cong C_p(A)\times C_p(X;A)$.*

(3) *Let X_1 and X_2 be closed subsets of a space X such that $X=X_1\cup X_2$, $X_0=X_1\cap X_2$ is a neighborhood retract of X and $C_p(X_0)\cong C_p(X_0)\times C_p(X_0)$. Then $C_p(X)\cong C_p(X_1)\times C_p(X_2)$.*

(4) *$C_p(I)\times C_p(I)\cong C_p(I)$.*

3.2. THEOREM. *Every dendrite (=1-dimensional compact AR) with finite ramification points is l-equivalent to I.*

PROOF. By Theorem 1.1 (2), we consider only a dendrite which is not a tree. Let X be a dendrite with ramification points $x_1, x_2, \cdots, x_n$. Let A be a tree in X which contains all x_i. Then by Lemma 3.1 (2) and (4),

$$\begin{aligned} C_p(X)&\cong C_p(A)\times C_p(X;A)\cong C_p(I)\times C_p(X/A;[A]) \\ &\cong C_p(I)\times R\times C_p(X/A;[A]) \\ &\cong C_p(I)\times C_p(X/A), \end{aligned}$$

where $[A]$ is the identification point of A in X/A. Since X/A is a dendrite with exactly one ramification point, by Lemma 3.1 (4), it suffices to show the case of dendrites with exactly one ramification point.

Let p be the pole (i.e., the origin) in the polar coordinate system in the plane R^2. Define in the polar coordinate (r, θ),

$$p_n=\left(\frac{1}{n}, \frac{1}{n}\right) \quad \text{for every } n\geqq 1,$$

and let

$$Y=\bigcup_{n\geqq 1}\overline{pp_n},$$

where $\overline{xy}$ stands for the straight line segment joining x and y. Now it is easily seen that every dendrite, which is not a tree and has exactly one ramification point, is homeomorphic to Y. Hence it suffices to prove that

(*) $C_p(Y)\cong C_p(I)$.

Let $S=\left\{0, 1, \frac{1}{2}, \frac{1}{3}, \cdots, \frac{1}{n}, \cdots\right\}$. Then by Lemma 3.1 (2),

$$C_p(I)\cong C_p(S)\times C_p(I\,;S)\cong R\times C_p(S\,;0)\times C_p(I\,;S)$$

We note that we can identify each $\alpha\in C_p(S\,;0)$ with the sequence $\{a_n\}_{n\geq 1}$ defined by $a_n=\alpha(1/n)$, which converges to 0. So for each $(\alpha, f)\in C_p(S\,;0)\times C_p(I\,;S)$, we define $\varphi(\alpha, f)\in C_p(Y\,;p)$ by the formula;

$$\varphi(\alpha, f)\left(r, \frac{1}{n}\right)=f\left(\frac{r+1}{n+1}\right)+nra_n \qquad \text{for each } r,\ 0\leqq r\leqq\frac{1}{n},\ n\geqq 1.$$

Namely, we have the continuous linear function $\varphi: C_p(S\,;0)\times C_p(I\,;S)\to C_p(Y\,;p)$. On the other hand, for each $g\in C_p(Y\,;p)$, $\psi_1(g)\in C_p(S\,;0)$ and $\psi_2(g)\in C_p(I\,;S)$ are defined as follows;

$$\psi_1(g)(t)=\begin{cases} g(p_n) & \text{if } t=\frac{1}{n} \text{ for some } n\geqq 1,\\ 0 & \text{if } t=0,\end{cases}$$

$$\psi_2(g)(t)=\begin{cases} g\left((n+1)t-1, \frac{1}{n}\right)+\{n-n(n+1)t\}g(p_n) & \\ \qquad\qquad \text{if } t\in\left[\frac{1}{n+1}, \frac{1}{n}\right] \text{ for some } n\geqq 1,\\ 0 \qquad\qquad \text{if } t=0.\end{cases}$$

Hence we have the continuous linear function $\psi: C_p(Y\,;p)\to C_p(S\,;0)\times C_p(I\,;S)$ given by $\psi(g)=(\psi_1(g), \psi_2(g))$. Then we can see that $\varphi\psi=1_{C_p(Y;p)}$ and $\psi\varphi=1_{C_p(S;0)\times C_p(I;S)}$. Hence $C_p(S\,;0)\times C_p(I\,;S)\cong C_p(Y\,;p)$. Therefore we have

(*) $C_p(I)\cong R\times C_p(S\,;0)\times C_p(I\,;S)\cong R\times C_p(Y\,;p)\cong C_p(Y)$.

3.3. COROLLARY. *Every* 1-*dimensional compact ANR with finite ramification points is l-equivalent to I.*

PROOF. Let X be a 1-dimensional compact ANR with finite ramification points. By Lemma 3.1 (4) and (3), we may assume that X is connected. We will prove by the induction on the number of simple closed curves in X. If there is no simple closed curve in X, then X is a dendrite. Hence by Theorem 3.2, the assertion holds.

Assume that the assertion holds for ANR's which has at most $n-1$ simple closed curves, where $n \geqq 1$. Let X be 1-dimensional compact ANR which has n simple closed curves. Take a simple closed curve L in X. Then X/L is a 1-dimensional compact ANR and has at most $n-1$ simple closed curves, because a 1-dimensional locally connected continuum with the finite Betti number is an ANR. Hence by the assumption, Theorem 1.1 (2) and Lemma 3.1,

$$\begin{aligned} C_p(X) &\cong C_p(L) \times C_p(X; L) \cong C_p(I) \times C_p(X/L; [L]) \\ &\cong C_p(I) \times C_p(X/L) \cong C_p(I) \times C_p(I) \\ &\cong C_p(I). \end{aligned}$$

Therefore X is also l-equivalent to I. The induction is completed.

3.4. COROLLARY. *Let X be a dendrite. If there exists an increasing finite sequence $X_0 \subset X_1 \subset \cdots \subset X_{n+1} = X$, $n \geqq 0$, of snbcontinua of X such that*

(1) *X has at most finite ramification points, and*

(2) *for $i=0, 1, \cdots, n$, the continuum X_{i+1}/X_i has at most finite ramification points,*

then X is l-equivalent to I.

Next, we will give other curves which are l-equivalent to I.

3.5. THEOREM. *Every continuum which is a one-to-one continuous image of $[0, \infty)$ is l-equivalent to I.*

PROOF. Let X be a continuum which admits a bijective map $f: [0, \infty) \to X$. Then by [9, Structure Theorem and its Remark], X can be written in the form $X = \alpha \cup C \cup L$, where α is an arc or a point, C is an arc-like continuum with at most two arc-components, L is an arc, $L \cap C$ is exactly the two non-cutpoints of L which are also opposite endpoints of C, and $\alpha \cap (C \cup L)$ is a single point of C which is a non-cutpoint of α and which, if C is not an arc (i.e., $C \cup L$ is not a simple closed curve), is the non-cutpoint not in $L \cap C$ of the arc-component of C which is an arc. In fact, by the proof, there are real numbers $0 \leqq a \leqq b < c$ such that $\alpha = f([0, a])$, $C = f([a, b]) \cup f([c, \infty))$ and $L = f([b, c])$.

If $a = b$, namely, $C \cup L$ is a simple closed curve, by Theorem 1.1 (2), X is l-equivalent to I. So we may assume that $a < b$. Let define

$$X_1 = \alpha \cup C,$$

and

$$X_2 = f([0, d]), \quad \text{where } d \text{ is an arbirary real number with } d > c.$$

Then by Lemma 3.1 (2) and (4),

$$\begin{aligned} C_p(X_1) &\cong C_p(f([0, b]))\times C_p(X_1/f([0, b]) ; [f([0, b])]) \\ &\cong C_p(I)\times C_p(I ; 0) \\ &\cong C_p(I) \end{aligned}$$

Note that $X=X_1\cup X_2$ and $X_0=X_1\cap X_2$ is a disjoint union of two arcs. Hence by Lemma 3.1 (3) and (4),

$$C_p(X)\cong C_p(X_1)\times C_p(X_2)\cong C_p(I)\times C_p(I)\cong C_p(I)$$

Therefore such a curve X is l-equivalent to I.

3.6. COROLLARY. *Every continuum which is a one-to-one continuous image of the real line R is l-equivalent to I.*

Curves described in Theorem 3.5 and Corollary 3.6 are called *half-real curves* and *real curves,* respcetively [10]. By Theorem 3.5 and Corollary 3.6, we see that the property of being l-equivalent to I does not imply even local connectivity. Hence Theorem 2.1 and Theorem 2.3 may be interesting properties. As mentioned in Introduction, for each $n\geqq 1$, there exist uncountable many n-dimensional compact AR's which are not l-equivalent to each ohther. Hence characterizatios of continua or compact AR's which are l-equivalent to I^n are important. In the case of curves we pose the following problem related to our result;

PROBLEM. *Characterize dendrites which are l-equivalent to I. Particularly, is the converse of Corollary* 3.4 *valid*?

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