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# ON COMPLETE HYPERSURFACES WITH HARMONIC CURVATURE IN A RIEMANNIAN MANIFOLD OF CONSTANT CURVATURE

Dedicated to Professor Morio Obata on his 60th birthday

By

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## 0. Introduction.

This paper is concerned with hypersurfaces with harmonic curvature in a Riemannian manifold of constant curvature. The classification of curvature-liked tensor fields on a Riemannian manifold has been studied by K. Nomizu [10], in which the Codazzi equation for the curvature-liked tensor played an important role. The subject is also treated by S. Y. Cheng and S. T. Yau [3] from the different point of view. A Riemannian curvature tensor is said to be *harmonic* if the Ricci tensor $S$ satisfies the Codazzi equation $\delta S=0$, namely, in local coordinates

$$R_{ijk}=R_{ikj}, \tag{0.1}$$

where $R_{ijk}$ denotes the covariant derivative of the Ricci tensor $R_{ij}$. Although the concept is closely related to a parallel Ricci tensor, it was shown by A. Derdziński [5] and A. Gray [6] that it is essentially weaker than the latter one. In the Yang-Mills theory the harmonic curvature is also weighty, and some studies for these topics are made. In particular, J. P. Bourguignon conjectured that on a 4-dimensional compact Riemannian manifold with harmonic curvature the Ricci tensor must be parallel. This is negatively answered by A. Derdziński [4], who gave an example of a 4-dimensional compact Riemannian manifold with harmonic curvature and non-parallel Ricci tensor. Certain kinds of Riemannian manifolds with harmonic curvature are investigated by J. P. Bourguignon [1], A. Derdziński [5], T. Kashiwada [7], S. Tachibana [13] and so on. In particular, A. Derdziński [5] gave also other examples of higher dimensional Riemannian manifolds.

On the other hand, hypersurfaces with parallel Ricci tensor in a Riemannian manifold of constant curvature are studied by H. B. Lawson Jr. [8] and I. Mogi

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and one of the present authors [9], and hypersurfaces with harmonic curvature in a Riemannian manifold of constant curvature are recently investigated by E. Ômachi [11] and one of the present authors [15], who determined the situation of the principal curvatures, provided that the mean curvature is constant. Especially, one of the present authors [15] treated also them without the assumption that the mean curvature is constant.

In this paper a class of hypersurfaces with harmonic curvature in a Riemannian manifold of constant curvature will be considered. The purpose is to classify completely hypersurfaces with harmonic curvature in the case where a multiplicity of each principal curvature is greater than one, and to show that there exist infinitely many hypersurfaces with harmonic curvature and non-parallel Ricci tensor.

## 1. Preliminaries.

In order to fix the notation, the theory of hypersurfaces in a Riemannian manifold of constant curvature is prepared for. Let $\bar{M}=M^{n+1}(c)$ be an $(n+1)$-dimensional Riemannian manifold of constant curvature $c$ and let $M$ be an $n$-dimensional connected Riemannian manifold. By $\phi$ the isometric immersion of $M$ into $\bar{M}$ is denoted. When the argument is local, $M$ need not be distinguished from $\phi(M)$ and therefore, to simplify the discussion a point $x$ in $M$ may be identified with the point $\phi(x)$ and a tangent vector $X$ at $x$ may be also identified with the tangent vector $d\phi(X)$ at $\phi(x)$ via the differential $d\phi$ of $\phi$.

To begin with, we choose an orthonormal local frame field $\{e_1, \cdots, e_n, e_{n+1}\}$ in $\bar{M}$ in such a way that, restricted to $M$, the vectors $e_1, \cdots, e_n$ are tangent to $M$ and hence the other $e_{n+1}$ is normal to $M$. With respect to this field of frames on $\bar{M}$, let $\{\bar{\omega}_1, \cdots, \bar{\omega}_n, \bar{\omega}_{n+1}\}$ be the dual field. Here and in the sequel, the following convention on the range of indices are adopted, unless otherwise stated:

$$A, B, \cdots=1, \cdots, n, n+1,$$
$$i, j, \cdots=1, \cdots, n.$$

Then, associated with the frame field $\{e_1, \cdots, e_n, e_{n+1}\}$ there exist differential 1-forms $\bar{\omega}_{AB}$ on $\bar{M}$, which are called *connection forms* on $\bar{M}$, so that they satisfy the following structure equations on $\bar{M}$:

$$d\bar{\omega}_A+\sum_B\bar{\omega}_{AB}\wedge\bar{\omega}_B=0, \qquad \bar{\omega}_{AB}+\bar{\omega}_{BA}=0, \tag{1.1}$$

$$d\bar{\omega}_{AB}+\sum_C\bar{\omega}_{AC}\wedge\bar{\omega}_{CB}=c\bar{\omega}_A\wedge\bar{\omega}_B. \tag{1.2}$$

By restricting these forms $\bar{\omega}_A$ and $\bar{\omega}_{AB}$ to $M$, they are denoted by $\omega_A$ and $\omega_{AB}$ without bar, respectively. Then

$$\omega_{n+1}=0. \tag{1.3}$$

The metric on $M$ induced from the Riemannian metric $\bar{g}$ on the ambient space $\overline{M}$ under the immersion $\phi$ is given by $g=2\sum_i\omega_i\omega_i$. Then $\{e_1, \cdots, e_n\}$ is an orthonormal local field with respect to the induced metric and $\{\omega_1, \cdots, \omega_n\}$ is the dual field, which consists of real valued, linearly independent 1-forms on $M$. They are called *canonical* forms on the hypersurface $M$. It follows from (1.3) and the Cartan lemma that

$$\omega_{n+1,i}=\sum_j h_{ij}\omega_j, \qquad h_{ij}=h_{ji}. \tag{1.4}$$

The quadratic form $\sum_{i,j}h_{ij}\omega_i\otimes\omega_j$ is called a *second fundamental form* of $M$. We call also a form $\sigma$ defined by

$$\sigma(X, Y)=\sum_{i,j}h_{ij}\omega_i(X)\omega_j(Y)e_{n+1}$$

for any vector fields $X$ and $Y$ a second fundamental form on $M$. A linear transformation $A$ on the tangent bundle $TM$ is defined by $g(AX, Y)=g(\sigma(X, Y), e_{n+1})$. Then $A$ is called a *shape operator* of $M$. By the structure equations (1.1), (1.2) and (1.3), the following structure equations on the hypersurface $M$ are given:

$$d\omega_i+\sum_j\omega_{ij}\wedge\omega_j=0, \qquad \omega_{ij}+\omega_{ji}=0, \tag{1.5}$$

$$d\omega_{ij}+\sum_k\omega_{ik}\wedge\omega_{kj}=\Omega_{ij}, \qquad \Omega_{ij}=-\frac{1}{2}\sum_{k,l}R_{ijkl}\omega_k\wedge\omega_l, \tag{1.6}$$

where $\omega_{ij}$ (resp. $\Omega_{ij}$ and $R_{ijkl}$) denotes a connection form (resp. a curvature form and a curvature tensor) on $M$. From (1.2) and (1.6) the Gauss equation

$$R_{ijkl}=c(\delta_{il}\delta_{jk}-\delta_{ik}\delta_{jl})+h_{il}h_{jk}-h_{ik}h_{jl} \tag{1.7}$$

is obtained, and the Ricci tensor $R_{ij}$ and the scalar curvature $R$ can be expressed as follows:

$$R_{ij}=(n-1)c\delta_{ij}+hh_{ij}-\sum_k h_{ik}h_{kj}, \tag{1.8}$$

$$R=n(n-1)c+h^2-\sum_{i,j}h_{ij}h_{ij}, \tag{1.9}$$

where $h$ is a function defined by $h=\sum_i h_{ii}$, namely, for the mean curvature $H$ it satisfies $h=nH$.

Now, the covariant derivative $h_{ijk}$ and $R_{ijk}$ of $h_{ij}$ and $R_{ij}$ are respectively defined by

$$\begin{aligned}\sum_k h_{ijk}\omega_k&=dh_{ij}-\sum_k h_{kj}\omega_{ki}-\sum_k h_{ik}\omega_{kj},\\ \sum_k R_{ijk}\omega_k&=dR_{ij}-\sum_k R_{kj}\omega_{ki}-\sum_k R_{ik}\omega_{kj}.\end{aligned} \tag{1.10}$$

Differentiating (1.4) exteriorly, we have the Codazzi equation on the hypersurface $M$

$$h_{ijk}-h_{ikj}=0\,, \tag{1.11}$$

since the ambient space $\bar{M}$ is of constant curvature, and by differentiating (1.8) exteriorly the covariant derivative $R_{ijk}$ satisfies

$$\sum_k R_{ijk}\omega_k=\sum_k(h_k h_{ij}+hh_{ijk}-\sum_l h_{ilk}h_{lj}-\sum_l h_{il}h_{ljk})\omega_k\,,$$

where $dh=\sum_k h_k\omega_k$, and hence

$$\sum_{j,k}R_{ijk}\omega_k\wedge\omega_j=\sum_{j,k}(h_k h_{ij}-\sum_l h_{ilk}h_{lj})\omega_k\wedge\omega_j\,. \tag{1.12}$$

A Riemannian curvature tensor is said to be *harmonic* if the Ricci tensor satisfies the Codazzi equation (0.1), namely, $R_{ijk}$ is symmetric with respect to all indices $i$, $j$ and $k$. It follows from (1.12) that it is necessary and sufficient for $M$ to be of harmonic curvature that it satisfies

$$h_k h_{ij}-h_j h_{ki}-\sum_l h_{ilk}h_{lj}+\sum_l h_{ilj}h_{lk}=0 \tag{1.13}$$

for any indices.

## 2. The gradient of the mean curvature.

Let $M$ be an $n$-dimensional hypersurface with harmonic curvature in $M^{n+1}(c)$ and let $H$ be the mean curvature on $M$. In this section, assume that the gradient of $H$ is an eigenvector associated with an eigenvalue 0 of the shape operator $A$. In other words, we shall assume that it satisfies

$$A\,\mathrm{grad}\,H=0\,,\quad\text{namely }\sum_j h_{ij}h_j=0 \tag{2.1}$$

holds true. In this assumption the case where $\mathrm{grad}\,H=0$ is included, that is, the situation that the mean curvature $H$ has critical points is admitted. For simplification, a tensor $h_{ij}{}^m$ and a function $h_m$ on $M$ for any integer $m$ are introduced as follows;

$$\begin{aligned} h_{ij}{}^m&=\sum_{i_1,\cdots,i_{m-1}}h_{ii_1}h_{i_1i_2}\cdots h_{i_{m-1}j}\\ h_m&=\sum_i h_{ii}{}^m\,, \end{aligned} \tag{2.2}$$

where $h_1=h=nH$. By taking account of the second Bianchi identity, it is easily seen that the scalar curvature is constant, and therefore the function $h^2-h_2$ is constant. This implies

$$dh_2=2hdh\,. \tag{2.3}$$

First of all, the generalization of (2.3) is requested. Namely, the relation

(2.4) $$dh_m = mh_{m-1}dh$$

is true for any integer $m$ ($\geqq 2$). In fact, the relation (2.4) is proved by induction on $m$. At first, (2.3) shows that the case where $m=2$ in (2.4) holds. By the property of derivations for the exterior differential, it is easily seen that the following equation

$$dh_m = \sum_{i,j} mh_{ij}{}^{m-1}dh_{ij}.$$

The definition (1.10) of the covariant derivative $h_{ijk}$ and the above equation imply

$$\begin{aligned} dh_m &= m\sum_{i,j,k}(h_{ijk}\omega_k + h_{kj}\omega_{ki} + h_{ik}\omega_{kj})h_{ij}{}^{m-1} \\ &= m\sum_{i,j,k}h_{ijk}h_{ij}{}^{m-1}\omega_k + 2m\sum_{i,j}h_{ij}{}^m\omega_{ij}. \end{aligned}$$

Hence

(2.5) $$dh_m = m\sum_{i,j,k}h_{ijk}h_{ij}{}^{m-1}\omega_k,$$

because $h_{ij}{}^m$ is symmetric with respect to $i$ and $j$ and the connection form $\omega_{ij}$ is skew-symmetric with respect to $i$ and $j$. This yields

$$\begin{aligned} dh_m &= m\sum_{i,j,k,l}h_{ijk}h_{jl}h_{li}{}^{m-2}\omega_k \\ &= m\sum_{i,k,l}(\sum_j h_{ijl}h_{jk} + h_k h_{il} - h_l h_{ik})h_{li}{}^{m-2}\omega_k \\ &= m(\sum_{i,j,k,l}h_{ijl}h_{li}{}^{m-2}h_{jk}\omega_k + h_{m-1}dh - \sum_{k,l}h_{kl}{}^{m-1}h_l\omega_k), \end{aligned}$$

where we have used (1.10) and (1.13). By the assumption (2.1) the last term in the right hand side vanishes identically. It follows from the case where $m-1$ in (2.5) and the supposition of the induction that we get

$$\sum_{i,j,l}h_{ijl}h_{il}{}^{m-2}\omega_j = \frac{1}{m-1}dh_{m-1} = h_{m-2}dh,$$

which yields

$$\sum_{i,l}h_{ijl}h_{il}{}^{m-2} = h_{m-2}dh(e_j).$$

This means that the first term in the right hand side of the above equation vanishes also identically, which completes the proof.

A function $H_m$ for any integer $m$ ($\geqq 2$) is next defined by

(2.6) $$H_m = \sum_{k=0}^{m}(-1)^k\binom{m}{k}h^{m-k}h_k, \qquad h_0 = 1.$$

By making use of (2.4) it follows from the straightforward calculation that

$$\begin{aligned} dH_m &= \sum_{k=0}^{m-1}(-1)^k\binom{m}{k}(m-k)h^{m-k-1}h_k dh + \sum_{k=1}^{m}(-1)^k k\binom{m}{k}h^{m-k}h_{k-1}dh \\ &= \sum_{k=0}^{m-1}(-1)^k\left\{(m-k)\binom{m}{k} - (k+1)\binom{m}{k+1}\right\}h^{m-k-1}h_k dh, \end{aligned}$$

which shows that $H_m$ is constant on $M$. Thus we have

LEMMA 2.1. *Let $M$ be a hypersurface with harmonic curvature in $M^{n+1}(c)$. If the shape operator $A$ of $M$ satisfies $A\,\mathrm{grad}\,H=0$, then $H_m$ is constant on $M$ for any integer $m$ ($\geqq 2$).*

By rewritting (2.6), the relation

$$h_m=h^m+\sum_{k=2}^{m}(-1)^k\binom{m}{k}H_k h^{m-k} \tag{2.7}$$

is true for any integer $m\geqq 2$. In fact, the equation is also verified by induction on $m$. At first, the case where $m=2$ in (2.6) is considered. Then it shows that (2.7) holds for $m=2$. Next, suppose that (2.7) holds for integers less than $m$. Since the constant $H_m$ is expressed as

$$H_m=\binom{m}{0}h^m-\binom{m}{1}h^{m-1}h+\sum_{k=2}^{m-1}(-1)^k\binom{m}{k}h^{m-k}h_k+(-1)^m h_m,$$

the supposition of the induction is applied to the third term in the right hand side, so it is reduced to

$$\begin{aligned}H_m&=(-1)^m h_m+\left\{\binom{m}{0}-\binom{m}{1}\right\}h^m\\&\quad+\sum_{k=2}^{m-1}(-1)^k\binom{m}{k}h^{m-k}\left\{h^k+\sum_{l=2}^{k}(-1)^l\binom{k}{l}H_l h^{k-l}\right\}\\&=(-1)^m h_m+\sum_{k=0}^{m-1}(-1)^k\binom{m}{k}h^m\\&\quad+\sum_{l=2}^{m-1}(-1)^l\left\{\sum_{k=l}^{m-1}(-1)^k\binom{m}{k}\binom{k}{l}\right\}H_l h^{m-l}.\end{aligned}$$

On the other hand, the binomial theorem $(1-x)^m=\sum_{k=0}^{m}(-1)^k\binom{m}{k}x^k$ and the derivative of $l$-order for variable $x$ yield the following relation for the binomial coefficients:

$$\sum_{k=l}^{m}(-1)^k\binom{m}{k}\binom{k}{l}=0.$$

Accordingly we have

$$\begin{aligned}H_m&=(-1)^m h_m+\left\{\sum_{k=0}^{m}(-1)^k\binom{m}{k}-(-1)^m\right\}h_m\\&\quad+\sum_{l=2}^{m-1}(-1)^l\left\{\sum_{k=l}^{m}(-1)^k\binom{m}{k}\binom{k}{l}-(-1)^m\binom{m}{l}\right\}H_l h^{m-l}\\&=(-1)^m h_m-(-1)^m h^m-(-1)^m\sum_{l=2}^{m-1}(-1)^l\binom{m}{l}H_l h^{m-l},\end{aligned}$$

which implies that (2.7) holds for any integer $m \geqq 2$.

### 3. No simple roots.

This section is devoted to the study the case where the hypersurfaces with harmonic curvature in $M^{n+1}(c)$ has principal curvatures all of whose multiplicities are greater than one. The second fundamental form may be diagonalized so that $\sum_{i,j} h_{ij}\omega_i \otimes \omega_j = \sum_i \lambda_i \omega_i \otimes \omega_i$. A principal curvature $\lambda_i$ is called a *simple root* at $x$ if the multiplicity at $x$ is equal to 1.

First of all, we prove

LEMMA 3.1. *Let $M$ be a hypersurface of $M^{n+1}(c)$ with harmonic curvature. If the shape operator has no simple roots on $M$, then $A$ grad $H=0$.*

PROOF. Since we have $h_{ij}=\lambda_i \delta_{ij}$ at a point $x$ on $M$, then equation (1.13) says that

$$\lambda_j h_k \delta_{ij} - \lambda_k h_j \delta_{ki} + (\lambda_k - \lambda_j) h_{ijk} = 0 \tag{3.1}$$

at $x$, where we have used

$$\sum_i h_i h_{ij} = \lambda_j h_j . \tag{3.2}$$

Because of the assumption that the second fundamental form $h_{ij}$ has no simple roots, for any fixed index $j$ there is an index $k$ different from $j$ such that $\lambda_j = \lambda_k$, and therefore (3.1) reduces to

$$\lambda_j (h_k \delta_{ij} - h_j \delta_{ki}) = 0$$

at the point $x$, which implies that if $x$ is not a zero point of the principal curvature $\lambda_j$, then we have $h_j = 0$ at $x$. From these data, we conclude, using (3.2), that $\sum_i h_i h_{ij} = 0$. This completes the proof of the lemma.

In the next place, using Lemma 3.1 we are going to prove that the mean curvature $H$ of $M$ is constant.

By taking account of (2.7), it is easily seen that

$$\begin{aligned} h_{n+1} - h h_n &= \sum_{k=2}^{n} (-1)^k \left\{ \binom{n+1}{k} - \binom{n}{k} \right\} H_k h^{n+1-k} + (-1)^{n+1} H_{n+1} \\ &= \sum_{k=2}^{n+1} (-1)^k \binom{n}{k-1} H_k h^{n+1-k} , \end{aligned} \tag{3.3}$$

which is a polynomial of degree $n-1$ with respect to $h$ with constant coefficient, because of Lemma 2.1. Since $\lambda_1, \cdots, \lambda_n$ are the principal curvatures of the second fundamental form $h_{ij}$, $h_m$ can be written as

(3.4) $$h_0=1\,,\quad h_1=h=\sum_{i=1}^n \lambda_i\,,\quad h_m=\sum_{i=1}^n \lambda_i{}^m\,,\qquad m=2,3,\cdots.$$

Now, let $f_1(\lambda), \cdots, f_n(\lambda)$ be elementary symmetric functions of $\lambda=(\lambda_1, \cdots, \lambda_n)$, namely,

(3.5) $$\begin{cases} f_1=f_1(\lambda)=(-1)\sum_i \lambda_i\,, \\ f_2=f_2(\lambda)=(-1)^2\sum_{i<j}\lambda_i\lambda_j\,, \\ \quad\vdots \\ f_n=f_n(\lambda)=(-1)^n\lambda_1\lambda_2\cdots\lambda_n\,. \end{cases}$$

Then it is well known that $f_1, \cdots, f_n$ and $h_1, \cdots, h_n, h_{n+1}$ are related by the Newton formulas (cf. [14]) as follows:

(3.6) $$\begin{cases} h_1+f_1=0\,, \\ h_2+f_1h_1+2f_2=0\,, \\ \quad\vdots \\ h_n+f_1h_{n-1}+\cdots f_{n-1}h_1+nf_n=0\,, \\ h_{n+1}+f_1h_n+\cdots f_{n-1}h_2+h_1f_n=0\,. \end{cases}$$

When these formulas are regarded as the linear homogeneous simultaneous equations with respect to $(1, f_1, \cdots, f_n)$, we see, using the principle of elimination, that the determinant of coefficients vanishes identically. If we take account of (3.3) and the Laplace expansion to this determinant, we can get

(3.7) $$((n+1)!/2)H_2h^{n-1}-((n-1)(n+1)!/3)H_3h^{n-2}+\cdots=0\,.$$

Therefore, it follows from (3.7) that $h_1$ is the root of the algebraic equation with constant coefficients unless all $H_m$ vanishes. According to Lemma 2.1, we have

LEMMA 3.2. *Let $M$ be a hypersurface with harmonic curvature in $M^{n+1}(c)$. If the shape operator $A$ of $M$ satisfies $A\,\mathrm{grad}\,H=0$, then the mean curvature of $M$ is constant, provided that there exists a nonzero $H_m$ defined by* (2.6).

On the other hand, if all $H_m$'s are zero and the shape operator of $M$ has no simple roots, then it is easily derived, by using (2.7), (3.5) and (3.6), that $h$ is also constant.

Combining this fact, Lemma 3.1 and Lemma 3.2, we have

PROPOSITION 3.3. *Let $M$ be a hypersurface with harmonic curvature in $M^{n+1}(c)$. If the shape operator of $M$ has no simple roots, then the mean curvature of $M$ is constant.*

Under the property of Proposition 3.3, (2.4) means that each principal curvature of $M$ is constant and hence, by means of Umehara's theorem [15] the number of distinct principal curvatures is at most two, say $\lambda$ and $\mu$, such that $c+\lambda\mu=0$, which is applied to the situation where the ambient space is a sphere, a Euclidean one or a hyperbolic one. So, in the case, the above result for the number of distinct principal curvatures is simply proved from a different point of view. In fact, $M$ is an isoparametric hypersurface in the sense of E. Cartan and the basic identity for principal curvatures shows that the above is true, provided that $c\leqq 0$ [2]. If $c>0$, then it is evident in [11]. Moreover, the second fundamental form of $M$ is parallel.

By the way, we shall here give a model of hypersurfaces with parallel Ricci tensor in a hyperbolic space $H^{n+1}(c)$ (cf. Lawson [8]). $H^{n+1}(c)$ is covered by a coordinate system $\{x_1, \cdots, x_{n+1}\}$ such that the Riemannian metric $ds^2$ of $H^{n+1}(c)$ is given by

$$ds^2=\sum_{\alpha=1}^{n+1} dx_\alpha^2-(\sum_{\alpha=1}^{n+1} x_\alpha dx_\alpha)^2/(r^2+\sum_{\alpha=1}^{n+1} x_\alpha{}^2)\,,$$

where $r^2=-1/c$. The space $H^{n+1}(c)$ is a complete and simply connected Riemannian manifold of constant negative curvature $c$. A family of hypersurfaces $M(s)$ in $H^{n+1}(c)$ is defined by

$$M(s)=\{x\in H^{n+1}(c): \sum_{\alpha=1}^{n+1} x_\alpha{}^2=s^2-r^2\}$$

for $s>r$. Then a hypersurface $M(s)$ for a fixed $s$ is a space of constant curvature $c_1=1/(s^2-r^2)$ in $H^{n+1}(c)$, which is totally umbilic. As another family of hypersurfaces $M(t)$, the following subject is defined:

$$M(t)=\{x\in H^{n+1}(c): x_1=t\geqq 0\}\,.$$

The hypersurface $M(t)$ for an arbitrary fixed $t$ is totally umbilic and hence it is a hyperbolic space of constant curvature $c_1=-1/(r^2+t^2)$. A flat hypersurface $F^n$ is constructed as follows:

$$F^n=\{x\in H^{n+1}(c): \sum_{i=1}^{n} x_i{}^2=2rx_{n+1}\}\,.$$

Then $F^n$ is covered by one coordinate system $\{x_1, \cdots, x_n\}$ such that the Riemannian metric induced from the Riemannian metric in $H^{n+1}(c)$ is given by $ds^2=\sum_{i=1}^{n} dx_i{}^2$. Accordingly, $F^n$ is flat. Lastly, a family of product hypersurfaces $S^k(c_1)\times H^{n-k}(c_2)$ in $H^{n+1}(c)$ is considered. They are defined by

$$S^k(c_1)\times H^{n-k}(c_2)=\{x\in H^{n+1}(c): \sum_{i=1}^{k+1} x_i{}^2=1/c_1\}\,,$$

where $c_1$ is positive constant and $1/c_1+1/c_2=1/c$, and $1\leqq k\leqq n-1$. Any hypersurface of the family is the product manifold of a sphere of constant curvature

$c_1$ and a hyperbolic space of constant curvature $c_2$ and consequently it has exactly two distinct principal curvatures $(c_1-c)^{1/2}$ and $(c_2-c)^{1/2}$ of multiplicity $k$ and $n-k$, respectively.

Combining Proposition 3.3 together with Umehara's theorem, we can see the following

THEOREM 3.4. *Let $M$ be an $n$ ($\geqq 3$)-dimensional complete and simply connected Riemannian manifold with harmonic curvature and let $\phi$ be an isometric immersion of $M$ into an $(n+1)$-dimensional complete and simply connected Riemannian manifold of constant curvature $c$. If the multiplicity of each principal curvature is greater than one, then $\phi(M)$ is isometric to one of the following spaces:*

(1) *The case where $c>0$. The great sphere, the small sphere and $S^k(c_1)\times S^{n-k}(c_2)$, where $2\leqq k\leqq n-2$ and $1/c_1+1/c_2=1/c$. In particular, $\phi$ is an imbedding.*
(2) *The case where $c=0$. The sphere, the Euclidean space and $S^k\times R^{n-k}$.*
(3) *The case where $c<0$. The sphere, the hyperbolic space, the flat space $F^n$ and $S^k(c_1)\times H^{n-k}(c_2)$, where $2\leqq k\leqq n-2$ and $1/c_1+1/c_2=1/c$. In particular, $\phi$ is an imbedding.*

## 4. Hypersurfaces with harmonic curvature and non-parallel Ricci tensor.

This section is devoted to the investigation of examples of hypersurfaces with harmonic curvature and non-parallel Ricci tensor in $M^{n+1}(c)$. By taking account of Theorem 3.4, it is seen that at least one principal curvatures ought to be of multiplicity 1.

Let $M$ be a hypersurface immersed in $M^{n+1}(c)$, and assume that the principal curvatures $\lambda_i$ on $M$ satisfy

$$\left\{\begin{array}{l}\lambda_1=\cdots=\lambda_{n-1}=\lambda\neq 0\,,\\ \lambda_n=\mu\,,\end{array}\right. \tag{4.1}$$

such that $\lambda\neq\mu$. Without loss of generality, we may suppose that $\lambda>0$. As is well known, the distribution of the space of eigenvectors corresponding to the eigenvalue $\lambda$ is completely integrable, because the multiplicity of each principal curvature is constant. Now, since $\lambda$ and $\mu$ are smooth functions on $M$, we have, using the covariant derivative $h_{ijk}$,

$$d\lambda=d\lambda_a=h_{aaa}\omega_a+\textstyle\sum_{b\neq a}h_{aab}\omega_b+h_{aan}\omega_n\,, \tag{4.2}$$

where indices $a, b, \cdots$ run over the range $\{1, \cdots, n-1\}$. Because of $\omega_{n+1\,a}=\lambda_a\omega_a$, we have

$$d\omega_{n+1\,a}=d\lambda_a\wedge\omega_a+\lambda_a d\omega_a$$
$$=d\lambda\wedge\omega_a+\lambda(-\sum_b\omega_{ab}\wedge\omega_b-\omega_{an}\wedge\omega_n).$$

On the other hand, the structure equation (1.2) yields

$$d\omega_{n+1\,a}=-\sum_k\omega_{n+1\,k}\wedge\omega_{ka}$$
$$=-\lambda\sum_b\omega_b\wedge\omega_{ba}-\mu\omega_n\wedge\omega_{na}.$$

Combining with above two equations, we have

$$\sum_b\lambda_{,b}\omega_b\wedge\omega_a+\{(\mu-\lambda)\omega_{an}-\lambda_{,n}\omega_a\}\wedge\omega_n=0 \tag{4.3}$$

for a fixed index $a$, where $d\lambda=\sum_b\lambda_{,b}\omega_b+\lambda_{,n}\omega_n$. This implies

$$\begin{cases}\lambda_{,a}=0,\\ (\mu-\lambda)\omega_{an}-\lambda_{,n}\omega_a=\sigma_a\omega_n\end{cases} \tag{4.4}$$

for any index $a$, where $\sigma_a$ is a function on $M$. From (4.2) and the first equation of (4.4) it follows that we have

$$h_{aaa}\omega_a+\sum_{b\neq a}h_{aab}\omega_b+h_{aan}\omega_n=\lambda_{,n}\omega_n,$$

and hence

$$h_{aaa}=0,\quad h_{aab}=0\ (b\neq a),\quad h_{aan}=\lambda_{,n}.$$

Similarly, for the other $\mu$ we have

$$d\mu=\sum_b h_{nnb}\omega_b+h_{nnn}\omega_n.$$

Because of $\omega_{n+1\,n}=\mu\omega_n$, by the same argument as that of $\lambda$ we have

$$d\omega_{n+1\,n}=-\lambda\sum_b\omega_{nb}\wedge\omega_b=d\mu\wedge\omega_n-\mu\sum_b\omega_{nb}\wedge\omega_b,$$

and hence

$$d\mu\wedge\omega_n+(\lambda-\mu)\sum_b\omega_{nb}\wedge\omega_b=0.$$

This together with (4.4) implies

$$\mu_{,a}=\sigma_a \quad \text{for any index } a. \tag{4.5}$$

On the other hand, for distinct indices $a$ and $b$, we have

$$h_{abk}=0. \tag{4.6}$$

In the case where $M$ is with harmonic curvature, principal curvatures $\lambda_j$ satisfy (3.1) and because of $h=(n-1)\lambda+\mu$ and $dh=\sum_k h_k\omega_k$, we see

$$h_k=(n-1)\lambda_{,k}+\mu_{,k}$$

for any index $k$. Considering the case where $j=a$ and $k=n$ in (3.1), one gets

$$\lambda h_n\delta_{ai}-\mu h_a\delta_{ni}-(\lambda-\mu)h_{ani}=0$$

for any indices $a$ and $i$. This means that it follows from the above equation and (4.5) that

$$(4.7)\qquad \begin{cases} \{(n-2)\lambda+\mu\}\lambda_{,n}+\lambda\mu_{,n}=0\,, \\ \mu h_a+(\lambda-\mu)h_{ann}=0\,. \end{cases}$$

Consequently, making use of the above relations, we have $h_a=\mu_{,a}=h_{ann}$ and $\lambda h_{ann}=0$, namely

$$h_a=0\,,\quad \sigma_a=0\,.$$

Thus, by (4.4) we have

$$(4.8)\qquad \begin{aligned} &h_{nnn}=\mu_{,n}\,, \\ &\omega_{na}=\frac{\lambda_{,n}}{\lambda-\mu}\omega_a\,. \end{aligned}$$

Accordingly, in order for $M$ to be with harmonic curvature, principal curvatures $\lambda$ and $\mu$ must satisfy (4.7) and (4.8). Moreover we have $d\omega_n=0$, which shows that we may put

$$(4.9)\qquad \omega_n=dv\,.$$

Thus we have

$$(4.10)\qquad \omega_{na}=\frac{\lambda'}{\lambda-\mu}\omega_a\,,$$

where the prime denotes the derivative with respect to $v$. This means that the integral submanifold $M^{n-1}(v)$ corresponding to $\lambda$ and $v$ is umbilic in $M$ and hence in $M^{n+1}(c)$.

By the simple calculation the following properties for the Ricci tensor are obtained:

$$R_{abc}=0\,,\quad R_{ann}=0\,,\quad R_{abn}=[\{2(n-2)\lambda+\mu\}\lambda_{,n}+\lambda\mu_{,n}]\delta_{ab}\,,$$
$$R_{nnn}=(n-1)(\lambda\mu_{,n}+\mu\lambda_{,n})\,.$$

Therefore, in order for $M$ to be with parallel Ricci tensor, it is necessary and sufficient that $\lambda$ and $\mu$ are both constant.

EXAMPLE. $M=S^{n-1}(c_1)\times S^1(c_2)\subset \boldsymbol{R}^n\times\boldsymbol{R}^2$ such that $1/c_1+1/c_2=1$. The principal curvatures $\lambda_j$ are given by

$$\lambda_1=\cdots=\lambda_{n-1}=\lambda=\pm(c_1-1)^{1/2}\,,$$
$$\lambda_n=\mu=\mp(c_2-1)^{1/2}\,.$$

Actually $M$ is with harmonic curvature and parallel Ricci tensor. In particular, when $c_1=n/(n-2)$ and $c_2=n/2$, $\lambda$ and $\mu$ are given by $\lambda=\pm(2/(n-2))^{1/2}$ and

$\mu=\mp((n-2)/2)^{1/2}$ and moreover they satisfy $(n-2)\lambda+\mu=0$. In the latter case, the scalar curvature $R$ is equal to $n(n-1)$.

Now, substituting (4.10) into the structure equation

$$d\omega_{na}+\sum_b\omega_{nb}\wedge\omega_{ba}=(c+\lambda\mu)\omega_n\wedge\omega_a\,,$$

we have

$$d\Big(\frac{\lambda'}{\lambda-\mu}\omega_a\Big)=-\frac{\lambda'}{\lambda-\mu}\sum_b\omega_b\wedge\omega_{ba}+(c+\lambda\mu)\omega_n\wedge\omega_a\,.$$

Since the left hand side is reduced to

$$\Big(\frac{\lambda'}{\lambda-\mu}\Big)'\omega_n\wedge\omega_a+\frac{\lambda'}{\lambda-\mu}(-\sum_b\omega_{ab}\wedge\omega_b-\omega_{an}\wedge\omega_n)\,,$$

the following equation is obtained:

$$\Big(\frac{\lambda'}{\lambda-\mu}\Big)'-\Big(\frac{\lambda'}{\lambda-\mu}\Big)^2-(c+\lambda\mu)=0\,,$$

and hence we have

$$\lambda''(\lambda-\mu)-\lambda'(\lambda'-\mu')-\lambda'^2-(c+\lambda\mu)(\lambda-\mu)^2=0\,. \tag{4.11}$$

Furthermore, under the condition (4.7) we have

$$\{(n-2)\lambda+\mu\}\lambda'+\lambda\mu'=0\,. \tag{4.12}$$

Thus the distinct principal curvatures $\lambda$ and $\mu$ satisfy a system of ordinary differential equations (4.11) and (4.12) of order 2. (4.12) is however equivalent to

$$\{(n-2)\lambda^2+2\lambda\mu\}'=0\,,$$

which yields

$$(n-2)\lambda^2+2\lambda\mu=c_1\,,$$

where $c_1$ is the integral constant. Then the scalar curvature $R$ is given by $R=n(n-1)c+(n-1)c_1$, and by taking account of (4.12), the ordinary differential equation of order 2 for $\lambda$ is given by

$$\begin{aligned}&4\lambda(n\lambda^2-c_1)\lambda''-4\{(n+2)\lambda^2+c_1\}\lambda'^2\\&-(n\lambda^2-c_1)^2\{2c+c_1-(n-2)\lambda^2\}=0\,,\end{aligned} \tag{4.13}$$

where $n\lambda^2-c_1\neq 0$. Putting $\omega=\lambda^{-2/n}$, (4.13) can be replaced by

$$\frac{d^2\omega}{dv^2}+\frac{(n+1)c_1\omega^{n-1}}{n-c_1\omega^n}\omega'^2+\frac{\omega}{2n}(n-c_1\omega^n)\Big(2c+c_1-\frac{n-2}{\omega^n}\Big)=0\,.$$

Integrating the differential equation of degree 2, we obtain

$$\Big(\frac{d\omega}{dv}\Big)^2=(n-c_1\omega^n)^{2(n+1)/n}\Big\{c_2-\frac{1}{n}\int\omega(n-c_1\omega^n)^{-(n+2)/n}\Big(2c+c_1-\frac{n-2}{\omega^n}\Big)d\omega\Big\}\,,$$

where $c_2$ is the integral constant. In the case where $c_1=0$, this is reduced to

$$\left(\frac{d\omega}{dv}\right)^2+\frac{1}{\omega^{n-2}}+c\omega^2=c_2,$$

which is the differential equation similar to that treated by T. Otsuki [12]. Thus there exist infinitely many hypersurfaces with harmonic curvature in $M^{n+1}(c)$ corresponding to the constants $c_1$ and $c_2$, and the hypersurfaces have non-parallel Ricci tensor and the scalar curvatures are equal to $n(n-1)c+(n-1)c_1$.

By the same method as that of Otsuki's theory, we have the following construction theorem concerning for hypersurfaces with harmonic curvature.

THEOREM 4.1. *Let $M$ be an $n$ ($\geqq 3$)-dimensional hypersurface with scalar curvature $n(n-1)c$ and the harmonic curvature immersed in $M^{n+1}(c)$. If it has exactly two distinct principal curvatures, one's multiplicity of which is equal to* 1, *and the other has no zero points, then the following assertions are true:*

(1) *$M$ is a locus of moving $(n-1)$-dimensional submanifold $M^{n-1}(v)$ along which the principal curvature $\lambda$ of multiplicity $n-1$ is constant and which is umbilic in $M$ and of constant curvature $(d/dv(\log(n\lambda^2-c_1)^{1/n}))^2+\lambda^2+c$, where $v$ is the arc length of an orthogonal trajectory of the family $M^{n-1}(v)$, and $\lambda=\lambda(v)$ satisfies the ordinary differential equation* (4.13) *of order* 2.

(2) *If $\overline{M}=S^{n+1}(c)\subset \boldsymbol{R}^{n+2}$, then $M^{n-1}(v)$ is contained in an $(n-1)$-dimensional sphere $S^{n-1}(v)=E^n(v)\cap S^{n+1}$ of the intersection of $S^{n+1}$ and an $n$-dimensional linear subspace $E^n(v)$ in $\boldsymbol{R}^{n+2}$ which is parallel to a fixed $E^n$. The center $q$ moves on a plane curve in a plane $\boldsymbol{R}^2$ through the origin of $\boldsymbol{R}^{n+2}$ and orthogonal to $E^n$.*

COROLLARY. *There exist infinitely many hypersurfaces with harmonic curvature and non-parallel Ricci tensor in $M^{n+1}(c)$, which is not congruent to each other in it.*

In the next place, the condition under which the plane curve figured with the center $q$ is controlled will be required. Since the matter discussed in [12, section 4] can be completely applied to this case, the necessary subjects for the explanation of the statement of the theorem are only quoted from [12], and the precise argument is omitted. The sphere $S^{n+1}$ is regarded as $S^{n+1}\subset \boldsymbol{R}^{n+2}=\boldsymbol{R}^n\times \boldsymbol{R}^2$, and $\{\bar{e}_1, \cdots, \bar{e}_n\}$ denotes the orthonormal frame in $\boldsymbol{R}^n$ at the origin. Let $C$ be a plane curve in $\boldsymbol{R}^2$ with a given surporting function $h(\theta)$, then the generic point $q(\theta)$ of $C$ is given by

$$q(\theta)=e^{i(\theta-\pi/2)}(h(\theta)+ih'(\theta)) \tag{4.14}$$

by considering $\boldsymbol{R}^2$ as the complex plane. The Frenet formula of $C$ at $q(\theta)$ is given by $\bar{e}_{n+1}=e^{i\theta}$ and $\bar{e}_{n+2}=e^{i(\theta+\pi/2)}$. Suppose that the curve $C$ is contained in the unit circle. Then a positive function $\rho$ can be defined by $\rho^2=1-\|q\|^2$, and a hypersurface $M$ is defined in $S^{n+1}(1)$ by

$$p=q+\rho\bar{e}_n\,. \tag{4.15}$$

A unit vector $e_n$ is defined by

$$e_n=(\rho'\bar{e}_n+(h+h'')\bar{e}_{n+1})/((\rho')^2+(h+h'')^2)^{1/2}\,.$$

If the hypersurface $M$ in $S^{n+1}(1)$ is with harmonic curvature and $R=n(n-1)$, then the function $h$ satisfies the following ordinary differential equation

$$nh(1-h^2)\frac{d^2h}{d\theta^2}+2\left(\frac{dh}{d\theta}\right)^2+(1-h^2)(nh^2-2)=0\,. \tag{4.16}$$

Conversely, if a function $h(\theta)$ satisfying (4.16) gives a plane curve by the equation (4.14) in $\boldsymbol{R}^2$ contained in the unit circle, then a hypersurface $M$ with harmonic curvature and $R=n(n-1)$ is obtained by (4.15). The hypersurfaces depend completely on properties of $h(\theta)$.

THEOREM 4.2. *Any complete hypersurface $M$ with harmonic curvature and $R=n(n-1)$ in $S^{n+1}(1)$ of the type of Theorem 4.1 is given by the following method.*

(1) *$C$ is a plane curve in $\boldsymbol{R}^2$ given by*

$$q(\theta)=e^{i(\theta-\pi/2)}(h(\theta)+ih'(\theta))\,,$$

*where $h(\theta)$ is a solution of the differential equation (4.16) with $0<h(0)\leqq(2/n)^{1/2}$ and $h'(0)=0$.*

(2) *$M\ni p=(1-h(\theta)^2-h'(\theta)^2)^{1/2}\bar{e}_n+q(\theta)$, where $\bar{e}_n\in\boldsymbol{R}^n$, $\|\bar{e}_n\|=1$ and $S^{n+1}\subset\boldsymbol{R}^n\times\boldsymbol{R}^2$.*

*There exist countable number of compact hypersurfaces of this type, and the special case $S^{n-1}(n/(n-2))\times S^1(n/2)$ corresponds to $h(0)=(2/n)^{1/2}$ and $h'(0)=0$.*

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