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SUBMANIFOLDS WITH HARMONIC CURVATURE

By

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Dedicated to Professor Mun-Gu Sohn on his sixtieth birthday

0. Introduction

A Riemannian curvature is said to be *harmonic* if the Ricci tensor S satisfies the so-called Codazzi equation $\delta S=0$. Riemannian manifolds with harmonic curvature are studied by A. Derziński [2] and A. Gray [4], who required a sufficient condition for the manifolds to be Einstein and constructed examples of non-parallel Ricci tensor. On the other hand, hypersurfaces with harmonic curvature in a Riemannian manifold of constant curvature are recently investigated by E. Ômachi [9], M. Umehara [12] and the authors [5], who determined completely the manifold structures provided that the mean curvature is constant, or provided that the shape operator has no simple roots. The purpose of this paper is to investigate submanifolds with harmonic curvature in a Riemannian manifold of constant curvature.

1. Submanifolds

Let $\bar{M}=M^{n+p}(c)$ be an $(n+p)$-dimensional connected Riemannian manifold of constant curvature c and ϕ an isometric immersion of an n-dimensional connected Riemannian manifold M into $\bar{M}$. When the argument is local, M need not be distinguished from $\phi(M)$. We choose a local field of orthonormal frames $\{e_1,\cdots, e_n, e_{n+1},\cdots, e_{n+p}\}$ in $\bar{M}$, in such a way that, restricted to M, the vectors $e_1,\cdots, e_n$ are tangent to M and hence the others are normal to M. Let $\{\bar{\omega}_1,\cdots, \bar{\omega}_n, \bar{\omega}_{n+1},\cdots, \bar{\omega}_{n+p}\}$ be the field of dual frames with respect to the above frame field. Here and in the sequel the following convention on the range of indices are used, unless otherwise stated:

$$\begin{aligned} &A, B,\cdots=1,\cdots, n, n+1,\cdots, n+p,\\ &i, j,\cdots=1,\cdots, n,\\ &\alpha, \beta,\cdots=n+1,\cdots, n+p. \end{aligned}$$

Then the structure equations of $\bar{M}$ are given by

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$$d\bar{\omega}_A+\Sigma_B\bar{\omega}_{AB}\wedge\bar{\omega}_B=0,$$
$$d\bar{\omega}_{AB}+\Sigma_C\bar{\omega}_{AC}\wedge\bar{\omega}_{CB}=c\bar{\omega}_A\wedge\bar{\omega}_B, \tag{1.1}$$

where $\bar{\omega}_{AB}$ denote connection forms on $\bar{M}$. By restricting these forms $\bar{\omega}_A$ and $\bar{\omega}_{AB}$ to M, they are simply denoted by ω_A and ω_{AB} without bar, respectively. Then we have

$$\omega_\alpha=0. \tag{1.2}$$

The metric on M induced from the Riemannian metric $\bar{g}$ on the ambient space $\bar{M}$ under the immersion ϕ is given by $g=2\Sigma_i\omega_i\omega_i$. Then $\{e_1,\cdots,e_n\}$ becomes a field of orthonormal frames on M with respect to this metric and $\{\omega_1,\cdots,\omega_n\}$ are the canonical forms on M. It follows from (1.1), (1.2) and Cartan's lemma that

$$\omega_{\alpha i}=\Sigma_j h_{ij}^\alpha\omega_j,\quad h_{ij}^\alpha=h_{ji}^\alpha. \tag{1.3}$$

The quadratic form $\Sigma_{i,j}h_{ij}^\alpha\omega_i\omega_j$ is called a *second fundamental form* on M in the direction of e_α and the second fundamental form σ on M can be written as

$$\sigma(X,Y)=\Sigma_{\alpha,i,j}h_{ij}^\alpha\omega_i(X)\omega_j(Y)e_\alpha$$

for any tangent vectors X and Y. For the canonical forms $\{\omega_i\}$ and the connection forms $\{\omega_{ij}\}$, the following equations on M are given:

$$d\omega_i+\Sigma_j\omega_{ij}\wedge\omega_j=0,$$
$$d\omega_{ij}+\Sigma_k\omega_{ik}\wedge\omega_{kj}=\Omega_{ij}, \tag{1.4}$$
$$\Omega_{ij}=-\frac{1}{2}\Sigma_{k,l}R_{ijkl}\omega_k\wedge\omega_l,$$

where Ω_{ij} and R_{ijkl} denote the curvature form and the Riemannian curvature tensor on M respectively. Moreover the forms $\{\omega_{\alpha\beta}\}$ which are called *normal connection forms* in the normal bundle $N(M)$ of M satisfy

$$d\omega_{\alpha\beta}+\Sigma_\gamma\omega_{\alpha\gamma}\wedge\omega_{\gamma\beta}=\Omega_{\alpha\beta},$$
$$\Omega_{\alpha\beta}=-\frac{1}{2}\Sigma_{k,l}R_{\alpha\beta kl}\omega_k\wedge\omega_l, \tag{1.5}$$

where $\Omega_{\alpha\beta}$ and $R_{\alpha\beta kl}$ are called the *normal curvature form* and the *normal curvature tensor* on M. By means of the above structure equations on M and $\bar{M}$, the Gauss equation of the submanifold is obtained as

$$R_{ijkl}=c(\delta_{il}\delta_{jk}-\delta_{ik}\delta_{jl})+\Sigma_\alpha(h_{il}^\alpha h_{jk}^\alpha-h_{ik}^\alpha h_{jl}^\alpha). \tag{1.6}$$

Now, the covariant derivative h_{ijk}^α of h_{ij}^α are defined as follows:

$$\Sigma_k h_{ijk}^\alpha\omega_k=dh_{ij}^\alpha-\Sigma_k h_{kj}^\alpha\omega_{ki}-\Sigma_k h_{ik}^\alpha\omega_{kj}+\Sigma_\beta h_{ij}^\beta\omega_{\alpha\beta}. \tag{1.7}$$

By differentiating (1.3) exteriorly and by making use of (1.1), (1.4) and (1.7), the equation

$$d\omega_{\alpha i}=\Sigma_j dh_{ij}^\alpha\wedge\omega_j+\Sigma_j h_{ij}^\alpha d\omega_j$$

is reduced to

$$\Sigma_{j,k} h^{\alpha}_{ijk} \omega_k \wedge \omega_j = 0,$$

from which the Codazzi equation on M

$$h^{\alpha}_{ijk} - h^{\alpha}_{ikj} = 0 \tag{1.8}$$

is yielded. By taking account of the structure equation (1.1) of the ambient space, the normal curvature form on M is also given by

$$\Omega_{\alpha\beta} = -\Sigma_i \omega_{\alpha i} \wedge \omega_{i\beta},$$

which means

$$R_{\alpha\beta kl} = \Sigma_i (h^{\alpha}_{il} h^{\beta}_{ik} - h^{\alpha}_{ik} h^{\beta}_{il}). \tag{1.9}$$

This is called the Ricci equation of the submanifold M.

A smooth section in the normal bundle $N(M)$ of M is called a *normal* vector field on M. When a normal vector field ξ on M is given, its covariant derivative with respect to the normal connection means the normal vector field $D\xi$, which is defined as follows: If $\xi = \Sigma_\alpha V^\alpha e_\alpha$, then

$$D\xi = \Sigma_\alpha D V^\alpha e_\alpha, \quad D V^\alpha = d V^\alpha + \Sigma_\beta \omega_{\alpha\beta} V^\beta.$$

It is easily seen that this is well defined, namely it is independent of the choice of the normal frames on M. By means of this normal connection D and the shape operator $A_\alpha = A_{e\alpha}$ for the normal vector e_α which is defined by $g(A_\alpha X, Y) = \bar{g}(\sigma(X, Y), e_\alpha)$, the normal curvature $R_{\alpha\beta kl}$ is given by

$$\begin{aligned} R_{\alpha\beta kl} &= g([A_\alpha, A_\beta](e_k), e_l) \\ &= g((D_k D_l - D_l D_k - D([e_k, e_l]))e_\alpha, e_\beta), \end{aligned} \tag{1.10}$$

where $D(X)Y = D_X Y$ and $D_k = D(e_k)$.

A given normal vector field ξ on M is said to be *parallel* in the normal bundle if it satisfies $D\xi = 0$ for the normal connection D [7]. For a parallel normal vector field ξ, we put $\xi = a e_{n+1}$, where $a = \|\xi\|$ is constant. Then a local field of orthonormal frames $\{e_{n+1}, \cdots, e_{n+p}\}$ such that e_{n+1} is parallel may be chosen. In this case, the fact that ξ is parallel and (1.10) show

$$\omega_{n+1\beta} = 0, \quad R_{n+1\beta kl} = 0. \tag{1.11}$$

2. Parallel mean curvature vector

Let M be an n-dimensional submanifold with harmonic curvature in $M^{n+p}(c)$. This section is devoted to the investigation of submanifolds with parallel mean curvature vector. The covariant derivative of the Ricci tensor satisfies

(2.1) $$R_{ijk}=R_{ikj}.$$

Let τ be the mean curvature vector field. Namely, it is defined by

$$\tau=\Sigma_i\sigma(e_i, e_i)/n=\Sigma_\alpha h^\alpha e_\alpha/n,$$

where $h^\alpha=\Sigma_i h_{ii}^\alpha$, which is independent of the choice of the local field of orthonormal frames $\{e_\alpha\}$. Let us assume that the mean curvature vector is parallel, and we may choose a local field $\{e_\alpha\}$ in such a way that $\tau=ae_{n+1}$. Because of the choice of the local field, the parallelism of τ yields

(2.2) $$h^\alpha=0,\quad \alpha\geqq n+2,$$
$$h^{n+1}=n\|\tau\|.$$

From Gauss and Codazzi equations and the definition of harmonic curvature it follows that

(2.3) $$\Sigma_{\alpha,r}h_{ijr}^\alpha h_{rk}^\alpha=\Sigma_{\alpha,r}h_{ikr}^\alpha h_{rj}^\alpha.$$

By means of the Ricci eq. (1.10), the normal curvature on M implies

$$[A_{n+1}, A_\alpha]=0$$

for any index α, which yields

(2.4) $$\Sigma_r h_{ir}h_{rj}^\alpha=\Sigma_r h_{jr}h_{ri}^\alpha,$$

where $h_{ij}=h_{ij}^{n+1}$. By the straightforward calculation of the exterior derivative of the above equation, we have

(2.5) $$\Sigma_r(h_{irk}h_{rj}^\alpha+h_{ir}h_{rjk}^\alpha)=\Sigma_r(h_{jrk}h_{ri}^\alpha+h_{jr}h_{rik}^\alpha),$$

from which it follows

$$\Sigma_{\alpha,r,s}(h_{irk}h_{rs}^\alpha h_{sj}^\alpha-h_{rsk}h_{sj}^\alpha h_{ri}^\alpha)=\Sigma_{\alpha,r,s}(h_{irk}^\alpha h_{rs}h_{sj}^\alpha-h_{rsk}^\alpha h_{ir}h_{sj}^\alpha).$$

By the properties (2.3) and (2.4) the second term in the right hand side is deformed as follows:

$$-\Sigma_{\alpha,r,s}h_{rsk}^\alpha h_{ir}h_{sj}^\alpha=-\Sigma_{\alpha,r,s}h_{jsk}^\alpha h_{ir}h_{rs}^\alpha=-\Sigma_{\alpha,r,s}h_{jsk}^\alpha h_{sr}h_{ri}^\alpha.$$

This means that the right hand side is skew-symmetric with respect to indices i and j and therefore it turns out that

(2.6) $$\Sigma_{\alpha,r,s}h_{ijk}h_{rsk}h_{ri}^\alpha h_{sj}^\alpha=\Sigma_{\alpha,r,s}h_{ijk}h_{irk}h_{rs}^\alpha h_{sj}^\alpha.$$

On the other hand, for fixed indices k and α $\Sigma_r(h_{irk}h_{rj}^\alpha-h_{ir}^\alpha h_{rjk})$ can be regarded as a square matrix of order n. By (2.6) the norm of this matrix with respect to the usual inner product vanishes identically, which implies

(2.7) $$\Sigma_r h_{irk}h_{rj}^\alpha=\Sigma_r h_{ir}^\alpha h_{rjk}$$

for any indices α, i, j and k. The eqs. (2.5) and (2.7) show

(2.8) $$\Sigma_r h_{ir} h^{\alpha}_{rjk} = \Sigma_r h_{jr} h^{\alpha}_{rik}.$$

Since the matrix h_{ij} is diagonalizable, the local field $\{e_i\}$ can be specialized so that $h_{ij} = \lambda_i \delta_{ij}$. Then, for the eigenvalues λ_i the following result is proved.

LEMMA 2.1. *Each eigenvalue λ_i is constant on M.*

PROOF. In the case where $\alpha = n+1$ in (2.8), we get

(2.9) $$\Sigma_r h_{ir} h_{rjk} = \Sigma_r h_{jr} h_{rik} = \Sigma_r h_{kr} h_{rij}.$$

This shows that a formula similar to that given in the case of hypersurfaces with harmonic curvature can be derived. Namely, it is easy that $dh_2 = 0$, where the function h_2 on M is defined by $h_2 = \Sigma_{i,j} h_{ij} h_{ij}$. When a function h_m for any integer $m \geqq 2$ is defined by $h_m = \Sigma_{i,j,\cdots,l} h_{ij} h_{jk} \cdots h_{li}$, m-times, it is easily seen that

$$dh_{m+1}(X)/(m+1) = dh_m(A_{n+1}(X))/m$$

can be derived by using the eq. (2.9). This implies inductively the fact that the function h_m for any integer $m \geqq 2$ is constant on M. This means that the assertion is verified.

By $\mu_1, \cdots, \mu_k$ mutually distinct eigenvalues of the shape operator A_{n+1} are denoted. Let $n_1, \cdots, n_k$ be their multiplicities. Since each eigenvalue μ_a $(a = 1, \cdots, k)$ is constant, the smooth distribution T_a which consists of all eigenspaces associated with the eigenvalue μ_a can be defined. By using the notation $[i] = \{j : \lambda_j = \lambda_i\}$ the distribution T_a is given by $T_a = \{\omega_i = 0 \text{ for } i \notin [a]\}$. For $i \notin [a]$ the structure eq. (1.4) shows

$$d\omega_i = -\Sigma_k \omega_{ik} \wedge \omega_k \equiv -\Sigma_{k \in [a]} \omega_{ik} \wedge \omega_k \qquad (\text{mod. } \omega_j : j \notin [a]),$$

which implies that the distribution T_a is completely integrable, provided that $\omega_{ik} \equiv 0$ (mod. ω_j; $j \notin [a]$) for any index $k \in [a]$. In particular, the distribution T_a is said to be *parallel* if the connection forms ω_{ij} satisfy $\omega_{ik} = 0$ for $i \notin [a]$ and $k \in [a]$. The parallelism of the distribution means geometrically that the covariant derivative of the vector field belonging to the distribution belongs also to itself.

LEMMA 2.2. *Distributions T_a are mutually orthogonal and parallel.*

PROOF. Mutual orthogonality is trivial. Since the second fundamental form h_{ij} can be diagonalized, we have by (2.4) and (2.8) $(\lambda_i - \lambda_i) h^{\alpha}_{ij} = 0$ and $(\lambda_i - \lambda_j) h^{\alpha}_{ijk} = 0$, which show that

$$h^{\alpha}_{ij} = 0, \quad h^{\alpha}_{ijk} = 0$$

for any index α provived that $\lambda_i \neq \lambda_j$. Accordingly we have

(2.10) $$h^{\alpha}_{ij} = 0, \quad h^{\alpha}_{ijk} = 0 \quad \text{for} \quad i \notin [a], \quad j \in [a],$$

from which the definition of h_{ijk} gives

$$(\lambda_i - \lambda_j)\omega_{ji} = 0 \quad \text{for} \quad i \notin [a], \quad j \in [a],$$

because the eigenvalues are all constant. This concludes the proof.

By means of Lemma 2.2 and the local decomposition theorem (cf. [6]) the above discussion is summerized in the following way.

PROPOSITION 2.3. *Let $\bar{M}$ be an $(n+p)$-dimensional Riemannian manifold of constant curvature c, and M an n-dimensional submanifold with harmonic curvature in $\bar{M}$. If the mean curvature vector of M is parallel in the normal bundle, then M is locally a product of Riemannian manifolds.*

In the case where the ambient space is a Euclidean space, the theorem due to B. Smyth [10] is completely applied to the situation given above. Thus we have

THEOREM 2.4. *Let M be a compact simply connected Riemannian manifold with harmonic curvature and ϕ the isometric immersion of M into $\boldsymbol{R}^{n+p}$. If the mean curvature vector is parallel in the normal bundle, then M is a product of Riemannian manifolds $M_1 \times \cdots \times M_k$, and ϕ is a product of minimal immersions of their factors into spheres.*

3. Flat normal connection

This section is concerned with the study of submanifolds with flat normal connection. Let $\bar{M}$ be an $(n+p)$-dimensional Riemannian manifold of constant curvature c and M an n-dimensional submanifold with harmonic curvature in $\bar{M}$. The normal connection of M is said to be *flat* if the normal curvature form $\Omega_{\alpha\beta}$ vanishes identically. As is well known [1], the normal connection is flat if and only if there exist p mutually orthogonal unit normal vector fields e_α such that each of the e_α is parallel in the normal bundle. Of course, all of the shape operators A_α can be simultaneously diagonalizable. These facts imply that we may choose a local field of orthonormal frames $\{e_i, e_\alpha\}$ such that

$$\omega_{\alpha\beta}=0, \quad [A_\alpha, A_\beta]=0. \tag{3.1}$$

In addition, assume that the mean curvature vector τ is parallel in the normal bundle. It is easily seen that the function h^α is constant for any index α on M. Accordingly, under these situations all of calculations which were done for the parallel mean curvature vector in the previous section are considered. Consequently we have

LEMMA 3.1. *The second fundamental form σ on M is parallel.*

PROOF. Using (3.1) we have $\Sigma_r h^\alpha_{ir} h^\beta_{rj} = \Sigma_r h^\alpha_{jr} h^\beta_{ri}$, from which it follows

$$\Sigma_r h^\alpha_{irk} h^\beta_{rj} = \Sigma_r h^\alpha_{jrk} h^\beta_{ri} \tag{3.2}$$

by the similar argument to that of (2.7). Therefore it turns out that

$$\Sigma_r h^\beta_{jr} h^\alpha_{irk} = \Sigma_r h^\beta_{ir} h^\alpha_{rjk} = \Sigma_r h^\beta_{kr} h^\alpha_{rji}.$$

By differentiating the equation exteriorly and by making use of the Ricci formula, the straightforward calculation gives rise to

$$\Sigma_r (h^{\beta}_{ljr} h^{\alpha}_{kir} + h^{\alpha}_{ljr} h^{\beta}_{kir} - h^{\beta}_{ijr} h^{\alpha}_{lkr} - h^{\alpha}_{ijr} h^{\beta}_{lkr}) = \Sigma_{r,s} (R_{liks} h^{\alpha}_{sr} h^{\beta}_{rj} - R_{lirs} h^{\beta}_{sj} h^{\alpha}_{rk} - R_{lirs} h^{\beta}_{rk} h^{\alpha}_{sj} - R_{lijs} h^{\alpha}_{sr} h^{\beta}_{rk}).$$

Because all shape operators A_α are simultaneously diagonalizable, a local field of orthonormal frames $\{e_i\}$ may be chosen such that $h^{\alpha}_{ij} = \lambda^{\alpha}_i \delta_{ij}$. This shows

$$\Sigma_r (h^{\alpha}_{ljr} h^{\beta}_{kir} + h^{\beta}_{ljr} h^{\alpha}_{kir} - h^{\alpha}_{ijr} h^{\beta}_{lkr} - h^{\beta}_{ijr} h^{\alpha}_{lkr}) = R_{likj} (\lambda^{\alpha}_j - \lambda^{\alpha}_k)(\lambda^{\beta}_j - \lambda^{\beta}_k) \tag{3.3}$$

for any indices α, β, i, j, k and l. When $l=j$, $k=i$ and $\alpha=\beta$ in (3.3), it is reduced to

$$R_{ijji} (\lambda^{\alpha}_j - \lambda^{\alpha}_i)^2 = 2\, \Sigma_r\, (h^{\alpha}_{iir} h^{\alpha}_{jjr} - h^{\alpha}_{ijr} h^{\alpha}_{ijr}). \tag{3.4}$$

On the other hand, (3.2) is equivalent to $(\lambda^{\beta}_j - \lambda^{\beta}_i) h^{\alpha}_{ijk} = 0$, which yields that for any indices β and k

$$h^{\beta}_{ijk} = 0, \tag{3.5}$$

provided that there exist indices i and j such that $\lambda^{\alpha}_i \neq \lambda^{\alpha}_j$. Under this condition, (3.4) is deformed as $R_{ijji} (\lambda^{\alpha}_j - \lambda^{\alpha}_i)^2 = 0$ for any indices. In fact, for a fixed α, the same notation $[i]$ as that in §2, that is, $[i] = \{k: \lambda^{\alpha}_k = \lambda^{\alpha}_i\}$ is adapted. Then $\Sigma_r h^{\alpha}_{ijr} h^{\alpha}_{ijr}$ vanishes identically, because of $\Sigma_r = \Sigma_{r\in[i]} + \Sigma_{r\in[j]} + \Sigma_{r\notin[i]\cup[j]}$. This means $R_{ijji} = 0$ if $\lambda^{\alpha}_i \neq \lambda^{\alpha}_j$. Summing up for i, j and α in (3.4) we have

$$-2\, \Sigma_{\alpha,i,j,k} h^{\alpha}_{ijk} h^{\alpha}_{ijk} = \Sigma_{i,j} R_{ijji}\, \Sigma_\alpha\, (\lambda^{\alpha}_j - \lambda^{\alpha}_i)^2.$$

By coming back together with above two equations, the fact that the second fundamental form of M is parallel is asserted. According to the decomposition theorem of J. Erbacher [3], K. Yano and S. Ishihara [13] and M. Takeuchi [11], we can prove the following

THEOREM 3.2. *Let $\bar{M}$ be an $(n+p)$-dimensional complete simply connected Riemannian manifold of constant curvature c, and let M be an n-dimensional Riemannian submanifold with harmonic curvature in $\bar{M}$. Assume that the mean curvature vector is parallel in the normal bundle and the normal connection is flat. Then the second fundamental form is parallel and moreover if M is complete, then the following properties are asserted:*

(a) *When $c \geqq 0$, M is a product of Riemannian manifolds $M_1 \times \cdots \times M_k$, where each M_a is a small n_a-dimensional sphere of $\bar{M}$, except that one of M_a is a great sphere.*

(b) *When $c<0$, M is a product of Riemannian manifolds $M^{n_0}(c_0) \times M_1 \times \cdots \times M_k \subset M^{n_0}(c_0) \times M^{n+p-n_0-1}(c') \subset M^{n+p}(c)$ with $c_0<0$, $c'>0$, $1/c_0 + 1/c' = 1/c$, where $M_1 \times \cdots \times M_k \subset M^{n+p-n_0-1}(c')$ is a submanifold as the one in the case where $c>0$ and the second inclusion is the natural one.*

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