

## A SEQUENCE ASSOCIATED WITH THE ZEROS OF THE RIEMANN ZETA FUNCTION

By

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Let  $\zeta(s)$  be the Riemann zeta function, let  $C_n$  denote the generalized Euler constants associated with  $\zeta(s)$ , i. e.,

$$C_n = \lim_{N \rightarrow \infty} \left\{ \sum_{k=1}^N \log^n k/k - \log^{n+1} N/(n+1) \right\},$$

and define the numbers  $\delta_n (n \geq 1)$  by

$$\delta_1 = 1 + \frac{1}{2} C_0 - \frac{1}{2} \log \pi - \log 2,$$

and, for  $n \geq 2$ ,

$$(1) \quad \delta_n = 1 - (1 - 2^{-n})\zeta(n) + n \sum_{h=1}^n \frac{1}{h} \sum_{\substack{j_1 + \dots + j_h = n-h \\ j_1 > 0, \dots, j_h > 0}} \prod_{b=1}^h \frac{C_{j_b}}{j_b!}.$$

Then we have, for  $n \geq 1$ ,

$$(2) \quad \delta_n = \sum_{\rho} \rho^{-n},$$

where the sum  $\sum_{\rho}$  is taken over all complex zeros  $\rho$  of  $\zeta(s)$  (see [2]).

In this paper we shall study the sequence  $\{\delta_n\}$  and prove four theorems. In Theorem 1 we shall give an expression of  $\delta_n$ , and, using it, we shall derive, in Theorem 2, a necessary and sufficient condition for the truth of the Riemann hypothesis. The expression of  $\delta_n$  will be specified in Theorem 3 under the Riemann hypothesis, and in the final Theorem 4, we shall give an upper bound for the quantity  $|\delta_n|$ .

We start with the following lemma.

LEMMA. *Let  $m$  be a positive integer, let  $a_j (j=1, 2, \dots, m)$  be real numbers, and let  $b_j (j=1, 2, \dots, m)$  be mutually distinct positive numbers  $< \pi$ . If*

$$\sum_{j=1}^m a_j \cos b_j n \rightarrow 0,$$

as an integer  $n \rightarrow \infty$ , then  $a_j = 0$  for all  $j = 1, 2, \dots, m$ .

PROOF. Suppose that  $a_k \neq 0$  for some integer  $k$  with  $1 \leq k \leq m$ , and let  $\varepsilon$  be a positive number  $< \frac{1}{4} |a_k|$ . Then, from the condition of the lemma, there exists a positive integer  $n_0$  such that the inequality

$$(3) \quad \left| \sum_{j=1}^m a_j \cos b_j n \right| < \varepsilon$$

holds for all integers  $n \geq n_0$ . Let  $N$  be an arbitrary positive integer. If we multiply both sides of (3) by  $|2 \cos b_k n|$ , and add them with respect to  $n$  from  $n_0 + 1$  to  $n_0 + N$ , then we get

$$\sum_{j=1}^m a_j \sum_{n=n_0+1}^{n_0+N} \{ \cos (b_k - b_j)n + \cos (b_k + b_j)n \} = r, \text{ say,}$$

with  $|r| < 2\varepsilon N$ . It follows that

$$(4) \quad a_k \sum_{n=n_0+1}^{n_0+N} \cos 2b_k n + \sum_{\substack{j=1 \\ j \neq k}}^m a_j \sum_{n=n_0+1}^{n_0+N} \{ \cos (b_k - b_j)n + \cos (b_k + b_j)n \} = r - Na_k.$$

Since  $b_j$  are mutually distinct positive numbers  $< \pi$ ,  $2b_k \not\equiv 0 \pmod{2\pi}$  and  $b_k \pm b_j \not\equiv 0 \pmod{2\pi}$  for any pair of  $k$  and  $j$  with  $k \neq j$ , so that the left hand side of (4) is  $o(N)$ , as  $N \rightarrow \infty$ . On the other hand, the absolute value of the right hand side is greater than  $\frac{1}{2} |a_k| N$ , and hence the equation (4) yields a contradiction. This completes the proof.

THEOREM 1. For any  $n \geq 1$ ,  $\delta_n$  can be uniquely expressed in the form

$$(5) \quad \delta_n = 2 \sum_{k=1}^{\infty} r_k^{-n} \sum_{j=1}^{m_k} a_{kj} \cos b_{kj} n,$$

where  $\{r_k\}$  is an increasing sequence of positive numbers,  $m_k (k \geq 1)$  and  $a_{kj} (j = 1, 2, \dots, m_k)$  are positive integers, and  $b_{kj} (j = 1, 2, \dots, m_k)$  are mutually distinct positive numbers  $< \frac{1}{2} \pi$  such that  $r_k \cos b_{kj} < 1$ . The constants  $r_k, m_k, a_{kj}$  and  $b_{kj}$  are independent of  $n$ .

PROOF. If we put  $\rho = \beta + i\gamma$  in (2), we get

$$\delta_n = 2 \sum_{\gamma > 0} (\beta^2 + \gamma^2)^{-\frac{1}{2}n} \cos \left( n \arctan \frac{\gamma}{\beta} \right),$$

noting that the complex conjugate of  $\rho$  is also the zero of  $\zeta(s)$ . If we make use of the fact that  $0 < \beta < 1$ , we get  $0 < \arctan (\gamma/\beta) < \frac{1}{2} \pi$ , and  $(\beta^2 + \gamma^2)^{1/2} \cos \{ \arctan (\gamma/\beta) \} < 1$ . Hence, we can obtain the expression (5).

It remains to show the uniqueness. Suppose that  $\delta_n$  has two expressions

$$(6) \quad \delta_n = 2 \sum_{k=1}^{\infty} r_k^{-n} \sum_{j=1}^{m_k} a_{kj} \cos b_{kj} n = 2 \sum_{k=1}^{\infty} R_k^{-n} \sum_{j=1}^{M_k} A_{kj} \cos B_{kj} n.$$

Then we can get

$$(7) \quad r_1^{-n} \sum_{j=1}^{m_1} a_{1j} \cos b_{1j}n + O(r_2^{-n}) = R_1^{-n} \sum_{j=1}^{M_1} A_{1j} \cos B_{1j}n + O(R_2^{-n}).$$

To prove (7), we make use of the estimate ([4], Theorem 9. 4.)

$$N(T) = \frac{T}{2\pi} \log T - \frac{1}{2\pi} (1 + \log 2\pi) T + O(\log T),$$

where  $N(T)$  is the number of the zeros of  $\zeta(\sigma + it)$  in the region  $0 \leq \sigma \leq 1, 0 < t \leq T$ . Then we obtain

$$\sum_{T \leq \left(\frac{r_k^2 - 1}{4}\right)^{1/2} < 2T} r_k^{-n} \sum_{j=1}^{m_k} a_{kj} = \sum_{T \leq \gamma < 2T} |\rho|^{-n} < T^{-n} \{N(2T) - N(T)\} = O(T^{-n+1} \log T),$$

so that

$$\sum_{T \leq \left(\frac{r_k^2 - 1}{4}\right)^{1/2}} r_k^{-n} \sum_{j=1}^{m_k} a_{kj} = O(T^{-n+1} \log T).$$

It follows that, for sufficiently large  $T$ ,

$$\begin{aligned} \sum_{k=1}^{\infty} r_k^{-n} \sum_{j=1}^{m_k} a_{kj} \cos b_{kj}n &= r_1^{-n} \sum_{j=1}^{m_1} a_{1j} \cos b_{1j}n + O\left(\sum_{\substack{\left(\frac{r_k^2 - 1}{4}\right)^{1/2} < T \\ k \neq 1}} r_k^{-n} \sum_{j=1}^{m_k} a_{kj}\right) \\ &\quad + O\left(\sum_{T \leq \left(\frac{r_k^2 - 1}{4}\right)^{1/2}} r_k^{-n} \sum_{j=1}^{m_k} a_{kj}\right) \\ &= r_1^{-n} \sum_{j=1}^{m_1} a_{1j} \cos b_{1j}n + O(r_2^{-n} T \log T) + O(T^{-n+1} \log T); \end{aligned}$$

which leads to (7). Now, if we compare the orders of both sides of (7), we get, using the lemma,  $r_1 = R_1$ . Thus we have

$$\sum_{j=1}^{m_1} a_{1j} \cos b_{1j}n - \sum_{j=1}^{M_1} A_{1j} \cos B_{1j}n = O(r_1^n r_2^{-n} + R_1^n R_2^{-n}),$$

where we may assume that  $b_{11} < b_{12} < \dots < b_{1m_1}$  and  $B_{11} < B_{12} < \dots < B_{1M_1}$ . If the equation above holds, then we see that

$$\sum_{j=1}^{m_1} a_{1j} \cos b_{1j}n - \sum_{j=1}^{M_1} A_{1j} \cos B_{1j}n \rightarrow 0,$$

as the integer  $n \rightarrow \infty$ . We therefore have by the lemma

$$(8) \quad m_1 = M_1, a_{1j} = A_{1j}, b_{1j} = B_{1j}.$$

We next consider the case of  $k=2$ . We have from (6) and (8)

$$r_2^{-n} \sum_{j=1}^{m_2} a_{2j} \cos b_{2j}n + O(r_3^{-n}) = R_2^{-n} \sum_{j=1}^{M_2} A_{2j} \cos B_{2j}n + O(R_3^{-n}),$$

which implies that  $r_2=R_2$ ,  $m_2=M_2$ ,  $a_{2j}=A_{2j}$  and  $b_{2j}=B_{2j}$  ( $j=1, 2, \dots, m_2$ ). Repeating this, we get, for all  $k$ ,  $r_k=R_k$ ,  $m_k=M_k$ ,  $a_{kj}=A_{kj}$  and  $b_{kj}=B_{kj}$  ( $j=1, 2, \dots, m_k$ ). This proves the uniqueness for the expression (5) of  $\delta_n$ , and the proof is completed.

We next prove the following theorem.

**THEOREM 2.** *Let the notations be as in Theorem 1. Then the Riemann hypothesis holds if and only if  $m_k=1$ , and  $r_k \cos b_{k1}=\frac{1}{2}$  for all  $k$ .*

**PROOF.** If the Riemann hypothesis is true, then we get  $m_k=1$  for all  $k$ . Because, if  $m_k>1$  for some  $k\geq 1$ , we have at least two different complex zeros of  $\zeta(s)$  on a circle with center 0 and radius  $r_k$  in the upper plane  $\text{Im } s>0$ ; a contradiction. We now consider the modulus of the complex zero  $\rho=\beta+iy$  of  $\zeta(s)$ . Then we get  $|\rho|=r_k$  for some  $k$ , and therefore we have  $r_k \cos b_{k1}=\frac{1}{2}$  under the hypothesis, since  $\beta=|\rho| \cos \{\arctan (y/\beta)\}=\frac{1}{2}$ . The uniqueness of the expression (5) completes the proof.

At present, we do not have a formula for  $\delta_n$  which enables us to determine  $r_k$ ,  $m_k$ ,  $a_{kj}$  and  $b_{kj}$ . However we have the following theorem. We define inductively the sequence  $\delta(n, k)$  with positive integers  $n$  and  $k$ . We put

$$\delta(n, 1)=\delta_n,$$

and define  $\delta(n, k)$  for  $k\geq 2$  by

$$\delta(n, k)=\delta(n, k-1)-2R_{k-1}^{-n} \sum_{\substack{v=0 \\ v \text{ even}}}^n (-1)^{v/2} \binom{n}{v} \left(\frac{1}{2} R_{k-1}^{-1}\right)^{n-v} \left(1-\frac{1}{4} R_{k-1}^{-2}\right)^{v/2}$$

with

$$R_{k-1}=\lim_{n\rightarrow\infty} \left\{ \frac{1}{2} \delta^2(n, k-1)-\frac{1}{2} \delta(2n, k-1) \right\}^{-1/2n}.$$

**THEOREM 3.** *Let the notations be as in Theorem 1. Suppose that the Riemann hypothesis is true, and all of the zeros of  $\zeta(s)$  are simple. Then, for all  $k$ ,  $m_k=1$ ,  $r_k=R_k$ ,  $a_{k1}=1$ , and  $b_{k1}=\arctan (R_k^2-\frac{1}{4})^{1/2}$ .*

**PROOF.** By Theorem 2, we get  $m_k=1$ ,  $b_{k1}=\arctan (r_k^2-\frac{1}{4})^{1/2}$ , and easily obtain  $a_{k1}=1$  from the fact that all of the zeros are simple. Hence, it is enough to prove that, for every  $k$ ,

$$(9) \quad \delta(n, k)=2 \sum_{h=k}^{\infty} r_h^{-n} \cos b_{h1}n, \quad r_k=R_k.$$

We prove this by induction on  $k$ . In case  $k=1$ , we have  $\delta(n, 1)=\delta_n$ , and hence we have only to show  $r_1=R_1$ . We get from the distribution of the zeros of  $\zeta(s)$

$$\delta(n, 1)=2r_1^{-n} \cos b_{11}n + O(r_2^{-n}).$$

It follows that

$$\frac{1}{2} \delta^2(n, 1) - \frac{1}{2} \delta(2n, 1) = r_1^{-2n} + O(r_2^{-n} r_1^{-n}),$$

since, for any  $\theta$ ,  $2 \cos^2 \theta - \cos 2\theta = 1$ . Therefore

$$\left\{ \frac{1}{2} \delta^2(n, 1) - \frac{1}{2} \delta(2n, 1) \right\}^{1/2n} = r_1^{-1} \left[ 1 + O \left\{ \frac{1}{n} \left( \frac{r_2}{r_1} \right)^{-n} \right\} \right],$$

and hence we get  $r_1 = R_1$ . We next assume that the condition (9) holds in case  $k-1$ . We then have from the definition of  $\delta(n, k)$

$$(10) \quad \delta(n, k) = 2 \sum_{h=k-1}^{\infty} r_h^{-n} \cos b_{h1} n - 2r_{k-1}^{-n} \sum_{\substack{v=0 \\ v \text{ even}}}^n (-1)^{v/2} \binom{n}{v} \left( \frac{1}{2} r_{k-1}^{-1} \right)^{n-v} \left( 1 - \frac{1}{4} r_{k-1}^{-2} \right)^{v/2},$$

since  $r_{k-1} = R_{k-1}$ . By Theorem 2, we get  $r_{k-1} \cos b_{k-1,1} = \frac{1}{2}$ , and hence

$$\begin{aligned} \cos b_{k-1,1} n &= \sum_{\substack{v=0 \\ v \text{ even}}}^n (-1)^{v/2} \binom{n}{v} \cos^{n-v} b_{k-1,1} \sin^v b_{k-1,1} \\ &= \sum_{\substack{v=0 \\ v \text{ even}}}^n (-1)^{v/2} \binom{n}{v} \left( \frac{1}{2} r_{k-1}^{-1} \right)^{n-v} \left( 1 - \frac{1}{4} r_{k-1}^{-2} \right)^{v/2}, \end{aligned}$$

which implies

$$2r_{k-1}^{-n} \cos b_{k-1,1} n = 2r_{k-1}^{-n} \sum_{\substack{v=0 \\ v \text{ even}}}^n (-1)^{v/2} \binom{n}{v} \left( \frac{1}{2} r_{k-1}^{-1} \right)^{n-v} \left( 1 - \frac{1}{4} r_{k-1}^{-2} \right)^{v/2}.$$

It follows from (10) that

$$\delta(n, k) = 2 \sum_{h=k}^{\infty} r_h^{-n} \cos b_{h1} n.$$

We can easily prove that  $r_k = R_k$  by means of the formula  $\frac{1}{2} \{ \delta^2(n, k) - \delta(2n, k) \} = r_k^{-2n} + O(r_k^{-n} r_{k+1}^{-n})$ , and the proof is completed.

Recently, we obtained an expression of  $\gamma_1$  ([3], Theorem 8), where  $\frac{1}{2} + i\gamma_1$  ( $\gamma_1 > 0$ ) is the first complex zero of  $\zeta(s)$ , which states that

$$\gamma_1 = \lim_{n \rightarrow \infty} \left\{ \left( \frac{1}{2} \delta_n^2 - \frac{1}{2} \delta_{2n} \right)^{-1/n} - \frac{1}{4} \right\}^{1/2}.$$

Let  $\gamma_k$  be the  $k$ th ordinate of the zeros of  $\zeta(s)$ . Noticing that  $r_k = |\frac{1}{2} + i\gamma_k|$  for all  $k$  under the assumptions of Theorem 3, we have, in general, the following corollary.

**COROLLARY 3.** *With the same assumptions as in Theorem 3, we have, for every  $k$ ,*

$$\gamma_k = \lim_{n \rightarrow \infty} \left[ \left\{ \frac{1}{2} \delta^2(n, k) - \frac{1}{2} \delta(2n, k) \right\}^{-1/n} - \frac{1}{4} \right]^{1/2}.$$

It is desirable to have a precise estimate for  $\delta_n$  and  $\delta(n, k)$ . In this direction, we can show the following upper bound for the quantity  $|\delta_n|$ .

**THEOREM 4.**  $\delta_n = O\{(1 + C_0)^n\}$ .

**PROOF.** Let us recall an estimate for  $C_n$  ([3], Theorem 4) which states that  $|C_n| < 0.0001 \exp(n \log \log n)$  for all  $n \geq 10$ , and the numerical values for  $0 \leq n \leq 9$  (see [1]), that is,  $C_0 = 0.57721566$ ,  $C_1 = -0.07281584$ ,  $C_2 = -0.00969036$ ,  $C_3 = 0.00205387$ ,  $C_4 = 0.00232537$ ,  $C_5 = 0.00079332$ ,  $C_6 = -0.00023876$ ,  $C_7 = -0.00052728$ ,  $C_8 = -0.00035212$  and  $C_9 = -0.00003439$ . Then we can prove that  $|C_n|/n!$  has the maximum  $|C_0|/0! = C_0$  at  $n=0$ . Hence, for  $n \geq 2$ , we have from (1)

$$\begin{aligned} \delta_n &= 1 - (1 - 2^{-n})\zeta(n) + n \sum_{h=1}^n \frac{1}{h} \sum_{\substack{j_1 + \dots + j_n = n-h \\ j_1 \geq 0, \dots, j_n \geq 0}} \prod_{b=1}^h \frac{C_{j_b}}{j_b!} \\ &= O(1) + O\left\{ \sum_{h=1}^n \binom{n}{h} C_0^h \right\} = O\{(1 + C_0)^n\}, \end{aligned}$$

since  $1 - (1 - 2^{-n})\zeta(n) = O(1)$ , and  $\sum_j C_0^h = \binom{n-1}{h-1} C_0^h$ . This proves the theorem.

If we use Theorem 3 and Theorem 4, we obtain  $r_1 \geq (1 + C_0)^{-1}$ , so that the first complex zero  $\rho_1$  satisfies  $|\rho_1| \geq (1 + C_0)^{-1}$ , which leads to the following corollary.

**COROLLARY 4.**  $\gamma_1 > \left\{ (1 + C_0)^{-2} - \frac{1}{4} \right\}^{1/2}$ .

### References

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