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ON ARTINIAN QF-3, 1-GORENSTEIN RINGS

By

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(Dedicated to Professor Hisao TOMINAGA on his 60th birthday)

A noetherian ring is called 1-Gorenstein if it has the self-injective dimension at most one on both sides. A well known example of artinian QF-3, 1-Gorenstein rings is the triangular matrix ring over a QF ring, which is QF-2, that is, every indecomposable projective module has the simple socle. Conversely Sumioka [11] characterized such a ring as an artinian QF-3, 1-Gorenstein ring with QF maximal quotient ring. But an artinian QF-3, 1-Gorenstein ring has not necessarily the QF maximal quotient ring (see §4). On the other hand, Sumioka's result is a generalization of Harada's characterization of artinian QF-3 hereditary rings, which states that a connected artinian ring is QF-3 hereditary if and only if it is Morita equivalent to a triangular matrix ring over a division ring (cf. [3]). Our results in the present paper are closely related to their results mentioned above.

First we shall deal with artinian QF-3 hereditary rings, which were investigated by Harada [3] and Iwanaga [4]. Our result is as follows.

THEOREM I. *Let Λ be a connected artinian ring which is not a QF ring. Then the following conditions for Λ are equivalent.*

(1) *Λ is a QF-3 hereditary ring.*

(2) *Λ is Morita equivalent to a triangular matrix ring over a division ring.*

(3) *Λ is a (left and right) serial 1-Gorenstein ring.*

(4) *Λ is a QF-3, 1-Gorenstein ring with a simple projective left module.*

(5) *Λ is a QF-3, 1-Gorenstein ring with a simple injective left module.*

Next we shall deal with the following problem:

(*) To investigate the length of the socle of an indecomposable projective module over an artinian QF-3, 1-Gorenstein ring.

It is well known that every indecomposable projective module over Λ is distributive in the sense of [1] if Λ is a representation-finite algebra over an algebraically closed field (cf. [6]). So it seems that it is worth studying artinian QF-3, 1-Gorenstein rings over which every indecomposable projective module is distributive. Our answer to the problem (*) is given by the following theorem.

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THEOREM II. *Let Λ be an artinian QF-3, 1-Gorenstein ring, and P an indecomposable projective Λ-module. If P is distributive, then* $|\mathrm{soc}\,(P)|\leqq 2$. *In particular if* $|\mathrm{soc}\,(P)|=2$, *then P has the smallest loose waist X such that top* $(X)\cong\mathrm{soc}\,(E(P)/P)$.

Here a submodule X of a module P is called a loose waist if X satisfies the following properties:

(i) X is local and essential in P.

(ii) If a submodule Y of P is local and essential in P, then either $Y\subset X$ or $X\subset Y$ holds.

Note that we do not assume that X is non-trivial. By definition a local waist is a loose waist, but the converse does not hold even under the assumption of Theorem II (see §4). Moreover we shall construct a QF-3, 1-Gorenstein algebra with a non-distributive indecomposable module P such that $|\mathrm{soc}\,(P)|=3$ (see §4).

The proof of Theorem I will be given in §2. Theorem II will be deduced from a more general result which will be shown in §3. The final section §4 is devoted to some examples. In particular we shall construct examples of QF-3, 1-Gorenstein algebras whose maximal quotient rings have the self-injective dimensions equal to any given m for $2\leqq m\leqq\infty$.

Throughout the present paper, a ring means an artinian ring with identity whose radical is denoted by N, modules are always unitary, and an algebra means a finite dimensional algebra over a field unless otherwise specified. For a module M, we shall denote the injective hull of M by $E(M)$, the socle of M by soc (M) and the top of M by top (M).

§1. Preliminaries

In the present section we shall give general remarks which will be used in the following sections.

The following was obtained by Iwanaga [4, Theorem 1] and Sumioka [12, Theorem 5].

LEMMA 1.1. *Let Λ be an artinian ring. Then the following conditions are equivalent.*

(1) *Λ is QF-3 and 1-Gorenstein.*

(2) *$E({}_\Lambda\Lambda)$ is projective and $E({}_\Lambda\Lambda)\oplus[E({}_\Lambda\Lambda)/\Lambda]$ is an injective cogenerator.*

LEMMA 1.2. *Let Λ be an artinian QF-3, 1-Gorenstein ring, and P an indecomposable projective non-injective Λ-module. Then $E(P)/P$ is an indecomposable injective non-projective Λ-module, and the canonical surjection: $E(P)\to E(P)/P$ is the projective cover.*

PROOF. See [13, Lemma 8.1] and recall the definition of 1-Gorenstein rings.

Let Λ be an artinian QF-3, 1-Gorenstein ring, and $\{P_1,\cdots,P_n\}$ a complete set of non-isomorphic indecomposable projective left Λ-modules. Let $S_i=\mathrm{top}\,(P_i)$, and $I=\{1,\cdots,n\}$. We can define a map $\sigma=\sigma_\Lambda$ of I into I as follows:

If P_i is injective, then $\sigma(i)=j$ where $S_j\cong\mathrm{soc}\,(P_i)$.

If P_i is non-injective, then $\sigma(i)=j$ where $S_j\cong\mathrm{soc}\,(E(P_i)/P_i)$.

The following lemma will play an important role in §2.

LEMMA 1.3. *The map σ_Λ is a permutation for any artinian QF-3, 1-Gorenstein ring Λ.*

PROOF. Let $I'=\{i\in I|P_i=E(P_i)\}$ and $I''=I-I'$. By Lemma 1.2 we have $E(S_{\sigma(i)})\cong E(P_i)/P_i$ for $i\in I''$. If $\sigma(i)\in\sigma(I')\cap\sigma(I'')$, then $E(S_{\sigma(i)})$ is projective and we have a decomposition $E(P_i)\cong E(S_{\sigma(i)})\oplus P_i$, which is impossible. Therefore we see $\sigma(I')\cap\sigma(I'')=\phi$. In order to show our statement, it is sufficient to verify that the restriction maps $\sigma|I'$ and $\sigma|I''$ are injections. The verification for $\sigma|I'$ is trivial. By Lemma 1.2 it is clear that $\sigma|I''$ is an injection.

In the following sections we shall encounter the following type of exact sequences:

$$(*)\qquad 0\longrightarrow P\xrightarrow{\lambda} G_1\oplus\cdots\oplus G_n\xrightarrow{\pi} L\longrightarrow 0$$

So we shall let the following notation for the above sequence and we shall keep it throughout the present paper. Let $\lambda_i:P\to G_i$ be the composite map of λ and the canonical projection: $G_1\oplus\cdots\oplus G_n\to G_i$, and let $\pi_i:G_i\to L$ be the composite map of π and the canonical inclusion: $G_i\to G_1\oplus\cdots\oplus G_n$. Furthermore let $\psi_i=\lambda_i\pi_i$: $P\to L$, and $W(P)=\cap_{i=1}^n \operatorname{Ker}(\psi_i)$.

DEFINITION 1.4. We call $W(P)$ the negligible submodule of P with respect to the exact sequence $(*)$.

The following is a key lemma for the proofs of Theorem I and Theorem II.

PROPOSITION 1.5. *Let Λ be an artinian ring. Consider the following exact sequence of nonzero finitely generated Λ-modules.*

$$(*)\qquad 0\longrightarrow P\xrightarrow{\lambda} G_1\oplus\cdots\oplus G_n\xrightarrow{\pi} L\longrightarrow 0$$

where $n\geqq 2$ and λ is an essential monomorphism. Assume that P is local and L is colocal. Let $S=\operatorname{soc}(L)$, and $V=\cap_{i=1}^n (S\psi_i^{-1})$. Then the following statements hold.

(i) $\operatorname{soc}(P/W)=V/W\cong S^{(\mu)}$ for some $\mu\geqq 1$.

(ii) The sequence $(*)$ induces the following exact sequence.

$$0\to V/W\to (V\lambda_1/W\lambda_1)\oplus\cdots\oplus(V\lambda_n/W\lambda_n)\to S\to 0$$

Here we let $W=W(P)$.

PROOF. Let $W_i=W\lambda_i$, $V_i=V\lambda_i$, $\tilde{W}=\Sigma_{i=1}^n\oplus W_i$ and $\tilde{V}=\Sigma_{i=1}^n\oplus V_i$. Since P is local and λ is an essential monomorphism, we have immediately,

(1) $\psi_i\neq 0$ for each i.

By the definition of W and V, we have easily,

(2) $W\lambda = \tilde{W} = \tilde{W} \cap V\lambda = \tilde{W} \cap P\lambda$,

(3) $V\lambda = \tilde{V} \cap P\lambda$.

We have the epimorphism $\bar{\pi} = (\bar{\pi}_i)$ which makes the following diagram commutative.

$$\begin{tikzcd} G_1 \oplus \cdots \oplus G_n \arrow[rr, "\pi"] \arrow[dr, "can."'] & & L \\ & (G_1/W_1) \oplus \cdots \oplus (G_n/W_n) \arrow[ur, "\bar{\pi}"'] & \end{tikzcd}$$

Here *can.* means the canonical surjection. By (2) we have the exact sequence below.

$$0 \longrightarrow P/W \xrightarrow{\bar{\lambda}} (G_1/W_1) \oplus \cdots \oplus (G_n/W_n) \xrightarrow{\bar{\pi}} L \longrightarrow 0$$

Let $\bar{\pi}' : \tilde{V}/\tilde{W} \to L$ be the restriction map of $\bar{\pi}$. By the definition of V and by the assumption for L, we have $NV_i \subset W_i$ for each i where N denotes the radical of Λ. It follows from (1) that $(V_i/W_i)\bar{\pi}_i = S$ for some i. Therefore $\bar{\pi}' : \tilde{V}/\tilde{W} \to S$ is an epimorphism. By (2) and (3) we see Ker $(\bar{\pi}') = (P\lambda \cap \tilde{V})/\tilde{W} = V\lambda/W\lambda$. This shows our statement (ii).

It remains to show (i). Since L is colocal and $S = \mathrm{soc}\,(L)$, we have $0 \neq V/W = \mathrm{soc}\,(P/W)$ by (1). Let $\bar{\lambda} = (\bar{\lambda}_i)$. Then we have the following commutative diagram for each i.

$$\begin{tikzcd} P \arrow[r, "\lambda_i"] \arrow[d, "\rho"] & G_i \arrow[r, "\pi_i"] \arrow[d, "\rho_i"] & L \arrow[d, equal] \\ P/W \arrow[r, "\bar{\lambda}_i"'] & G_i/W_i \arrow[r, "\bar{\pi}_i"'] & L \end{tikzcd}$$

Here ρ and ρ_i are the canonical surjections. Suppose that V/W contains a simple submodule S' with $S' \not\cong S$. Let $X = S'\rho^{-1}$. Then $0 = S'\bar{\lambda}_i\bar{\pi}_i = X\lambda_i\pi_i = X\psi_i$ for each i. This shows $X \subset W$ and hence $S' = 0$, which is impossible. Thus we have just proved our statement (i).

§2. Proof of Theorem I

In the present section we shall give the proof of Theorem I stated in the introduction. Now Harada [3] established the equivalence of the conditions (1) and (2) in Theorem I. Iwanaga [4] proved the condition (3) implies the condition (1). Thus the conditions (1), (2) and (3) are equivalent. It is trivial that the condition (1) implies the conditions (4) and (5). Therefore we have only to show the implications (5) ⇒ (4) and (4) ⇒ (1). In the remainder of the present section we assume that Λ is a connected basic artinian ring which is QF-3 and 1-Gorenstein, and we denote the radical of Λ by N.

From Lemma 2.1 up to Lemma 2.4 we assume moreover that Λ has a simple injective left module $I_1 = S_1$.

LEMMA 2.1. *Let $P_1 \to I_1 \to 0$ be the projective cover. Then we have $Hom_\Lambda\,(P_1, Q) = 0$ for any indecomposable projective left module Q which is not isomorphic to P_1.*

PROOF. Obvious.

LEMMA 2.2. *Let I be an indecomposable direct summand of $E({}_{\Lambda}\Lambda)/\Lambda$, and $0\to P'\to P(I)\to I\to 0$ the projective cover. Then P' is indecomposable projective and $E(P')=P(I)$.*

PROOF. By [5, Proposition 1], P' is projective. Since I is a summand of $E(\Lambda)/\Lambda$, $P(I)$ is a summand of $E(\Lambda)$. Hence $P(I)$ is injective. Therefore we have $E(P')=P(I)$ and P' is indecomposable by Lemma 1.2.

LEMMA 2.3. *Consider the exact sequence below*

$$0\to S\to V\to I\to 0$$

where S is simple, V is colocal and I is injective indecomposable. If S is embedded into L, then S is projective.

PROOF. We have the following commutative diagram with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & S \arrow[r] \arrow[d, equal] & V \arrow[r] \arrow[d, "f"] & I \arrow[r] \arrow[d, "g"] & 0 \\
0 \arrow[r] & S \arrow[r] & E(S) \arrow[r] & E(S)/S \arrow[r] & 0
\end{tikzcd}$$

Since S is essential in V, f is a monomorphism and so is g. Since Λ is QF-3, $E(S)$ is projective. It follows from [13, Lemma 8.1] that $E(S)/S$ is indecomposable. Since I is injective indecomposable, g is an isomorphism and so is f. Thus V is projective. By [5, Proposition 1] we conclude that S is projective.

Now we define indecomposable projective modules P_i and indecomposable injective modules I_i inductively so long as $S_{i-1}=\mathrm{top}\ (P_{i-1})$ can not be embedded into Λ, as follows.

Let $I_1=S_1=E(S_1)$.

For $i>1$, we take the projective cover:

$$0\to P_i\to P(I_{i-1})\to I_{i-1}\to 0.$$

Let $I_i=E(S_i)$ where $S_i\cong\mathrm{top}\ (P_i)$.

In fact, P_i is indecomposable projective by Lemma 2.2, and hence S_i is simple and I_i is indecomposable injective. We assume that $\mathrm{Hom}_{\Lambda}\ (S_k, \Lambda)\neq 0$ and $\mathrm{Hom}_{\Lambda}\ (S_i, \Lambda)=0$ for each $i<k$. Then it follows from Lemma 1.3 that $P_1=P(I_1)$, $P_2,\cdots, P_k$ are not isomorphic each other, and we have $E(P_i)=P(I_{i-1})$ by Lemma 2.2 for $2\leqq i\leqq k$.

Next we define left modules V_2, $V_3,\cdots$, V_k as the pushout in the following diagrams.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & P_i & \longrightarrow & P(I_{i-1}) & \longrightarrow & I_{i-1} & \longrightarrow & 0 \\ & & {\scriptstyle\text{can.}}\downarrow & {\scriptstyle\text{p.o.}} & \downarrow & & \| & & \\ 0 & \longrightarrow & S_i & \longrightarrow & V_i & \longrightarrow & I_{i-1} & \longrightarrow & 0 \end{array}$$

Here can.: $P_i \to S_i$ is the canonical surjection.

LEMMA 2.4. *In the notation above, we have $P(I_i)=P_1$ for $1 \leqq i<k$, $V_i \cong I_i$ and $I_i/\mathrm{soc}\,(I_i) \cong I_{i-1}$ for $2 \leqq i<k$. Moreover V_i is uniserial for $i \leqq k$.*

PROOF. We assume that our statement holds for $1 \leqq i<k-1$. Then we have $P_{j+1}=NP_j$ for $j \leqq i$ by the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & P_{j+1} & \longrightarrow & P(I_j) & \longrightarrow & I_j & \longrightarrow & 0 \\ & & \downarrow & & \| & & \downarrow & & \\ 0 & \longrightarrow & P_j & \longrightarrow & P(I_{j-1}) & \longrightarrow & I_{j-1} & \longrightarrow & 0 \end{array}$$

Therefore $V_{i+1} \cong P_1/NP_{i+1}$ is uniserial, and hence we have $I_{i+1}=E(V_{i+1})$. Thus the epimorphism: $V_{i+1} \to I_i$ induces an epimorphism: $I_{i+1} \to I_i$, and hence P_1 is isomorphic to a direct summand of $P(I_{i+1})$.

Suppose that I_{i+1} is not local. Since $P(I_{i+1})$ is injective projective by Lemma 2.2, we have a decomposition

$$P(I_{i+1})=Q_1 \oplus \cdots \oplus Q_s \qquad (s \geqq 2)$$

where Q_i is indecomposable projective and $Q_1 \cong P_1$. Applying Proposition 1.5 to the exact sequence below:

$$0 \to P_{i+2} \to Q_1 \oplus \cdots \oplus Q_s \to I_{i+1} \to 0,$$

we see that $S_{i+1}(=\mathrm{soc}\,(I_{i+1}))$ can be embedded into $P_{i+2}/W(P_{i+2})$ where $W(P_{i+2})$ is the negligible submodule with respect to the above exact sequence. So we have a nonzero map $g: P_{i+1} \to P_{i+2}$. Let H be the pushout of g and the canonical inclusion $\kappa: P_{i+1} \to E(P_{i+1})$. Since $E(P_{i+1})=P(I_i)=P_1$ and $E(P_{i+2})=P(I_{i+1})$ by Lemma 2.2, we have the following diagram with exact rows.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & P_{i+1} & \xrightarrow{\kappa} & P_1 & \longrightarrow & I_i & \longrightarrow & 0 \\ & & {\scriptstyle g}\downarrow & {\scriptstyle\text{p.o.}} & \downarrow{\scriptstyle g'} & & \| & & \\ 0 & \longrightarrow & P_{i+2} & \xrightarrow[\kappa']{} & H & \longrightarrow & I_i & \longrightarrow & 0 \\ & & \| & & \downarrow{\scriptstyle \rho} & & \downarrow{\scriptstyle h} & & \\ 0 & \longrightarrow & P_{i+2} & \xrightarrow{\nu} & E(P_{i+2}) & \longrightarrow & I_{i+1} & \longrightarrow & 0 \end{array}$$

Since I_i has no composition factor which is isomorphic to $S_{i+1}=\mathrm{soc}\,(I_{i+1})$, we have $h=0$.

Thus there exists a map μ: $H \to P_{i+2}$ such that $\rho = \mu\nu$. So we have $\nu = \kappa'\mu\nu$. Since ν is a monomorphism, we have $\kappa'\mu = 1_{P_{i+2}}$. Thus we have a decomposition $H = (P_{i+2}\kappa') \oplus J$ where $J \cong I_i$. Then we have $\mathrm{Hom}_\Lambda(P_1, H) \cong \mathrm{Hom}_\Lambda(P_1, J)$ by Lemma 2.1, and hence $g\kappa' = \kappa g' = 0$. Since κ' is a monomorphism, we see $g = 0$, which is a contradiction. Therefore I_{i+1} is a local module.

On the other hand, $V_{i+1} \subset I_{i+1}$ induces a monomorphism: $I_i \cong V_{i+1}/S_{i+1} \to I_{i+1}/S_{i+1}$. Since I_{i+1}/S_{i+1} is local, we have $I_i \cong I_{i+1}/S_{i+1}$, which shows $I_{i+1} = V_{i+1}$. Furthermore we have $P_1 = P(I_{i+1})$ by the fact that I_{i+1} is local. It remains to show that V_k is uniserial. We have already shown $I_{k-1}/S_{k-1} \cong I_{k-2}$ and $P(I_{k-1}) \cong P(I_{k-2}) \cong P_1$. Thus we have the following commutative diagram with exact rows.

$$\begin{tikzcd}
0 \arrow[r] & P_k \arrow[r] \arrow[d] & P(I_{k-1}) \arrow[r] \arrow[d, equal] & I_{k-1} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & P_{k-1} \arrow[r] & P(I_{k-2}) \arrow[r] & I_{k-2} \arrow[r] & 0
\end{tikzcd}$$

Therefore we have $P_k \cong NP_{k-1}$. Since we have already shown $P_{i+1} \cong NP_i$ for $1 \leqq i \leqq k-2$, we see that $V_k \cong P_1/N^kP_1$ is uniserial.

Proof of the implication (5) $\Rightarrow$ (4).

We keep the notation in the preceding argument. Then it follows from Lemma 1.3 that there exists an index $k > 1$ such that $\mathrm{Hom}_\Lambda(S_k, \Lambda) \neq 0$ and $\mathrm{Hom}_\Lambda(S_j, \Lambda) = 0$ for $j < k$. By Lemma 2.4 we have the following exact sequence:

$$0 \to S_k \to V_k \to I_{k-1} \to 0$$

where V_k is colocal and I_{k-1} is indecomposable injective. Thus it follows from Lemma 2.3 that S_k is projective.

Next we shall that the condition (4) implies the condition (1). So in the remainder of the present section we assume that Λ is an artinian basic connected QF-3, 1-Gorenstein ring with a simple projective left module $P_1 = S_1$.

Proof of the implication (4) $\Rightarrow$ (1).

If P_1 is injective, we have ${}_\Lambda\Lambda = P_1$ because Λ is connected. So $\Lambda \cong \mathrm{End}(P_1)$ and it is a division ring. Hence we can assume that P_1 is not injective.

Let $\sigma = \sigma_\Lambda$ be the permutation in Lemma 1.3, and let $\sigma(i) = i+1$ for $i \leqq k-2$. Then we assume that we have a series of uniserial projective non-injective left modules $P_1, P_2, \cdots, P_{k-1}$ such that $NP_i \cong P_{i-1}$ for $2 \leqq i \leqq k-1$. Then $k-1 < n$ because Λ is QF-3 and Λ has an indecomposable injective projective module, where n is the number of non-isomorphic indecomposable projective modules. Let $k = \sigma(k-1)$. In other words we let $S_k \cong \mathrm{soc}(E(P_{k-1})/P_{k-1})$. Then $S_i \not\cong S_j$ for $1 \leqq i < j \leqq k$ by Lemma 1.3.

We shall prove $NP_k \cong P_{k-1}$, which shows that P_k is also uniserial. Now we take the pull back of the canonical surjection: $E(P_{k-1}) \to E(P_{k-1})/P_{k-1}$ and the canonical inclusion: $S_k \to E(P_{k-1})/P_{k-1}$.

(*)
$$\begin{tikzcd} 0 \arrow[r] & P_{k-1} \arrow[r, "\kappa"] \arrow[d, equal] & U_k \arrow[r] \arrow[d] & S_k \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & P_{k-1} \arrow[r] & E(P_{k-1}) \arrow[r] & E(P_{k-1})/P_{k-1} \arrow[r] & 0 \end{tikzcd}$$

If U_k is not uniserial, then there exists i, $2 \leqq i \leqq k-1$, such that $NP_i \cong P_{i-1}$ is uniserial and P_i is not uniserial. This gives a contradiction that $E(P_{i-1})/P_{i-1}$ is not colocal. Thus U_k is uniserial. Take the projective cover of U_k.

$$0 \longrightarrow W_k \longrightarrow P_k \xrightarrow{\pi} U_k \longrightarrow 0$$

Take the pull back of π and κ.

(**)
$$\begin{tikzcd} 0 \arrow[r] & W_k \arrow[r] \arrow[d, equal] & Y \arrow[r] \arrow[d] & P_{k-1} \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & W_k \arrow[r] & P_k \arrow[r] & U_k \arrow[r] & 0 \end{tikzcd}$$

Then $P_k/Y \cong U_k/P_{k-1} \cong S_k$ and hence $Y \cong NP_k$. Therefore $NP_k \cong W_k \oplus P_{k-1}$. Suppose $W_k \neq 0$. Then neither P_k nor W_k is injective because P_k is local. Hence $E(P_k)/P_k \neq 0$ and $E(W_k)/W_k \neq 0$. Moreover we have the following exact sequence:

$$0 \longrightarrow P_k/NP_k \xrightarrow{(\lambda_1, \lambda_2)} [E(W_k)/\mathrm{W}_k] \oplus [E(P_{k-1})/P_{k-1}] \longrightarrow E(P_k)/P_k \longrightarrow 0.$$

Since $E(P_k)/P_k$ is colocal and $P_k/NK_k \cong S_k$, it follows from Lemmas 1.2 and 1.3 that $\lambda_1 = 0$ and hence $S_k \cong E(P_{k-1})/P_{k-1}$. In view of the diagram (*), $U_k \cong E(P_{k-1})$ and it is projective. This shows $W_k = 0$, which is a contradiction. Thus we have $P_k \cong U_k$, and $NP_k \cong P_{k-1}$ by the definition of U_k. Since Λ has at least one indecomposable projective injective left module, we have by induction a series of uniserial projective left modules $P_1, \cdots, P_k$ with $P_{i-1} \cong NP_i (2 \leqq i \leqq k)$ such that $P_1, \cdots, P_{k-1}$ are non-injective and P_k is injective.

We shall show that $P_1 \oplus \cdots \oplus P_k$ forms a block of Λ. We have only to show $\mathrm{Hom}_\Lambda (P_i, Q) = 0$ and $\mathrm{Hom}_\Lambda (Q, P_i) = 0$ for any indecomposable projective module Q which is not isomorphic to P_i for $1 \leqq i \leqq k$. Suppose that there exists a nonzero map f: $P_i \to Q$. If $\mathrm{Ker}(f) \neq 0$, then $\mathrm{soc}(\mathrm{Im}(f)) \cong S_j$ for some $j > 1$. Since Λ is QF-3, $E(S_j)$ is projective. On the other hand, we see $S_j = \mathrm{soc}(E(P_{j-1})/P_{j-1})$ and hence $E(S_j)$ is a direct summand of $E(\Lambda)/\Lambda$. It is impossible by Lemma 1.3. Therefore such a map f is a monomorphism. Since we have shown $E(P_i) = P_k$, there exists a nonzero map g which makes the diagram below commutative.

$$\begin{array}{ccc} P_i & \xrightarrow{f} & Q \\ \downarrow & \swarrow g & \\ P_k = E(P_i) & & \end{array}$$

Therefore we have only to show $\mathrm{Hom}_\Lambda (Q, P_i)=0$. Suppose $\mathrm{Hom}_\Lambda (Q, P_i)\neq 0$. Then we have $\mathrm{Hom}_\Lambda (Q, P_k)\neq 0$ because P_i is embedded into P_k. Let g be a nonzero map of Q into P_k. Since every nonzero submodule of P_k is isomorphic to some P_i, we have $Q\cong P_i$ for some $i\leqq k$ because Q is indecomposable. It contradicts our assumption $Q\ncong P_i$ for each i. Therefore we see that $P_1\oplus\cdots\oplus P_k$ is a block of Λ. Since Λ is connected, we have $\Lambda=P_1\oplus\cdots\oplus P_k$ which shows that Λ is left hereditary. Since Λ is an artinian ring, Λ is hereditary.

§3. Proof of Theorem II

It follows from Lemma 1.2 that Theorem II stated in Introduction is a special case of the following.

THEOREM II′. *Let Λ be an artinian ring. Let*

$$(*)\qquad 0 \longrightarrow P \xrightarrow{\lambda} G_1\oplus\cdots\oplus G_n \xrightarrow{\pi} L \longrightarrow 0$$

be an exact sequence of nonzero finitely generated Λ-modules satisfying the following properties.

(i) λ is an essential monomorphism.

(ii) P is local and distributive.

(iii) L is colocal.

(iv) G_i is colocal for each i.

Then we have $|\mathrm{soc}\,(P)|\leqq 2$. In particular if $|\mathrm{soc}\,(P)|=2$, then the following statements hold.

(1) P has the smallest loose waist X so that $\mathrm{top}\,(X)\cong\mathrm{soc}\,(L)$.

(2) P/W is isomorphic to a submodule of L where W is the negligible submodule of P with respect to the sequence $(*)$.

(3) W is the sum of all colocal submodules of P.

REMARK 3.1. As is easily seen by our proof, we have $n\leqq 2$ without the hypothesis (iv). In this setting, however, $|\mathrm{soc}\,(P)|\leqq 2$ need not hold.

In the sequel we keep the above setting and denote $\mathrm{soc}\,(G_i)$ by S_i and $\mathrm{soc}\,(L)$ by S. We let λ_i, π_i, ψ_i, W, V, W_i, V_i, $\tilde{W}$ and $\tilde{V}$ be the same ones in the proof of Proposition 1.5. In order to show our statements, we can assume $n\geqq 2$ and we have only to show $n=2$ and that our statements (1), (2) and (3) hold.

LEMMA 3.2. *$V/W=\mathrm{soc}\,(P/W)\cong S$.*

PROOF. Since P is distributive, we have $S\cong V/W=\mathrm{soc}\,(P/W)$ by Proposition 1.5 and [1, Theorem 1].

LEMMA 3.3. *$n=2$ and P/W can be embedded into L.*

PROOF. By Proposition 1.5, the following sequence is exact.

$$0 \longrightarrow V/W \xrightarrow{\bar{\lambda}} (V_1/W_1)\oplus\cdots\oplus(V_n/W_n) \xrightarrow{\bar{\pi}} S \longrightarrow 0$$

By Lemma 3.2 we can assume $V_1/W_1\cong V_2/W_2\cong S$ and $V_i/W_i=0$ for $i\geqq 3$. On the other hand we have the following commutative diagram.

$$\begin{array}{ccc} V/W & \longrightarrow & V\lambda_1/W\lambda_1 \\ \downarrow & & \downarrow \\ P/W & \longrightarrow & P\lambda_1/W\lambda_1 \end{array}$$

Here the vertical maps are canonical inclusions. Hence the map: $P/W\to P\lambda_1/W\lambda_1$ is an isomorphism. So we have $P/W\cong P\lambda_1/W\lambda_1\cong P\lambda_2/W\lambda_2$. We define a Λ-module L' and a Λ-map: $L'\to L$ in the following commutative diagram with exact rows in which $P\lambda_i/W\lambda_i\to G_i/W_i$ is the canonical inclusions for each i.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & P/W & \longrightarrow & (P\lambda_1/W\lambda_1)\oplus\cdots\oplus(P\lambda_n/W\lambda_n) & \longrightarrow & L' & \longrightarrow & 0 \\ & & \| & & \downarrow \quad\quad\quad\quad \downarrow & & \downarrow & & \\ 0 & \longrightarrow & P/W & \xrightarrow{\bar{\lambda}} & (G_1/W_1)\oplus\cdots\oplus(G_n/W_n) & \xrightarrow{\bar{\pi}} & L & \longrightarrow & 0 \end{array}$$

Then L' is nonzero because P is local and $P\lambda_i/W\lambda_i\neq 0$ for $i=1, 2$, and $L'\to L$ is a monomorphism. Therefore L' is colocal and $\mathrm{soc}\,(L')\cong\mathrm{soc}\,(L)=S$. Since we have

$$(P\lambda_1/W\lambda_1)\oplus\cdots\oplus(P\lambda_n/W\lambda_n)=(P/W)\bar{\lambda}\oplus(P\lambda_2/W\lambda_2)\oplus\cdots\oplus(P\lambda_n/W\lambda_n),$$

we have $L'\cong(P\lambda_2/W\lambda_2)\oplus\cdots\oplus(P\lambda_n/W\lambda_n)$. Since L' is colocal and $P\lambda_2/W\lambda_2\cong P/W\neq 0$, we have $L'\cong P\lambda_2/W\lambda_2\cong P/W$ and $P\lambda_i/W\lambda_i=0$ for each $i\geqq 3$. Since we have shown $\psi_i\neq 0$ for each i in Proposition 1.5, we have $P\lambda_i=W\lambda_i=0$ for each $i\geqq 3$. This completes the proof.

LEMMA 3.4 *W is the sum of all colocal submodules of P. In particula we have* $\mathrm{soc}\,(W)=\mathrm{soc}\,(V)=\mathrm{soc}\,(P)$.

PROOF. In the proof of Proposition 1.5, we have already shown that W is a sum of some colocal submodules of P. So we have only to show that every colocal submodule X of P is contained in W. It is obvious that $\mathrm{soc}\,(X)$ is isomorphic to S_1 or S_2 by Lemma 3.3. Let $\mathrm{soc}\,(X)\cong S_1$ for instance. Since P is distributive, it follows from [1, Theorem 1] that λ_i:

$X \to G_1$ is a monomorphism. Furthermore we have $X\lambda_2=0$. For, $S_2\lambda^{-1} \cong S_2$ because λ is an essential monomorphism, and hence $X\lambda_2 \neq 0$ implies $X+S_2\lambda^{-1}=X\oplus(S_2\lambda^{-1})$ in P, and consequently we have a monomorphism

$$S_2\oplus S_2 \to (X\oplus(S_2\lambda^{-1}))/(X\cap \mathrm{Ker}\,(\lambda_2)),$$

which is impossible because P is distributive. By the formulae $\psi_1+\psi_2=0$ and $X\lambda_2=0$, we have easily $W\supset X$.

LEMMA 3.5. *Take two elements x and y in V so that $x\notin W$, $y\notin W$* **and** top $(\Lambda x)\cong$ top $(\Lambda y)\cong S$. *Then $\Lambda x=\Lambda y$.*

PROOF. Suppose $\Lambda x\neq\Lambda y$. Then we can assume $\Lambda x\cap\Lambda y\subset Nx$ where N denotes the radical of Λ. Since V/W is simple by Lemma 3.2, we have $V=\Lambda x+W=\Lambda y+W$. Let $x=cy+w$ for some $c\in\Lambda$ and some $w\in W$. Since P is distributive, we have $W\cap(\Lambda x+\Lambda y)=(W\cap\Lambda x)+(W\cap\Lambda y)\subset Nx+Ny$. Hence we have

$$w=x-cy=-ax+by \quad \text{for some} \quad a\in N \quad \text{and} \quad b\in N.$$

Thus we have $x+ax=by+cy\in\Lambda x\cap\Lambda y\subset Nx$ and hence $x\in Nx$, which is impossible.

LEMMA 3.6. *Take an element x in V such that* top $(\Lambda x)\cong S$ *and $x\notin W$. When Λx is the smallest loose waist in P.*

PROOF. We assume $P\subset G_1\oplus G_2$ in the following discussion. Now $x\notin W$ implies $x\lambda_1\neq 0$ and $x\lambda_2\neq 0$. This shows that Λx is essential in P. Take any element $y\in P$ such that Λy is local and essential in P. We have only to show $\Lambda y\supset\Lambda x$. Now suppose $y\psi_1=0$. Then $y\in W$ because of the formula $\psi_1+\psi_2=0$. Hence we have

$$\Lambda y=\Lambda y\cap(W_1\oplus W_2)=(\Lambda y\cap W_1)\oplus(\Lambda y\cap W_2).$$

Since Λy is indecomposable, either $\Lambda y\cap W_1=0$ or $\Lambda y\cap W_2=0$ holds. But it is impossible because Λy is essential in P. Therefore $y\psi_1\neq 0$. Since L has the simple socle S, there exists an integer $h\geqq 0$ such that $N^h y\psi_1=S$. This shows that there exists an element a in N^h such that $ay\in V$, $ay\notin W$ and top $(\Lambda ay)\cong S$. By Lemma 3.5, we have $\Lambda x=\Lambda ay\subset\Lambda y$.

Theorem II′ is a conclusion of the above Lemmas from 3.2 up to 3.6.

§4. Remarks

In the present section we shall give some remarks and examples related to our theorems.

We have not yet had any examples of artinian left 1-Gorenstein ring which is not right 1-Gorenstein, that is, an artinian ring Λ such that $id({}_\Lambda\Lambda)=1$ and $id(\Lambda_\Lambda)=\infty$ (cf. [14, Lemma A]). In case of artin algebras we have the following, which is easily obtained by making use of elementary properties of the tilting theory. (See [2] for the tilting theory.)

PROPOSITION 4.1. *Let Λ be an artin algebra over a commutative ring R, and $D(\Lambda)$ $=\mathrm{Hom}_R(\Lambda, E_R(\mathrm{top}(R)))$. Then the following conditions are equivalent.*

(1) ${}_\Lambda D(\Lambda)_\Lambda$ *is a tilting module.*

(2) $id({}_\Lambda\Lambda)=1$.

(3) $id(\Lambda_\Lambda)=1$.

Compare the above proposition with Lemma 1.1 in the present paper.

We denote a connected basic serial ring with left admissible sequence $(a_1, \cdots, a_n)$ by Ser $(a_1, \cdots, a_n)$. A similar argument as in [10] shows the following, which will be applied in Examples 4.5 and 4.6.

PROPOSITION 4.2. *Let $\Gamma=\mathrm{Ser}\,(a_1, \cdots, a_n)$ with the properties that $a_1=2\leqq a_i\leqq 3=a_n$ and $a_i=3$ implies $a_{i+1}=2$. Let $\{i_0<i_1<\cdots<i_t\}=\{0\}\cup\{i\,|\,a_i=3\}$ and $m=max$ $\{i_j-i_{j-1}\,|\,1\leqq j\leqq t\}$. Then $\mathrm{gl.dim}\,\Gamma=m$.*

PROPOSITION 4.3. *For any given m, $2\leqq m\leqq\infty$, there exists a QF-3, 1-Gorenstein algebra with maximal quotient ring A such that $id({}_AA)=id(A_A)=\mathrm{gl.dim}\,A=m$.*

In fact we shall construct such examples in Examples 4.5, 4.6 and 4.7. We begin with

DEFINITION 4.4. Let Q be a bounden quiver. A vertex i in Q is called a node if $\beta\alpha=0$ for each arrow α: $j\rightarrow i$ and each arrow β: $i\rightarrow k$.

In the sequel let K be a field, and $K(Q)$ the bounden quiver algebra over K for a bounden quiver Q.

EXAMPLE 4.5. Let Q be the following bounden quiver:

$$\begin{tikzcd} 2 \arrow[r, shift left, "\alpha"] & 1 \arrow[l, shift left, "\gamma"] \arrow[r, shift left, "\beta"] & 3 \arrow[l, shift left, "\delta"] \end{tikzcd}$$

with the relations that the vertex 1 is a node and $\alpha\gamma=\delta\beta$. Then $K(Q)$ is a QF-2, 1-Gorenstein algebra whose maximal quotient ring A is Morita equivalent to the ring Ser (2, 3). Therefore gl.dim $A=2$ by Proposition 4.2.

EXAMPLE 4.6. Let Q be the following bounden quiver:

$$\begin{tikzcd}
& x \arrow[dr, "\gamma"] & & & & \\
1 \arrow[ur, "\alpha"] \arrow[dr, "\beta"'] & & 2 \arrow[r] & 3 \arrow[r] & \cdots \arrow[r] & n \arrow[lllll, bend left=60] \\
& y \arrow[ur, "\delta"'] & & & &
\end{tikzcd} \quad (n \geqq 2)$$

in which the vertices $1, 2, \cdots, n$ are all nodes and the commutative relation $\gamma\alpha = \delta\beta$ holds. Then $K(Q)$ is a QF-2, 1-Gorenstein algebra whose maximal quotient ring A is Morita equivalent to the ring Ser $(2, 2, \cdots, 2, 3)$ in which the term$=2$ occurs just n times. Therefore gl.dim $A = n+1$ by Proposition 4.2.

EXAMPLE 4.7. Let Q be the following bounden quiver:

$$\begin{tikzcd}
& 1 \arrow[dd, "\alpha"] & \\
3 \arrow[ur, "\delta"] & & 4 \arrow[ul, "\rho"'] \\
& 2 \arrow[ul, "\beta"] \arrow[ur, "\gamma"'] &
\end{tikzcd} ; \qquad \begin{aligned} &\delta\beta = \rho\gamma, \\ &\delta\beta\alpha = \alpha\delta\beta = \gamma\alpha\delta = \beta\alpha\rho = 0. \end{aligned}$$

Then $K(Q)$ is a QF-3, 1-Gorenstein algebra by Proposition 4.9 below, and its maximal quotient ring A coincides with the following bounden quiver.

$$\begin{tikzcd}
& 1 \arrow[dd, shift right, "\alpha"'] \arrow[dd, shift left, "\alpha'"] & \\
3 \arrow[ur, "\delta"] & & 4 \arrow[ul, "\rho"'] \\
& 2 \arrow[ul, "\beta"] \arrow[ur, "\gamma"'] &
\end{tikzcd} ; \qquad \begin{aligned} &\delta\beta = \rho\gamma, \\ &\delta\alpha = \beta\alpha' = \alpha'\delta = \alpha\rho = 0. \end{aligned}$$

Then it is easy to show $id({}_AA) = id({}_AAe_1) = \infty$.

As is well known, a finite poset can be regarded as an ordinary quiver in a natural way. Such a quiver is called a poset quiver in the present paper. Let G be a poset. An element $x \in G$ is said to be regular if x is comparable with any element in G. Otherwise x is said to be irregular.

DEFINITION 4.8. A finite poset G is said to be admissible if it satisfies the following conditions.

(1) For any element $x \in G$, there exists at most one element in G which is incomparable with x.

(2) Let $\{x, y\}$ be a pair of incomparable elements in G.

(2–1) There exists the least upper bound $x \cup y$ of $\{x, y\}$, which is regular.

(2–2) If there exists the largest lower bound $x \cap y$ of $\{x, y\}$, then it is regular. If there does not exist $x \cap y$, then both x and y are minimal elements.

PROPOSITION 4.9. *Let G be an admissible poset, and $\Gamma = K(G)$. Let $T = \Gamma \ltimes D(\Gamma)$ be the trivial extension of Γ by $D(\Gamma) = \mathrm{Hom}_K(\Gamma, K)$. We let $1 = 1_T = \Sigma_{x \in G}\, \varepsilon_x$, the decomposition of identity 1_T into a sum of primitive orthogonal idempotents. Let $\Lambda = T/\mathrm{soc}\,(\Sigma_{x \in G, x:\mathrm{regular}}\, T\varepsilon_x)$. Then the ring Λ is a QF-3, 1-Gorenstein algebra.*

PROOF. We denote the canonical surjection: $T \to \Lambda$ by Φ and let $\Phi(\varepsilon_x) = e_x$. We can view mod (Λ) as a full subcategory of mod (T) through Φ. If $x \in G$ is irregular, then Λe_x is identified with $T\varepsilon_x$ and hence ${}_\Lambda \Lambda e_x$ is injective because T is a symmetric algebra. When $x \in G$ is regular, we let $U(x) = \{y \in G \mid y > x\}$ and divide the situation into the following two cases:

(1) The case where $U(x)$ has the smallest element y.

(2) The case where $U(x)$ is empty or $U(x)$ has not the smallest element.

First we consider the case (1). Then Λe_x has the simple socle which is isomorphic to top (Λe_y). It is quite easy to show that $\Lambda e_x \subset E_\Lambda(\Lambda e_x) \subset E_T(\Lambda e_x)$ and $E_T(\Lambda e_x)/\Lambda e_x$ is a simple T-module where, for a Λ-module M, $E_\Lambda(M)$ or $E_T(M)$ is the injective hull of M as Λ-module or T-module respectively. Here $E_T(\Lambda e_x)$ cannot be a Λ-module by definition. Therefore $\Lambda e_x = E_\Lambda(\Lambda e_x)$ and hence it is an injective Λ-module.

Next we consider the case (2). Then, by considering the convering of Γ, we can assume $U(x) \neq \phi$ by replacing G into another admissible poset G' poset G' so that $T = K(G') \ltimes D(K(G'))$. So we can assume that $U(x)$ has two minimal elements a and b which are incomparable each other. By the definition of admissible posets, there exists $c = a \cup b$. Now we have soc $(P_x) \cong$ top $(P_a) \oplus$ top (P_b) where $P_y = \Lambda e_y$ for $y \in G$. Since both a and b are irregular, we have $E(\mathrm{top}\,(P_a)) \cong P_a$ and $E(\mathrm{top}\,(P_b)) \cong P_b$ and hence $E_\Lambda(P_x) \cong P_a \oplus P_b$. Combining results proved above, we see that Λ is QF-3. On the other hand we have immediately $P_x/\mathrm{soc}\,(P_a) \cong \mathrm{rad}\,(P_a)/\mathrm{soc}\,(P_a) \cong \mathrm{rad}\,(E_\Lambda(P_x)/P_x)$, which has the simple socle isomorphic to top (P_c). As is easily seen, $Y = E_\Lambda(P_x)/P_x$ can be embedded into $E_T(\mathrm{top}\,(P_c)) = T\varepsilon_c$ as T-module and $T\varepsilon_c/Y$ is a simple T-module. Since $T\varepsilon_c$ cannot be a Λ-module by definition, we see that $E_\Lambda(P_x)/P_x$ is an injective Λ-module. Therefore Λ is 1-Gorenstein.

Note that the algebra in Example 4.7 is obtained by means of the abobe proposition by letting G as follows.

$$G\colon \begin{tikzcd}[row sep=small] & & 3 \\ 1 \arrow[r] & 2 \arrow[ur] \arrow[dr] & \\ & & 4 \end{tikzcd}$$

REMARK 4.10. A triangular matrix ring over a QF ring has a QF maximal quotient ring, and we have shown in [8] that any 1-Gorenstein ring which is its own maximal quotient ring is QF.

REMARK 4.11. In contrast with Proposition 4.3, any QF-3, 1-Gorenstein ring with zero socle has the QF classical quotient ring (cf. [9]). Here a ring R is said to be left QF-3 if every finitely generated submodule of $E({}_RR)$ is torsionless. See [7], [8] and [12] for the related topics.

EXAMPLE 4.12. Theorem II does not necessarily hold without the assumption that P is distributive, as the following example shows.

Let Q be the following bounden quiver:

$$\begin{tikzcd}[column sep=huge, row sep=large]
1 \arrow[r, shift left, "a"] \arrow[d, "u"] & 2 \arrow[l, shift left, "c"] \arrow[r, shift left, "b"] \arrow[d, "v"] & 3 \arrow[l, shift left, "d"] \arrow[d, "w"] \\
4 \arrow[r, shift left, "a'"] & 5 \arrow[l, shift left, "c'"] \arrow[r, shift left, "b'"] \arrow[u, shift left=2, "y"] \arrow[u, shift right=2, "x"'] & 6 \arrow[l, shift left, "d'"]
\end{tikzcd}$$

relations:

$$\begin{aligned}
&va=a'u,\ vd=d'w,\ ac=xv=db,\\
&uc=c'v,\ wb=b'v,\ a'c'=d'b'=vx,\\
&ba=cd=ca=bd=0=b'a'=c'd'=c'a'=b'd',\\
&yv=vy=by=ya'=0=cx=xd'=xvx.
\end{aligned}$$

Then it is verified that $K(Q)$ is a QF-3, 1-Gorenstein algebra with a non-distributive indecomposable projective left module P_5 such that $|\mathrm{soc}\,(P_5)|=3$.

PROPOSITION 4.13. *Let A be a QF algebra over K, and B a QF-3, 1-Gorenstein algebra over K. Then $A\otimes_K B$ is a QF-3, 1-Gorenstein algebra.*

PROOF. An easy exercise.

EXAMPLE 4.14. Smallest loose waists are not necessarily waists, as the following example shows.

Let Λ be the K-algebra in Example 4.7, and $B=\mathrm{Ser}\,(2, 2)$ as K-algebra. Then it follows from Proposition 4.13 that $\Gamma=B\otimes_K\Lambda$ is a QF-3, 1-Gorenstein algebra. Let e_i be the primitive idempotent of Λ corresponding to the vertex i in the quiver in Example 4.7. Also let f_i be the primitive idempotent of B. As is easily shown, every indecomposable projective Γ-module is distributive. But the smallest loose waist in $\Gamma(f_1 \otimes e_1)$ is not a waist in it.

EXAMPLE 4.15. There arises an analogous question to Theorem I whether an artinian 1-Gorenstein ring with a simple projective or an injective module is hereditary or not. The following example tells us that the statement mentioned above does not hold.

Let Q be the bounden quiver below.

$$\begin{tikzcd} & & 3 \arrow[dd, "\gamma;"] \\ Q\colon 1 \arrow[r, "\alpha"] & 2 \arrow[ur, "\beta"] & \\ & & 4 \arrow[ul, "\delta"] \end{tikzcd} \qquad \gamma\beta = \delta\gamma = \beta\delta = 0.$$

Then $K(Q)$ is a left serial 1-Gorenstein algebra with a simple injective left module and with a simple projective right module, which is neither QF-3 nor hereditary.

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