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NOTE ON LEFT SERIAL ALGEBRAS

By

Manabu HARADA

(Dedicated to the memory of Professor Akira HATTORI)

Let R be a left and right artinian ring with identity. We have studied the condition $(*, n)$: every maximal submodule of direct sum of arbitrary n R-hollow modules is also a direct sum of hollow modules [1].

We shall study, in this short note, some left serial rings satisfying $(*, 1)$ for right R-module, and give a characterization of such a left serial algebra with $J^4=0$.

§1. Algebras of right local type

Let R be a left and right artinian ring with identity. We assume that every R-module is a unitary right R-module and we denote the Jacobson radical and the socle of an R-module M by $J(M)$ and Soc (M), respectively. We put $J=J(R)$, and $|M|$ means the length of a composition series of M. Following H. Tachikawa [5], R is called a ring of right local type, if every finitely generated right R-module is a direct sum of local (hollow) modules. We are sometimes interested in an algebra R over a field K with the following condition:

(A) *$eRe/eJe=eK+eJe$ for each primitive idempotent e,* (Condition II″ in [1], e.g., K is an algebraically closed field).

T. Sumioka found the following remarkable result for a left serial ring R [4]:

LEMMA 1. ([4], Corollary 4.2). *Let R be a left serial ring, then eJ^i is a direct sum of hollow modules as right R-modules for any i.*

On the other hand, if R satisfies $(*, 1)$, then eJ^i has the same structure from the definition (cf. [3], §1). Further we obtained

LEMMA 2. ([3], Theorem 4). *Let R be a right artinian ring.* Then R satisfies $(*, 1)$ *for any hollow module if and only if the following two conditions are fulfiled:*

1) *$eJ=\sum_{i=1}^{n(e)}\oplus A_i$, where e is any primitive idempotent in R and the A_i are hollow.*
2) *Let $C_i\supset D_i$ be two submodules of A_i such that C_i/D_i is simple. If f: $C_i/D_i\approx C_j/D_j$ for $i\neq j$, f or f^{-1} is extendible to an element in* Hom_R $(A_i/D_i, A_j/D_j)$ *or* Hom_R $(A_j/D_j, A_i/D_i)$.

We shall study a relationship between those lemmas in the next section.

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LEMMA 3. *Let R be a left serial algebra with (A), and $eJ^i=\sum_{j=1}^{n_i}\oplus A_{ij}$ with A_{ij} hollow (from Lemma 1). Then $\bar{A}_{ij}\not\approx\bar{A}_{ij'}$ for $j\neq j'$, where $\bar{A}_{ij}=A_{ij}/A_{ij}J$.*

PROOF. Assume $\bar{A}_{i1}\approx\bar{A}_{i2}\approx fR/fJ$: f is a primitive idempotent. Then $A_{ij}=a_{ij}R$; $a_{ij}f=a_{ij}\,(j=1, 2)$. Since Rf is uniserial, there exists x in eRe such that $a_{i1}=xa_{i2}$ (or $a_{i2}=xa_{i1}$). If $x\in eJe$, $a_{i1}\in eJ^{i+1}$. Hence $x\notin eJe$, and $x=ek+j$; $k\in K$, $j\in eJe$ from (A). $a_{i1}=(ek+j)a_{i2}=eka_{i2}+ja_{i2}\equiv a_{i2}k \pmod{eJ^{i+1}}$, contradiction.

THEOREM 1. *Assume that R is a left serial algebra with (A). Then the following are equivalent:*

1) *R is of right local type.*
2) *R satisfies (*, 2) and $|eJ/eJ^2|\leqslant 2$ for each e.*
3) *R satisfies (*, 3).*

PROOF. This is clear from Lemma 3, [3], Theorem 7 and [5].

THEOREM 2. *Let R be an algebra over a field. Assume that R is a left serial algebra. Then the following are equivalent:*

1) *R is of right local type.*
2) *R satisfies (*, 3).*

PROOF. This is clear from [1], Theorem 1, [2], Remark 2 and [5].

We give an example for Theorem 1, 2.

$$R=\begin{pmatrix} K & K & \cdots & K \\ & K & & \\ 0 & & \ddots & 0 \\ & & & K \end{pmatrix}$$

is a left serial algebra with (*, 2) and $|eJ/eJ^2|=n$.

§2. (*, 1)

We study, in this section, some left serial rings satisfying (*, 1). First we give

THEOREM 3. *Let R be a left serial ring. Then R satisfies (*, 1) if eJ is a direct sum of uniserial modules for each primitive idempotent e.*

PROOF. Let $C_i\supset D_i$ be submodules of A_i such that C_i/D_i is simple and $\bar{h}$: $C_1/D_1\approx C_2/D_2$. Since C_i is hollow, $C_1=x_1R$ and $C_2=h(x_1)R$, where $h(x_1)$ is a representation of $\bar{h}(x_1)$. We may assume that $x_1f=x_1$ and $h(x_1)f=h(x_1)$ for a primitive idempotent f, since C_i is hollow. Rf being uniserial, there exists y in R such that $x_1=yh(x_1)$ or $h(x_1)=yx_1$. Since $\bar{h}$ is an isomorphism, we may assume $h(x_1)=yx_1$, and $y\in eRe$. For any element d in D_1, $d=x_1r$; $r\in R$. Then $yd=yx_1r=h(x_1)r\in C_2$. Hence $h(x_1)r+D_2=\bar{h}(x_1)r+D_2=\bar{h}(x_1r)+D_2=\bar{h}(d)+D_2=D_2$, and so $yd\in D_2$. Let p_2: $eJ\to A_2$ be the projection. Then $g=P_2y_l|A_1$ is in $\mathrm{Hom}_R(A_1, A_2)$ and $g(D_1)\subset P_2(D_2)=D_2$, where y_l is the left-sided multiplication of y. Hence $\bar{h}$ is extendible to g in $\mathrm{Hom}_R(A_1/D_1, A_2/D_2)$, and so $(*, 1)$ is satisfied by Lemma 2.

THEOREM 4. *Let R be a left serial algebra with (A) and put $J(eR)=\Sigma_{i=1}^{n(e)}\oplus A_i$, $J(A_i)=\Sigma_{j=1}^{n_i}\oplus B_{ij}$, where the A_i and B_{ij} are hollow. Assume that $J^4=0$. Then the following are equivalent:*

1) *R satisfies $(*, 1)$.*
2) *eR has the following structure: If $\bar{B}_{ij}\approx C_{i'j'}$, then $B_{i'j'}$ is unierial, where $\bar{B}_{ij}=B_{ij}/B_{ij}J$ and $C_{i'j'}$* is a simple submodule in $J(B_{i'j'})$, $(i\neq i')$.

PROOF. Assume that R satisfies $(*, 1)$ and $\bar{B}_{11}\approx C_{21}\subset eJ^3$. Put $D_1^*=J(B_{11})\oplus B_{12}\oplus\cdots\oplus B_{1n_1}$. Then f: $J(A_1)/D_1^*\approx\bar{B}_{11}\approx C_{21}$. Assume that f is extended to $g'\in\mathrm{Hom}_R(A_2, A_1/D_1^*)$. Since $A_2J^2\supseteq C_{21}$, $\bar{B}_{11}=f\ (C_{21})=v'(C_{21})\subseteq(A_1/D_1^*)J^2=0$. Hence f is extendible to g in $\mathrm{Hom}_R(A_1/D_1^*, A_2)$ by Lemma 2. Now, since A_1/D_1^* is uniserial and $g(\mathrm{Soc}\,(A_1/D_1^*))=C_{21}$, g is a monomorphism, and so $g(A_1/D_1^*)$ is a uniserial submodule of $J(A_2)$ which contains C_{21}, and $|g(A_1/D_1^*)|=2$. However $g(A_1/D_1^*)$ is a direct sum of two simple modules from the structure of $J(A_2)/C_{21}$ and the fact that $g(A/D_1^*)/C_{21}$ is simple, provided that B_{21} is not uniserial. Therefore B_{21} is uniserial. Conversely, if 1) is satisfied, then $(*, 1)$ is trivially satisfied. Assume that 2) and $C_i\supset D_i$ are submodules in A_i and h:$C_1/D_1\approx C_2/D_2$. Since we use Lemma 2, we may assume that $C_1\subset eJ^2$, because, if $C_1\not\subset eJ^2$, $C_1=A_1$ and $D_1=A_1J$. Since C_i/D_i is simple, $C_i/D_i=\bar{x}_iR$ and we may assume $x_1f=x_1$, where x_i is in C_i and f is a primitive idempotent. Further from Lemma 3 we may assume that either $x_1\notin eJ^3$ or $x_2\notin eJ^3$, since eJ^3 is semisimple. Let $x_1=b_1+\cdots b_{n_1}\notin eJ^3$; $b_i\in B_{1i}$. Since $x_1f=x_1$ and $x_1=b_1f+\cdots+b_{n_1}f$, $b_i=b_if$ and there exists k say 1 from Lemma 3 such that

$$(\#)\qquad b_jf\in eJ^3\quad\text{for}\quad j\neq 1\ (\text{actually } b_jf=0 \text{ except one } j').$$

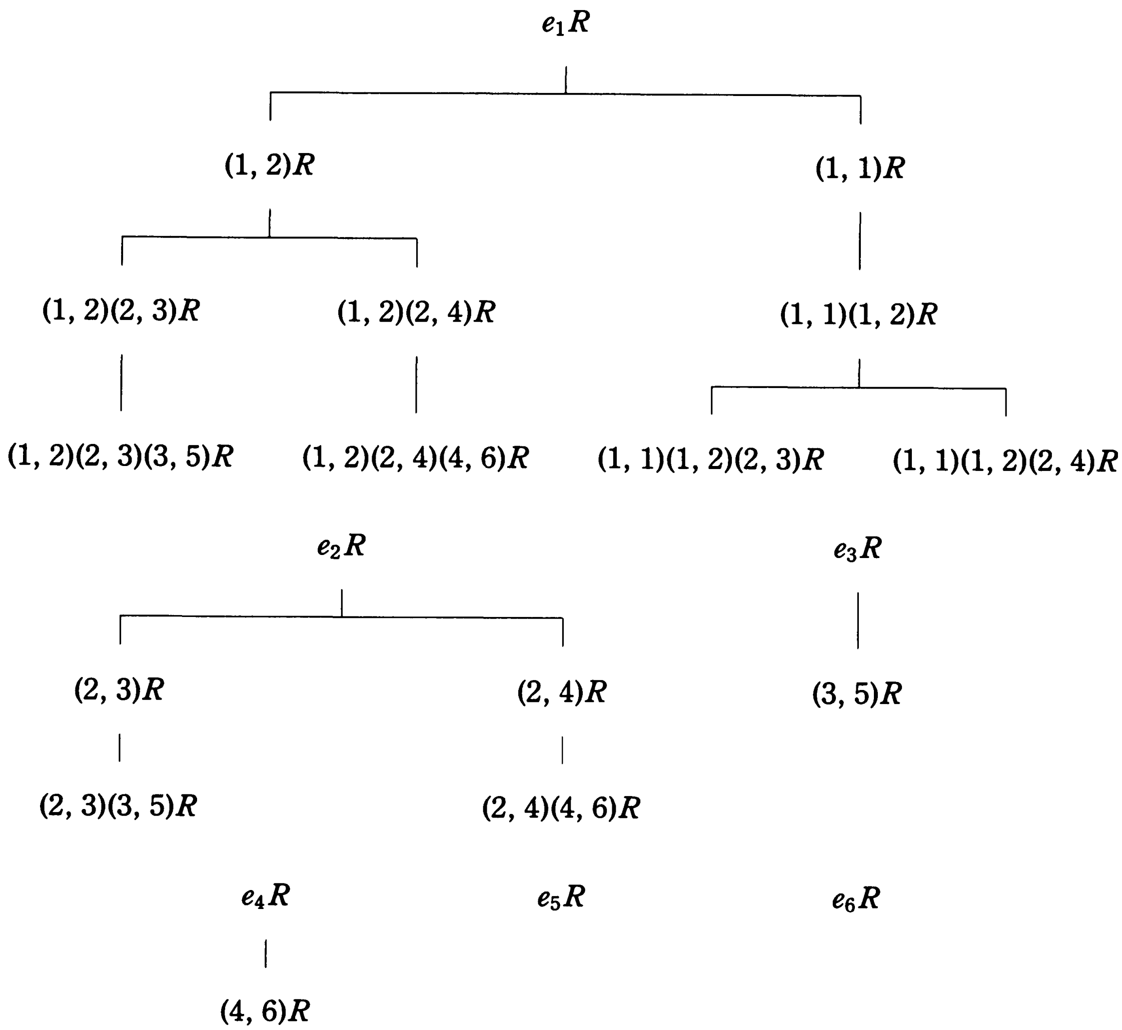
Let x_2' be a representation of $h(\bar{x}_1)$ such that $x_2'f=x_2'$. Put $x_2'=b_1'+\cdots+b_{n_2}'$; $b_i'\in B_{2i}$. If some b_i' is in $B_{2i}-J(B_{21})$, $\bar{B}_{11}=B_{11}/J(B_{11})=\bar{b}_1R\approx\bar{b}_i'R=\bar{B}_{2i}$, which is a contradiction by Lemma 3. Hence $b_i'\in J(B_{2i})$ for all i, and so $x_2'=b_t'$ for some t by Lemma 3 (cf. (#)). Then B_{2t} is uniserial from the assumption. Now $C_2=x_2'R\oplus D_2$, since $x_2'R=\bar{x}_2'R$ is a simple submodule of C_2 and there exists d in eJe such that $x_2'=dx_1$. Being d in eJe, $dA_1\subset eJ^2$. By p_2 we denote the projection of eJ^2 to B_{2t} and put $g=p_2d_l|A_1\in\mathrm{Hom}_R(A_1, A_2)$. We shall show $g(D_1)=0$. Assume contrarily $g(D_1)\neq 0$. Take an element z in D_1 such that $g(z)\neq 0$; $z=b_1''+\cdots+b_{n_1}''$; $b_i''\in B_{1i}$. If $b_1''\in eJ^3$, $0\neq p_2\,dz=p_2(db_2''+\cdots+db_{n_1}'')$ implies that, for some j, $0\neq p_2\,db_j''\in\mathrm{Soc}\,(B_{2t})=x_2'R(j\geqslant 2)$, since B_{2t} is uniserial. Further $b_j''\notin J(B_{1j})\subset eJ^3$ for $p_2\,db_j''\neq 0$. Hence $B_{1j}/B_{1j}J\approx x_2'R$, and so $B_{11}/B_{11}J\approx B_{1j}/B_{1j}J$, a contradiction. Accordingly, being

$b_1''R=B_{11}(b_1''\notin eJ^3)$, there exists r in R such that $b_1=b_1''r$. Put $x_1'=x_1-zr=b_2''+\cdots+b_{n_1}''$ $(\in C_1)$. Then $\bar{x}_1'$ is a generator of C_1/D_1. Further $x_1'f=b_2''f+\cdots+b_{n_1}''f$ is in eJ^3 from (#). Hence $x_1'fR$ is a semisimple submodule of A_1. $x_1'fRf\neq 0$ implies that $x_1'fR$ contains a simple submodule isomorphic to $x_2'R$, a contradiction. Therefore $g(D_1)=0$, and so g induces an element in $\mathrm{Hom}_R(A_1/D_1, A_2/D_2)$, which is an extension of h.

COROLLARY. *Let R be a left serial algebra with (A). If $J^3=0$, then (*, 1) is satisfied.*

Finally we give a left serial algebra with $J^4=0$ but (*, 1) is not satisfied.

Let R be a vector space over K with basis $\{e_1, (1, 2), (1, 2)(2, 3)\cdots\}$ given in the below, we define the product among the basis, $e_ie_j=e_i\delta_{ij}$, $e_i(k, s)e_j=(k, s)\delta_{ik}\delta_{sj}$ and products of any four elements (k, s) are zero. Then R is a left serial ring with $J^4=0$ and (A).



Put $A_1=(1, 2)R$, $D_1=(1, 2)(2, 4)K\oplus(1, 2)(2, 3)(3, 5)K\oplus(1, 2)(2, 4)(4, 6)K$, $C_1=(1, 2)(2, 3)K\oplus D_1$. $C_2=(1, 1)(1, 2)(2, 3)K$ and $D_2=0$. Then h: $C_1/D_1\approx C_2$. However $B_2=(1, 1)(1, 2)R$ is not uniserial. Hence R does not satisfy (*, 1).

References

[1] Harada, M., On maximal submodules of a finite direct sum of hollow modules III, Osaka J. Math. **22** (1985) 81–98.

[2] Asashiba, H. and Harada, M., ··· V, to appear.

[3] Harada, M., Generalizations of Nakayama ring III, to appear.

[4] Sumioka, T., Tachikawa's theorem on algebras of left colocal type, Osaka J. Math. **21** (1984) 624–648.

[5] Tachikawa, H., On rings for which every indecomposable right module has a unique maximal submodule, Math (1959) 200–222.

Department of Mathematics
Osaka City University
Sugimoto 3, Sumiyoshi-Ku
Osaka 558
Japan