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CLASSICAL QUOTIENT RINGS OF TRIVIAL EXTENSIONS

By

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Let R be a ring with identity. The right quotient ring of R, if it exists, is a ring Q which satisfies the following conditions:

(i) R is a subring of Q.

(ii) Every regular element of R is a unit of Q.

(iii) Every element q of Q is of the form ac^{-1} for some elements a and c of R with c regular.

An (R, R)-bimodule M is called to satisfy the right Ore condition with respect to a multiplicatively closed subset D of R if, given $m\in M$ and $d\in D$, there exist $m'\in M$ and $d'\in D$ such that $md'=dm'$. It is well-known that R has the classical right quotient ring if and only if R satisfies the right Ore condition with respect to D when D is the set of all regular elements of R.

Let M be an (R, R)-bimodule. The trivial extension $\Lambda=R\ltimes M$ of R by an (R, R)-bimodule M is the Cartesian product $R\ltimes M$ with addition componentwise and multiplication given by $(r, m)(r', m')=(rr', mr'+rm')$. In general, it is difficult to determine the form of regular elements of Λ. If c is a regular element of R, (c, m) is not always a regular element of Λ and vice versa. Let $\Gamma=\begin{pmatrix} R & 0 \\ R & R \end{pmatrix}$ be a generalized triangular matrix ring. In [3], Chatters remarked that H. Attarchi determined the form of a regular element of Γ under the assumption that $cR(Rc)$ is an essential right (left) ideal of R for each right (left) regular element c of R. In this case, $\begin{pmatrix} c_1 & 0 \\ x & c_2 \end{pmatrix}$ is regular in Γ if and only if both c_1 and c_2 are regular in R. For example, if both R_R and ${}_RR$ have finite Goldie dimension, the above assumption is satisfied. So, if we can find a suitable description of regular elements of Λ by those of R, we can investigate whether Λ has the classical right quotient ring. The main purpose of this paper is to give a necessary and sufficient condition for Λ to have the classical right quotient ring under the condition that ${}_RM_R$ is faithful or both ${}_RM$ and M_R have finite Goldie dimension.

In Section 1, we show that every regular element of Λ has the form of (c, m) with c regular in R. Let $C(R)$ denote the set of all regular elements of R and $D=\{c\in C(R)|cm\neq 0$ and $mc\neq 0$ for every $0\neq m\in M\}$. In Section 2, we show that Λ

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has the classical right quotient ring if and only if both R and M satisfy the right Ore condition with respect to D. When the above equivalent condition holds, D becomes a right denominator set and the classical right quotient ring of Λ has the form of $R[D^{-1}]\ltimes M[D^{-1}]$. So, we see that the classical right quotient ring of a trivial extension is also given by a trivial extension. As by-products of results in Sections 1 and 2, we exhibit some corollaries concerning generalized triangular matrix rings in the final Section 3.

Throughout this paper, unless otherwise specified, Λ denotes the trivial extension of R by an (R, R)-bimodule M. For a subset I of R, $\boldsymbol{l}_R(I)(\boldsymbol{r}_R(I))$ denotes the left (right) annihilator of I in R. Furthermore, let $C(R)$ denote the set of all regular elements of R and $D=\{c\in C(R)|cm\neq 0$ and $mc\neq 0$ for every $0\neq m\in M\}$. "The right quotient ring of R" means the classical right quotient ring of R.

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1. Regular elements in Λ.

LEMMA 1.1. *Assume that ${}_RM_R$ is faithful. Then $(c_0, m_0)\in C(\Lambda)$, $m_0\in M$ if and only if $c_0\in D$.*

PROOF. ($\Longrightarrow$). Let $(c_0, m_0)\in C(\Lambda)$, $r'\in \boldsymbol{r}_R(c_0)$ and $m\in M$. Since $(c_0, m_0)(0, r'm)=(0, c_0r'm)=0$ and $(c_0, m_0)\in C(\Lambda)$, we have $r'm=0$ for every $m\in M$. Since ${}_RM$ is faithful, we have $r'=0$. By the similar manner as above, we can prove that $\boldsymbol{l}_R(c_0)=0$. Thus we have $c_0\in C(R)$. Moreover, since $(c_0, m_0)\in C(\Lambda)$, we have $(c_0, m_0)(0, m)=(0, c_0m)\neq 0$ and $(0, m)(c_0, m_0)=(0, mc_0)\neq 0$ for every $0\neq m\in M$. Hence $c_0\in D$.

($\Longleftarrow$). Let $c_0\in D$ and $0\neq(r, m)\in\Lambda$. Since $c_0\in D$, we have $(c_0, m_0)(r, m)=(c_0r, c_0m+m_0r)\neq 0$ and $(r, m)(c_0, m_0)=(rc_0, mc_0+rm_0)\neq 0$. Thus $(c_0, m_0)\in C(\Lambda)$.

Recall that a right R-module X is called to have finite Goldie dimension if X_R contains no infinite independent families of non-zero submodules.

LEMMA 1.2. *Let r be an element of R such that $rm\neq 0$ for every $0\neq m\in M$. If M_R has finite Goldie dimension, then rM_R is an essential submodule of M_R.*

PROOF. Let M'_R be a submodule of M_R with $rM\cap M'=0$. Since $rM\cap M'=0$, $M'+rM'+r^2M'+\cdots+r^nM'+\cdots$ is a direct sum. Since M_R has finite Goldie dimension, $r^nM'=0$ for some n. Therefore, we obtain $M'=0$ by assumption on R.

LEMMA 1.3. *Assume that both M_R and ${}_RM$ have finite Goldie dimension. Then*

$(c_0, m_0) \in C(\Lambda)$, $m_0 \in M$ *if and only if* $c_0 \in D$.

PROOF. This proof is a slight modification of [3, Remarks (2), p. 189].

($\Longrightarrow$). Let $(c_0, m_0) \in C(\Lambda)$. We put $K=\{r \in R \mid m_0 r \in c_0 M\}$. Let $r_1 \in \boldsymbol{r}_R(c_0) \cap K$. Then there exists $m_1 \in M$ such that $m_0 r = -c_0 m_1$. Moreover, since $(c_0, m_0)(r_1, m_1) = 0$ and $(c_0, m_0) \in C(\Lambda)$, we have $(r_1, m_1)=0$. Therefore, we get $\boldsymbol{r}_R(c_0) \cap K = 0$. Since $(c_0, m_0)(0, m)=(0, c_0 m) \neq 0$ for every $0 \neq m \in M$, and M_R has finite Goldie dimension, $c_0 M$ is an essential submodule of M_R by Lemma 1.2. Furthermore, it is easily verified that K is an essential right ideal of R. Therefore, we have $\boldsymbol{r}_R(c_0)=0$. By the similar argument as above, we can show that $\boldsymbol{l}_R(c_0)=0$. Thus $c_0 \in D$.

($\Longleftarrow$). This can be proved by the similar manner as in the proof ($\Longleftarrow$) of Lemma 1.1.

By a slight modification of the proof of [4, (1, 36)], we have the following.

LEMMA 1.4. *Assume that* R *has the right ring of fractions* $R[D^{-1}]$. *If* $c_1, \cdots, c_k \in D$, *then there exist* $c, d_1, \cdots, d_k \in D$ *such that* $c_i^{-1} = d_i c^{-1}$ $(i=1, \cdots, k)$.

2. Quotient rings of Λ.

In this section, we assume that ${}_R M_R$ is faithful or both M_R and ${}_R M$ have finite Goldie dimension.

THEOREM 2.1. *The following conditions are equivalent.*

(1) *Λ has the right quotient ring.*

(2) *R and M satisfy the right Ore condition with respect to D.*

PROOF. In this case, we note that $(c, m) \in C(\Lambda)$, $m \in M$ if and only if $c \in D$ in view of Lemmas 1.1 and 1.3.

(2)$\Longrightarrow$(1). It suffices to show that Λ satisfies the right Ore condition with respect to $C(\Lambda)$. Let $(r, m) \in \Lambda$ and $(c_0, m_0) \in C(\Lambda)$. Since R satisfies the right Ore condition with respect to D, there exist $r_1 \in R$ and $c_1 \in D$ such that $rc_1 = c_0 r_1$. Moreover, since M satisfies the right Ore condition with respect to D, there exist $c_1' \in D$ and $m_1 \in M$ such that $(mc_1 - m_0 r_1)c_1' = c_0 m_1$. Thus we have $(r, m)(c_1 c_1', 0)=(c_0, m_0)\cdot(r_1 c_1', m_1)$ with $(c_1 c_1', 0) \in C(\Lambda)$. Hence Λ has the right quotient ring.

(1)$\Longrightarrow$(2). Let $c \in D$, $(0, m) \in \Lambda$ and $(r, 0) \in \Lambda$. Then $(c, 0) \in C(\Lambda)$. Since Λ satisfies the right Ore condition with respect to $C(\Lambda)$, there exist $(r_i, m_i) \in \Lambda$ $(i=1,2)$ and $(c_i, m_i') \in C(\Lambda)$ $(i=1,2)$ such that $(r, 0)(c_1, m_1')=(c, 0)(r_1, m_1)$ and $(0, m)(c_2, m_2')=(c, 0)(r_2, m_2)$, from which it follows that $rc_1 = cr_1$ and $mc_2 = cm_2$ with $c_1, c_2 \in D$. Hence both R and M satisfy the right Ore condition with respect to D.

REMARK. The following example indicates that the equivalence of (1) and (2) in Theorem 2.1 does not hold in general, if we do not suppose the standing assumption.

EXAMPLE 2.2 [3, Example 2.1]. Let T be a right Noetherian domain which is not left Ore. Let u be an indeterminate which commutes with the elements of T and C denotes the set of all elements of the polynomial ring $T[u]$ which have non-zero constant term. Let $V=T[u][C^{-1}]$ and W the right quotient division ring of T. We can make W into a right V-module by identifying W with V/uV, i.e. by setting $Wu=0$. We set

$$S=\begin{pmatrix} T[u] & 0 \\ T & T \end{pmatrix}, \quad Q=\begin{pmatrix} V & 0 \\ W & W \end{pmatrix}.$$

Then Q is the right quotient ring of S. Since ${}_TT$ does not have finite Goldie dimension, ${}_SS$ does not have finite Goldie dimension. Let

$$\Gamma=\begin{pmatrix} S & 0 \\ S & S \end{pmatrix}$$

be a 2×2 lower triangular matrix ring over S. Then Γ does not have the right quotient ring, but S has the right quotient ring Q. We put $R=S\oplus S$. Since S is regarded as an (R, R)-bimodule in the natural way, Γ is isomorphic to $R\ltimes S$. Note that, in this case, ${}_RS_R$ is not faithful and ${}_RS$ does not have finite Goldie dimension.

A right R-module X is called D-torsion-free if $xd\neq 0$ for every $0\neq x\in X$ and $d\in D$.

If R satisfies the right Ore condition with respect to D, then $R[D^{-1}]$, the right of fractions and $M[D^{-1}]$, the right module of fractions exist.

THEOREM 2.3. *If Λ has the right quotient ring, then the following* (1) *and* (2) *hold.*

(1) *$M[D^{-1}]$ has an $(R[D^{-1}], R[D^{-1}])$-bimodule structure.*

(2) *The canonical embedding $\Lambda \longrightarrow R[D^{-1}]\ltimes M[D^{-1}]$ gives the right quotient ring $Q(\Lambda)$ of Λ.*

PROOF. It is to be noted that $(c, m)\in C(\Lambda)$, $m\in M$ if and only if $c\in D$ in view of Lemmas 1.1 and 1.3.

(1) Let $c\in D$ and $m\in M$. Since M satisfies the right Ore condition with respect to D, there exist $m_1\in M$ and $c_1\in D$ such that $mc_1=cm_1$. We define a left multiplication on $M[D^{-1}]$ by an element of $R[D^{-1}]$ via $c^{-1}\cdot m=m_1c_1^{-1}$. If there exist other

element $c_2 \in D$ and $m_2 \in M$ satisfying $mc_2 = cm_2$, then we have $m = cm_1c_1^{-1} = cm_2c_2^{-1}$, from which it follows that $m_1c_1^{-1} = m_2c_2^{-1}$, for ${}_RM$ is D-torsion-free. Therefore, this multiplication is well-defined. Moreover, it is easily seen that $M[D^{-1}]$ has an $(R[D^{-1}], R[D^{-1}])$-bimodule structure.

(2) Since M_R is D-torsion-free, M can be considered as a submodule of $M[D^{-1}]$. Therefore, we can suppose that $\Lambda \subsetneq R[D^{-1}] \ltimes M[D^{-1}]$. Let $(c_0, m_0) \in C(\Lambda)$. Since $(c_0, m_0)(c_0^{-1}, -c_0^{-1}m_0c_0^{-1}) = (1, 0)$ and $(c_0^{-1}, -c_0^{-1}m_0c_0^{-1})(c_0, m_0) = (1, 0)$ with $(c_0^{-1}, -c_0^{-1} m c_0^{-1}) \in R[D^{-1}] \ltimes M[D^{-1}]$, (c_0, m_0) is a unit of $R[D^{-1}] \ltimes M[D^{-1}]$. Let $(rc_1^{-1}, mc_2^{-1}) \in R[D^{-1}] \ltimes M[D^{-1}]$. Since there exist $c, d_1, d_2 \in D$ such that $c_i^{-1} = d_ic^{-1}$ $(i = 1, 2)$ by Lemma 1.4, we have $(rc_1^{-1}, mc_2^{-1})(c, 0) = (rd_1c^{-1}, md_2c^{-1})(c, 0) = (rd_1, md_2) \in \Lambda$ with $(c, 0) \in C(\Lambda)$. Hence we conclude that the canonical embedding $\Lambda \longrightarrow R[D^{-1}] \ltimes M[D^{-1}]$ gives the right quotient ring $Q(\Lambda)$ of Λ.

Corollary 2.4. *Let $\Lambda = R \ltimes R$. Then the following are equivalent.*

(1) *Λ has the right quotient ring $Q(\Lambda)$.*

(2) *R has the right quotient ring $Q(R)$.*

In this case, the canonical embedding $\Lambda \longrightarrow Q(R) \ltimes Q(R)$ gives the right quotient ring $Q(\Lambda)$ of Λ.

Proof. This directly follows from Theorems 2.1 and 2.3.

We exhibit the following example for which R does not satisfy the right Ore condition with respect to D, but R satisfies the right Ore condition with respect to $C(R)$.

Example 2.5 [5, Example 5.5]. Let

$$R = \begin{pmatrix} \mathbf{Z} & 2\mathbf{Z} \\ \mathbf{Z} & \mathbf{Z} \end{pmatrix} \supset {}_RI_R = \begin{pmatrix} 2\mathbf{Z} & 2\mathbf{Z} \\ \mathbf{Z} & \mathbf{Z} \end{pmatrix}$$

and $\Lambda = R \ltimes R/I$. Then both $(R/I)_R$ and ${}_R(R/I)$ have finite Goldie dimension and $D = \left\{ \begin{pmatrix} z_1 & 2z_3 \\ z_2 & z_4 \end{pmatrix} \in C(\Lambda) \mid z_1 \notin 2\mathbf{Z} \right\}$. Since R does not satisfy the right Ore condition with respect to D, Λ does not have the right quotient ring in view of Theorem 2.1.

3. Generalized triangular matrix rings.

Let

$$\Gamma = \begin{pmatrix} S & 0 \\ U & T \end{pmatrix}$$

be a generalized triangular matrix ring, where S and T are rings with identity and U a (T, S)-bimodule. We put $R=S\oplus T$. Since U is regarded as an (R, R)-bimodule in the natural way, Γ is isomorphic to $R\ltimes U$. Let $D_1=\{d_1\in C(S)|ud_1\neq 0$ for every $0\neq u\in U\}$ and $D_2=\{d_2\in C(T)|d_2u\neq 0$ for every $0\neq u\in U\}$. It is clear that ${}_RU_R$ satisfies the right Ore condition with respect to a subset (D_1, D_2) of R if, given $u\in U$ and $d_2\in D_2$, there exist $u'\in U$ and $d_1\in D_1$ such that $ud_1=d_2u'$. (In this case, ${}_TU_S$ is called to satisfy the right Ore condition with respect to D_1-D_2). It is to be noted that ${}_RU_R$ is unfaithful, whenever S or T is non-zero. So, we consider only in case both U_R and ${}_RU$ have finite Goldie dimension. If we apply Theorems 2.1 and 2.3 to Γ, then we have the following.

COROLLARY 3.1. *Assume that both U_S and ${}_TU$ have finite Goldie dimension. Then the following are equivalent.*

(1) *Γ has the right quotient ring.*

(2) *S, T and ${}_TU_S$ satisfy the right Ore condition with respect to D_1, D_2, and D_1-D_2, respectively.*

COROLLARY 3.2. *In the same situation as in the preceding corollary, the right quotient ring of Γ has the form of*

$$\begin{pmatrix} S[D_1^{-1}] & 0 \\ U[D_1^{-1}] & T[D_2^{-1}] \end{pmatrix}$$

COROLLARY 3.3. *Let $T_n(R)$ be the ring of $n\times n$ lower triangular matrices over R. Assume that both R_R and ${}_RR$ have finite Goldie dimension and that R has the right quotient ring $Q(R)$. Then $T_n(R)$ has the right quotient ring isomorphic to $T_n(Q(R))$.*

PROOF. We prove by induction on n. If $n=1$, then it is obvious. We suppose that $T_{n-1}(R)$ has the right quotient ring iomorphic to $T_{n-1}(Q(R))$. Since $T_n(R)$ can be considered as

$$\begin{pmatrix} T_{n-1}(R) & \begin{matrix} 0 \\ \vdots \\ 0 \end{matrix} \\ R\cdots R & R \end{pmatrix},$$

${}_R(R\cdots R)_{T_{n-1}(R)}$ satisfies the right Ore condition with respect to $C(R)-C(T_{n-1}(R))$ and both ${}_R(R\cdots R)$ and $(R\cdots R)_{T_{n-1}(R)}$ have finite Goldie dimension, $T_n(R)$ has the right quotient ring isomorphic to $T_n(Q(R))$ by Corollary 3.2.

REMARK. It is well-known that, if R has the right quotient ring $Q(R)$ and

$Q(R)_{Q(R)}$ has finite Goldie dimension, then R_R has finite Goldie dimension. Therefore, Corollary 3.3 holds under weaker conditions than in [1, Corollary 3.6].

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