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MAGNETOHYDRODYNAMIC APPROXIMATION OF THE COMPLETE EQUATIONS FOR AN ELECTROMAGNETIC FLUID

By

Shuichi KAWASHIMA and Yasushi SHIZUTA

§ 1 Introduction.

In this paper, we give a singular limit theorem for the system of equations describing an electromagnetic fluid in two space dimensions, which was studied in [2]. The magnetohydrodynamic equations are obtained as the limit of the complete equations for the electromagnetic fluid at the vanishing of the dielectric constant. It is customary to regard the limit equations as an approximation to the complete equations. This approximation is usually referred to as the magnetohydrodynamics, and is equivalent to the neglect of the displacement current.

The system of equations for an electromagnetic fluid in three space dimensions consists of 14 equations in 12 unknowns, namely, the mass density ρ, the velocity $\boldsymbol{u}=(u^{1}, u^{2}, u^{3})$, the absolute temperature θ, the electric field $\boldsymbol{E}=(E^{1}, E^{2}, E^{3})$, the magnetic flux density $\boldsymbol{B}=(B^{1}, B^{2}, B^{2})$ and the electric change density ρ_{e}. We refer the reader to [2] for the explicit form of this system.

We restrict ourselves to the study of two-dimensional motion of the electromagnetic fluid. Unfortunately, our method is not applicable to the three-dimensional problem. The reason is as follows: When the hydrodynamic quantities $(\rho, \boldsymbol{u}, \theta)$ are regarded as known functions, the equations for the electromagnetic quantities $(\boldsymbol{E}, \boldsymbol{B}, \rho_{e})$ form a first order hyperbolic system, which is neither symmetric hyperbolic nor strictly hyperbolic in the three-dimensional case. For this reason, we assume that all the unknowns $(\rho, \boldsymbol{u}, \theta, \boldsymbol{E}, \boldsymbol{B}, \rho_{e})$ are independent of the third component of the space variable (x_{1}, x_{2}, x_{3}) and that

$$\boldsymbol{u}=(u^{1}, u^{2}, 0)\,, \qquad \boldsymbol{E}=(0,0, E^{3})\,, \qquad \boldsymbol{B}=(B^{1}, B^{2}, 0)\,.$$

In this case, we have $\rho_{e}=0$. We consider therefore the following symmetric system of 8 equations in 7 unknowns (ρ, u, θ, E, B), where $u=(u^{1}, u^{2})$, $E=E^{3}$ and $B=(B^{1}, B^{2})$:

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$$\text{(1.1)}\quad \begin{cases} \rho_t+\operatorname{div}(\rho u)=0\,, \\ \rho(u_t+(u\cdot\nabla)u)+\nabla p=\operatorname{div}(2\mu P+\mu' I\operatorname{div}u)+J\times B\,, \\ \rho e_\theta(\theta_t+u\cdot\nabla\theta)+\theta p_\theta\operatorname{div}u=\operatorname{div}(\kappa\nabla\theta)+\Psi+J(E+u\times B)\,, \\ \varepsilon E_t-(1/\mu_0)\operatorname{rot}B+J=0\,, \\ B_t+\operatorname{rot}E=0\,, \end{cases}$$

$$\text{(1.2)}\quad \operatorname{div}B=0\,,$$

with the initial condition

$$\text{(1.3)}\quad (\rho, u, \theta, E, B)(0, x)=(\rho_0^{\varepsilon}, u_0^{\varepsilon}, \theta_0^{\varepsilon}, E_0^{\varepsilon}, B_0^{\varepsilon})(x)\,,$$

where $x=(x_1, x_2)\in \boldsymbol{R}^2$. Here and in the sequel we use the notations for two-dimensional vectors. (See $(1.10)_{1,2}$ below.) The pressure p and the internal energy e are known functions of (ρ, θ). We write $p_\theta=\partial p/\partial\theta$ and $e_\theta=\partial e/\partial\theta$. The deformation tensor P and the viscous dissipation function Ψ are given by

$$P=(P_{ij})_{1\le i,j\le 2} \quad\text{with}\quad P_{ij}=\frac{1}{2}(u^i_{x_j}+u^j_{x_i})\,,$$

$$\Psi=2\mu\sum_{i,j=1}^{2}(P_{ij})^2+\mu'(\operatorname{div}u)^2\,,$$

respectively. For an electrically conducting fluid, Ohm's law applies, and hence the current density J is given by the relation

$$J=\sigma(E+u\times B)\,,$$

since $\rho_e=0$. The viscosity coefficients μ and μ', the heat conductivity coefficient κ and the electric conductivity coefficient σ are known functions of (ρ, θ). The dielectric constant ε and the magnetic permeability constant μ_0 are positive constants.

The assumptions for the system (1.1), (1.2) are stated as follows: Let $\mathfrak{D}=\{(\rho, \theta)\,;\,\rho>0, \theta>0\}$. Let $\nu=2\mu+\mu'$.

(1.4) p and e are smooth functions on $\mathfrak{D}$. Both $p_\rho=\partial p/\partial\rho$ and $e_\theta=\partial e/\partial\theta$ are positive on $\mathfrak{D}$.

(1.5) μ, μ' and κ are smooth functions on $\mathfrak{D}$. Furthermore, one of the following four conditions is valid on $\mathfrak{D}$.

(i) $\mu, \nu, \kappa>0$, (ii) $\mu=\nu=0$, $\kappa>0$,

(iii) $\mu, \nu>0$, $\kappa=0$, (iv) $\mu=\nu=\kappa=0$.

(1.6) σ is smooth and positive on $\mathfrak{D}$.

We remark that, under these conditions, the system (1.1) (or (1.1), (1.2)) is regarded as a symmetric system of hyperbolic-parabolic type in each case (i)-(iii) of (1.5), or hyperbolic type in the last case (iv).

Let $\varepsilon=0$ in (1.1), (1.2). Then it follows from the equation for the electric field that $J=(1/\mu_0)\operatorname{rot} B$. Combining this with Ohm's law, we obtain

$$E=E(\rho, u, \theta, B)\equiv -u\times B+(1/\sigma\mu_0)\operatorname{rot} B\,. \tag{1.7}$$

Eliminating E from (1.1), (1.2) by using (1.7), we get the reduced system of equations:

$$\begin{cases} \rho_t+\operatorname{div}(\rho u)=0\,, \\ \rho(u_t+(u\cdot\nabla)u)+\nabla p-(1/\mu_0)\operatorname{rot} B\times B=\operatorname{div}(2\mu P+\mu' I\operatorname{div} u)\,, \\ \rho e_\theta(\theta_t+u\cdot\nabla\theta)+\theta p_\theta\operatorname{div} u=\operatorname{div}(\kappa\nabla\theta)+\Psi+(1/\sigma\mu_0^2)(\operatorname{rot} B)^2\,, \\ B_t-\operatorname{rot}(u\times B)=-\operatorname{rot}\{(1/\sigma\mu_0)\operatorname{rot} B\}\,, \end{cases} \tag{1.8}$$

$$\operatorname{div} B=0\,. \tag{1.9}$$

These equations are called the magnetohydrodynamic equations in two space dimensions. We can see that, under the conditions (1.4), (1.5), (1.6), this system is reduced to a symmetric system of hyperbolic-parabolic type in every case (i)–(iv) of (1.5). The electric field E is obtained from (1.7), which is regarded as a defining equation.

Our aim is to show the convergence of the solutions of the complete equations (1.1), (1.2) to the solutions of the magnetohydrodynamic equations (1.8), (1.9) at the vanishing of the dielectric constant ε. The results obtained are stated as follows: We assume that the initial data $(\rho_0, u_0, \theta_0, E_0, B_0)(x)$ are smooth and independent of $\varepsilon\in(0,1]$, and satisfy $\operatorname{div} B_0=0$. Then the system (1.1), (1.2) for the electromagnetic fluid has a unique smooth solution $(\rho^\varepsilon, u^\varepsilon, \theta^\varepsilon, E^\varepsilon, B^\varepsilon)(t, x)$ on a region $[0, T]\times\boldsymbol{R}^2$, whose time length is independent of ε. As $\varepsilon\to 0$, the solution converges on $[0, T]\times\boldsymbol{R}^2$ to a function $(\rho^0, u^0, \theta^0, E^0, B^0)(t, x)$ with the rate $O(\varepsilon^{1/2})$ in an appropriate norm. This limit function satisfies (1.7). Moreover, the limit function $(\rho^0, u^0, \theta^0, B^0)(t, x)$ excepting $E^0(t, x)$ is a unique smooth solution of the magnetohydrodynamic equations (1.8), (1.9) with the initial condition $(\rho^0, u^0, \theta^0, B^0)(0, x)=(\rho_0, u_0, \theta_0, B_0)(x)$. If, furthermore, the initial data for the complete equations satisfy (1.7), then the rate of convergence is $O(\varepsilon)$. This is explained by the absence of the initial layer for the electric field in this case. The proof is based on an energy inequality for quasilinear symmetric systems of hyperbolic-parabolic type. The uniform estimates in ε are obtained by using (1.6). Although the results are valid for each case (i)–(iv) mentioned in (1.5), we give the proofs only for the case (i) and omit the detailed discussion for the other three.

In § 2, we provide energy estimates for the linealized equations of (1.1) and show the existence of solutions. Then in § 3 we construct, in an appropriate function space, a subset invariant under the mapping whose fixed point gives a solution of (1.1). Although the map depends on the parameter ε, the invariant subset is independent of ε except for the electric field. It is shown in § 4 that a solution of (1.1), (1.2) is obtained as the limit in uniform convergence of sequences of approximating functions defined on $[0, T]\times \boldsymbol{R}^2$, where T is a positive constant not depending on ε. In § 5, we discuss the convergence of the solution of (1.1), (1.2) constructed in § 4 to the solution of (1.8), (1.9) as $\varepsilon\to 0$.

A paper by Milani [4] appeared, while the present paper is in preparation. A similar singular limit theorem is proved there for the Maxwell equations.

Notations

In this paper, we use the following notations for two-dimensional vectors in addition to the ordinary ones: Let $v=(v^1, v^2)$ and $w=(w^1, w^2)$. Let ϕ be a scalar. We write

$$(1.10)_1 \qquad \begin{cases} v\times w=-w\times v=v^1w^2-v^2w^1\,, \\ v\times\phi=-\phi\times v=(\phi v^2,\ -\phi v^1)\,, \end{cases}$$

$$(1.10)_2 \qquad \begin{cases} \operatorname{rot} w=\nabla\times w=w^2_{x_1}-w^1_{x_2}\,, \\ \operatorname{rot}\phi=\nabla\times\phi=(\phi_{x_2},\ -\phi_{x_1})\,. \end{cases}$$

We enumerate some function spaces used in the following. L^2 is the space of square integrable functions on $\boldsymbol{R}^2$, whose norm is denoted by $\|\cdot\|$. For an integer l, H^l stands for the L^2-Sobolev space (on $\boldsymbol{R}^2$) of order l, with the norm $\|\cdot\|_l$. Let k be a nonnegative integer and T be a positive constant. Then $C^k(0, T; H^l)$ denotes the space of k-times continuously differentiable functions on $[0, T]$ with values in H^l. $L^2(0, T; H^l)$ (resp. $L^\infty(0, T; H^l)$) is the space of square integrable (resp. bounded measurable) functions on $[0, T]$, valued in H^l.

§ 2. Linearized equations

In the following argument, we shall assume (i) of (1.5). The other cases listed in (1.5) can be treated similarly. We consider the system of linearized equations for (1.1), which is as follows.

$$(2.1) \qquad \begin{cases} \hat\rho_t+u\cdot\nabla\hat\rho=f_1\,, \\ \rho\hat u_t-\mu\Delta\hat u-(\mu+\mu')\nabla\operatorname{div}\hat u=\rho(f_2+g_2)\,, \\ \rho e_\theta\hat\theta_t-\kappa\Delta\hat\theta=\rho e_\theta(f_3+g_3)\,, \end{cases}$$

$$(2.2) \qquad \begin{cases} \varepsilon\hat E_t-(1/\mu_0)\operatorname{rot}\hat B+\sigma\hat E=f_4\,, \\ \hat B_t+\operatorname{rot}\hat E=0\,. \end{cases}$$

Here $\varepsilon\in(0,1]$ and $\mu_0>0$ are constants, and e, μ, μ', κ and σ are given functions of $(\rho,\theta)\in\mathfrak{D}$. The functions (ρ, u, θ), f_j $(j=1,\cdots,4)$ and g_j $(j=2, 3)$ are regarded as known functions of $(t, x)\in[0, T]\times\boldsymbol{R}^2$.

First we define several families of functions for later use. Let $(\bar\rho,\bar\theta)\in\mathfrak{D}$ and $\bar B\in\boldsymbol{R}^2$ be arbitrarily fixed constants. Let $s\geq 2$ be an integer and T be a positive constant.

DEFINITION 2.1. For positive constants m_0, M_0, M_1 and M_2, we define $V^s(T) = V^s(T; m_0, M_0, M_1, M_2)$ to be the set of all functions $(\rho, u, \theta)(t, x)$ satisfying the following conditions.

$$(2.3)\qquad \begin{cases} \partial_t^j(\rho-\bar{\rho})\in C^0(0,T;H^{s-j}). \\ \partial_t^j(u,\theta-\bar{\theta})\in C^0(0,T;H^{s-2j})\cap L^2(0,T;H^{s+1-2j}) \qquad \text{for } j=0,1, \end{cases}$$

$$(2.4)_1\qquad m_0\le\rho(t,x),\ \theta(t,x)\le M_0 \qquad \text{for } (t,x)\in[0,T]\times\mathbf{R}^2,$$

$$(2.4)_2\qquad \|(\rho-\bar{\rho},u,\theta-\bar{\theta})(t)\|_s^2+\int_0^t\|(u,\theta-\bar{\theta})(\tau)\|_{s+1}^2d\tau\le M_1^2,$$

$$(2.4)_3\qquad \int_0^t\|\partial_t(\rho,u,\theta)(\tau)\|_{s-1}^2d\tau\le M_2^2 \qquad \text{for } t\in[0,T].$$

DEFINITION 2.2. $\tilde{V}^s(T)=\tilde{V}^s(T;m_0,M_0,M_1,M_2)$ is the set of all functions satisfying (2.3) with $C^0(0,T;H^l)$ replaced by $L^\infty(0,T;H^l)$ $(l=s-j, s-2j)$, and also the estimates $(2.4)_{1,2,3}$.

DEFINITION 2.3. Let $\varepsilon\in(0,1]$. Then, for positive constants N_0, N_1 and N_2, and for an exponent $\eta\in[0,1]$, we define $W_\varepsilon^s(T)=W_\varepsilon^s(T;N_0,N_1,N_2,\eta)$ to be the set of all functions $(E,B)(t,x)$ satisfying the following conditions.

$$(2.5)\qquad \partial_t^j(E,B-\bar{B})\in C^0(0,T;H^{s-j}) \qquad \text{for } j=0,1,$$

$$(2.6)_1\qquad \|E(t)\|_{s-1}\le N_0,$$

$$(2.6)_2\qquad \|(\varepsilon^{1/2}E,B-\bar{B})(t)\|_s^2+\int_0^t\|E(\tau)\|_s^2d\tau\le N_1^2,$$

$$(2.6)_3\qquad \|\partial_t(\varepsilon^{1/2}E,B)(t)\|_{s-1}^2+\int_0^t\|\partial_tE(\tau)\|_{s-1}^2d\tau\le\varepsilon^{-\eta}N_2^2 \qquad \text{for } t\in[0,T].$$

DEFINITION 2.4. $\tilde{W}_\varepsilon^s(T)=\tilde{W}_\varepsilon^s(T;N_0,N_1,N_2,\eta)$ is the set of all functions satisfying (2.5) with $C^0(0,T;H^{s-j})$ replaced by $L^\infty(0,T;H^{s-j})$, and also the estimates $(2.6)_{1,2,3}$.

Now we give energy inequalities for (2.1), (2.2).

LEMMA 2.1. *Suppose* (1.4) *and* (i) *of* (1.5). *Let* $s\ge3$ *and* $l\in[1,s]$ *be integers, and let* T *be a positive constant. We assume that* $(\rho,u,\theta)\in\tilde{V}^s(T;m_0,M_0,M_1,M_2)$, $(f_1,f_2,f_3)\in L^\infty(0,T;H^{l-1})\cap L^2(0,T;H^l)$ *and* $(g_2,g_3)\in L^\infty(0,T;H^{l-1})$. *Let* $(\hat{\rho},\hat{u},\hat{\theta})(t,x)$ *be a solution of* (2.1) *such that* $\partial_t^j\hat{\rho}\in L^\infty(0,T;H^{l-j})$ *and* $\partial_t^j(\hat{u},\hat{\theta})\in L^\infty(0,T;H^{l-2j})$ *for* $j=0,1$. *Then we have*

$$(2.7)\qquad \hat{\rho}\in C^0(0,T;H^l),\qquad (\hat{u},\hat{\theta})\in C^0(0,T;H^l)\cap L^2(0,T;H^{l+1}).$$

Moreover, there exist constants $C_1=C_1(m_0,M_0)>1$ *and* $C_2=C_2(m_0,M_0,M_1)>0$ *such that the following inequality holds for any* $\alpha\in(0,1]$ *and* $t\in[0,T]$.

$$(2.8)\qquad \|(\hat\rho,\hat u,\hat\theta)(t)\|_l^2+\int_0^t\|(\hat u,\hat\theta)(\tau)\|_{l+1}^2d\tau$$
$$\le C_1^2e^{\alpha^{-1}C_2t}\Big\{\|(\hat\rho,\hat u,\hat\theta)(0)\|_l^2+t\int_0^t\|f_1(\tau)\|_l^2d\tau$$
$$+\alpha\int_0^t\|(f_2,f_3)(\tau)\|_l^2d\tau+\int_0^t\|(g_2,g_3)(\tau)\|_{l-1}^2d\tau\Big\}.$$

Here the constants C_1 and C_2 are independent of $\alpha\in(0,1]$.

PROOF. The first equation of (2.1) is regarded as a single hyperbolic equation for $\hat\rho$. While, under the condition of the lemma, the second equation of (2.1) can be regarded as a symmetric system of strongly parabolic type for $\hat u$. Similarly, the last equation of (2.1) is a single strongly parabolic equation for $\hat\theta$. Therefore, (2.7) and (2.8) are shown by standard arguments. See [3] for details.

LEMMA 2.2. *Suppose* (1.6). *Let $s\ge 3$ and $l\in[0,s]$ be integers, and let T be a positive constant. We assume that $(\rho,u,\theta)\in\tilde V^s(T;m_0,M_0,M_1,M_2)$ and $f_4\in L^\infty(0,T;H^{l-1})\cap L^2(0,T;H^l)$. Let $(\hat E,\hat B)(t,x)$ be a solution of* (2.2) *such that $\partial_t^j(\hat E,\hat B)\in L^\infty(0,T;H^{l-j})$ for $j=0,1$. Then we have*

$$(2.9)\qquad (\hat E,\hat B)\in C^0(0,T;H^l).$$

Moreover, there exists a constant $C_3=C_3(m_0,M_0,M_1)>1$ not depending on $\varepsilon\in(0,1]$ such that the following inequality holds for $t\in[0,T]$.

$$(2.10)\qquad \|(\varepsilon^{1/2}\hat E,\hat B)(t)\|_l^2+\int_0^t\|\hat E(\tau)\|_l^2d\tau$$
$$\le C_3^2\Big\{\|(\varepsilon^{1/2}\hat E,\hat B)(0)\|_l^2+\int_0^t\|f_4(\tau)\|_l^2d\tau\Big\}.$$

PROOF. The equations (2.2) can be regarded as a symmetric hyperbolic system for $(\hat E,\hat B)$. Therefore (2.9) is shown by standard arguments. Here we prove (2.10). Since the argument using Friedrichs mollifier is applicable, it suffices to prove (2.10) by assuming that $f_4\in L^\infty(0,T;H^l)$ and $\partial_t^j(\hat E,\hat B)\in L^\infty(0,T;H^{l+1-j})$ for $j=0,1$. Let us apply $D_x^k=\{(\partial/\partial x)^\alpha;|\alpha|=k\}$, $k=0,1,\cdots,l$, to both members of (2.2). Then we get

$$(2.11)\qquad \begin{cases}\varepsilon\hat E_t^k-(1/\mu_0)\operatorname{rot}\hat B^k+\sigma\hat E^k=F_4^k.\\ \hat B_t^k+\operatorname{rot}\hat E^k=0,\end{cases}$$

where $(\hat E^k,\hat B^k)=D_x^k(\hat E,\hat B)$ and $F_4^k=-[D_x^k,\sigma]\hat E+D_x^kf_4$. The bracket [,] denotes the commutator. Noting that (2.11) is a symmetric hyperbolic system for $(\hat E^k,\hat B^k)$, we multiply the first equation by $\hat E^k$ and take the inner product of the second equation with $(1/\mu_0)\hat{B}^k$. We sum up these equations and integrate over $[0, t]\times \boldsymbol{R}^2$. Then we obtain, by virtue of (1.6),

$$\|(\varepsilon^{1/2}\hat{E}^k, \hat{B}^k)(t)\|^2+\int_0^t \|\hat{E}^k(\tau)\|^2 d\tau \leq C\left\{\|(\varepsilon^{1/2}\hat{E}^k, \hat{B}^k)(0)\|^2+\int_0^t \|F_4^k(\tau)\|^2 d\tau\right\}. \tag{2.12}$$

Here $C=C(m_0, M_0)$ is a constant independent of $\varepsilon\in(0, 1]$. Observe that $\|F_4^0\|=\|f_4\|$ and that $\|F_4^k\|=C\|\hat{E}\|_{k-1}+\|D_x^k f_4\|$ for $1\leq k\leq l$, where $C=C(m_0, M_0, M_1)$ is a constant not depending on ε. Hence, combining these inequalities with (2.12) and then using the induction for $k=0, 1, \cdots, l$, we get the desired estimate (2.10). This completes the proof of Lemma 2.2.

Next we state the results concerning the existence of solutions of the systems (2.1) and (2.2).

PROPOSISION 2.3. *Suppose* (1.4) *and* (i) *of* (1.5). *Let* $s\geq 3$ *and* $l\in[2, s]$ *be integers, and let* T *be a positive constant. Assume that* $(\rho, u, \theta)\in V^s(T; m_0, M_0, M_1, M_2)$, $(f_1, f_2, f_3)\in C^0(0, T; H^{l-1})\cap L^2(0, T; H^l)$ *and* $(g_2, g_3)\in C^0(0, T; H^{l-1})$. *(Compare these conditions with those of Lemma* 2.1) *If the initial data* $(\hat{\rho}, \hat{u}, \hat{\theta})(0)\in H^l$, *then the Cauchy problem for* (2.1) *has a unique solution* $(\hat{\rho}, \hat{u}, \hat{\theta})(t, x)$ *such that*

$$\begin{cases} \partial_t^j\hat{\rho}\in C^0(0, T; H^{l-j}), \\ \partial_t^j(\hat{u}, \hat{\theta})\in C^0(0, T; H^{l-2j})\cap L^2(0, T; H^{l+1-2j}) \end{cases} \tag{2.13}$$

for $j=0, 1$. *Furthermore, the energy inequality* (2.8) *holds.*

PROPOSITION 2.4. *Suppose* (1.6). *Let* $s\geq 3$ *and* $l\in[1, s]$ *be integers, and let* T *be a positive constant. Assume that* $(\rho, u, \theta)\in V^s(T; m_0, M_0, M_1, M_2)$ *and* $f_4\in C^0(0, T; H^{l-1})\cap L^2(0, T; H^l)$. *(Compare these conditions with those of Lemma* 2.2.) *If the initial data* $(\hat{E}, \hat{B})(0)\in H^l$, *then the Cauchy problem for* (2.2) *has a unique solution* $(\hat{E}, \hat{B})(t, x)$ *such that*

$$\partial_t^j(\hat{E}, \hat{B})\in C^0(0, T; H^{l-j}) \tag{2.14}$$

for $j=0, 1$. *Furthermore, the energy inequality* (2.10) *holds.*

We remark that Propositions 2.3 and 2.4 can be proved by means of Theorem II of Kato [1] (pp. 658). We omit the details and refer the reader to [3].

§ 3. Invariant subset with respect to interactions

In order to solve the Cauchy problem (1.1), (1.3) by iterations, let us consider the following linear system:

$$(3.1)\qquad \begin{cases} \hat{\rho}_t+u\cdot\nabla\hat{\rho}=F_1\,, \\ \rho\hat{u}_t-\mu\Delta\hat{u}-(\mu+\mu')\nabla\operatorname{div}\hat{u}=\rho G_2^*\,, \\ \rho e_\theta\hat{\theta}_t-\kappa\Delta\hat{\theta}=\rho e_\theta G_3^*\,, \\ \varepsilon\hat{E}_t-(1/\mu_0)\operatorname{rot}\hat{B}+\sigma\hat{E}=F_4\,, \\ \hat{B}_t+\operatorname{rot}\hat{E}=0\,, \end{cases}$$

with the initial condition

$$(3.2)\qquad (\hat{\rho},\hat{u},\hat{\theta},\hat{E},\hat{B})(0,x)=(\rho,u,\theta,E,B)(0,x) \equiv(\rho_0^\varepsilon,u_0^\varepsilon,\theta_0^\varepsilon,E_0^\varepsilon,B_0^\varepsilon)(x)\,.$$

Here the functions on the right hand side of (3.1) are given as follows.

$$(3.3)\qquad \begin{cases} F_1(\rho,u)=-\rho\operatorname{div}u\,, \\ F_2(\rho,u,\theta,E,B)=(\sigma/\rho)(E+u\times B)\times B\,, \\ F_3(\rho,u,\theta,E,B)=(\sigma/\rho e_\theta)(E+u\times B)^2\,, \\ F_4(\rho,u,\theta,B)=-\sigma u\times B\,, \end{cases}$$

$$(3.4)\qquad \begin{cases} G_2(\rho,u,\theta)=-\{(u\cdot\nabla)u+(p_\rho/\rho)\nabla\rho+(p_\theta/\rho)\nabla\theta\}+(1/\rho)(2\nabla\mu\cdot P+\nabla\mu'\cdot\operatorname{div}u)\,, \\ G_3(\rho,u,\theta)=-\{u\cdot\nabla\theta+(\theta p_\theta/\rho e_\theta)\operatorname{div}u\}+(1/\rho e_\theta)(\nabla\kappa\cdot\nabla\theta+\Psi)\,, \end{cases}$$

$$(3.5)\qquad G_j^*(\rho,u,\theta,E,B)=G_j(\rho,u,\theta)+F_j(\rho,u,\theta,E,B)\,,\qquad j=2,3\,.$$

In (3.1), (ρ,u,θ,E,B) are regarded as given functions on $[0,T]\times\boldsymbol{R}^2$. Note that $\varepsilon\in(0,1]$ and $\mu_0>0$ are constants. Also, p, e, μ, μ', κ and σ are known functions of (ρ,θ).

Let $s\geq3$ be an integer, and let $(\bar{\rho},\bar{\theta})\in\mathfrak{D}$ and $\bar{B}\in\boldsymbol{R}^2$ be arbitrarily fixed constants independent of $\varepsilon\in(0,1]$. We assume that the initial data $(\rho_0^\varepsilon,u_0^\varepsilon,\theta_0^\varepsilon,E_0^\varepsilon,B_0^\varepsilon)(x)$ may depend on $\varepsilon\in(0,1]$, so far as the following conditions are satisfied:

$$(3.6)\qquad (\rho_0^\varepsilon-\bar{\rho},u_0^\varepsilon,\theta_0^\varepsilon-\bar{\theta},E_0^\varepsilon,B_0^\varepsilon-\bar{B})\in H^s \quad\text{and}\quad \inf_x\{\rho_0^\varepsilon(x),\theta_0^\varepsilon(x)\}>0 \qquad \text{for every } \varepsilon.$$

$$(3.7)_1\qquad \inf_\varepsilon\inf_x\{\rho_0^\varepsilon(x),\theta_0^\varepsilon(x)\}=k_0>0 \quad\text{and}\quad \sup_\varepsilon\sup_x\{\rho_0^\varepsilon(x),\theta_0^\varepsilon(x)\}=K_0<+\infty\,.$$

$$(3.7)_2\qquad \sup_\varepsilon\|(\rho_0^\varepsilon-\bar{\rho},u_0^\varepsilon,\theta_0^\varepsilon-\bar{\theta},\varepsilon^{1/2}E_0^\varepsilon,B_0^\varepsilon-\bar{B})\|_s=K_1<+\infty\,.$$

Moreover, we assume the following conditions: There exist numbers $\beta\geq0$ and $\beta'\in[0,1/2]$, both independent of ε, such that

$$(3.8)_1\qquad \sup_\varepsilon\varepsilon^{-\beta}\|E_0^\varepsilon-E(\rho_0^\varepsilon,u_0^\varepsilon,\theta_0^\varepsilon,B_0^\varepsilon)\|_{s-1}=K_2<+\infty\,,$$

$$(3.8)_2\qquad \sup_\varepsilon\varepsilon^{(1/2-\beta')}\|\operatorname{rot}E_0^\varepsilon\|_{s-1}=K_3<+\infty\,,$$

where $E(\rho,u,\theta,B)$ is the function defined in (1.7). We give here some remarks on these conditions. First we note that $(3.7)_2$ implies $(3.8)_2$ with $\beta'=0$ and $K_3=K_1$. Also, if $(3.7)_{1,2}$ and $(3.8)_1$ are true, then we have

$$\|E_0^\varepsilon\|_{s-1} \leq CK_1 + K_2\,, \tag{3.9}$$

where $C=C(k_0, K_0, K_1)$ is a constant independent of ε. Next, we consider the simple case where the initial daat $(\rho_0^\varepsilon, u_0^\varepsilon, \theta_0^\varepsilon, E_0^\varepsilon, B_0^\varepsilon)(x)$ are independent of ε. In this case, (3.6) implies $(3.7)_{1,2}$, $(3.8)_1$ with $\beta=0$ and $(3.8)_2$ with $\beta'=1/2$. If, in addition, the initial data satisfy the relation (1.7), i.e., the initial layer for the electric field is absent, then $(3.8)_1$ holds for any $\beta \geq 0$.

Now our aim is to show, under these conditions, the following: For a suitable choice of positive constants T, m_0, M_0, M_1, M_2, N_0, N_1, N_2, and an exponent $\eta \in [0, 1]$, the set $V^s(T; m_0, M_0, M_1, M_2) \times W_\varepsilon^s(T; N_0, N_1, N_2, \eta)$ is invariant under the mapping $(\rho, u, \theta, E, B) \rightarrow (\hat{\rho}, \hat{u}, \hat{\theta}, \hat{E}, \hat{B})$ defined by (3.1), (3.2) with $\varepsilon \in (0, 1]$. The precise statement of the fact is given in the following proposition.

PROPOSITION 3.1. *Suppose* (1.4), (i) *of* (1.5) *and* (1.6). *Let* $s \geq 3$ *be an integer and let the initial data* $(\rho_0^\varepsilon, u_0^\varepsilon, \theta_0^\varepsilon, E_0^\varepsilon, B_0^\varepsilon)(x)$ *satisfy* (3.6), $(3.7)_{1,2}$ *and* $(3.8)_{1,2}$. *Then there exist positive constants* T, m_0, M_0, M_1, M_2, N_0, N_1, N_2, *and an exponent* $\eta \in [0, 1]$, *which are independent of* $\varepsilon \in (0, 1]$, *such that if*

$$(\rho, u, \theta, E, B) \in V^s(T; m_0, M_0, M_1, M_2) \times W_\varepsilon^s(T; N_0, N_1, N_2, \eta)\,,$$

the Cauchy problem (3.1), (3.2) *has a unique solution* $(\hat{\rho}, \hat{u}, \hat{\theta}, \hat{E}, \hat{B})(t, x)$ *in the same set* $V^s(T; m_0, M_0, M_1, M_2) \times W_\varepsilon^s(T; N_0, N_1, N_2, \eta)$.

REMARK 3.1. (i) The parameters T, m_0, M_j, N_j $(j=0, 1, 2)$ and η are chosen as follows: $m_0=k_0/2$, $M_0=2K_0$. M_1, M_2 and N_1 are determined in terms of k_0, K_0 and K_1. Also, T, N_0 and N_2 are determined by means of k_0, K_0, K_1, K_2 and K_3. The exponent η can be taken as

$$\eta = \max\{1-2\beta, 1-2\beta'\}\,. \tag{3.10}$$

(ii) In the simple case where the initial data are independent of ε, we can take $\eta=1$. If, in addition, the initial data satisfy (1.7), i.e., the initial layer for the electric field is absent, we have $\eta=0$.

PROOF of Proposition 3.1. The existence and uniqueness of solution follows from Propositions 2.3 and 2.4 at once. Therefore it suffices to show the estimates. We apply the energy inequality (2.8) with $l=s$ to the equations for $(\hat{\rho}-\bar{\rho}, \hat{u}, \hat{\theta}-\bar{\theta})$ in (3.1). From the explicit form of the functions F_1, G_2^* and G_3^*, and the assumption that $(\rho, u, \theta, E, B) \in V^s(T) \times W_\varepsilon^s(T)$ (we use the abbreviation in notation), we obtain

$$\begin{cases} \|F_1\|_s \leq C(M_1 + \|D_x u\|_s)\,, \qquad \|F_1\|_{s-1} \leq CM_1\,, \\ \|(G_2^*, G_3^*)\|_{s-1} \leq C'(M_1 + N_0 + N_1)\,. \end{cases} \tag{3.11}$$

Here $C=C(m_0, M_0, M_1)$ and $C'=C'(m_0, M_0, M_1, N_0, N_1)$ are constants independent of ε. Combining these estimates with (2.8) (with $l=s$ and $\alpha=1$) leads to

$$\|(\hat{\rho}-\bar{\rho}, \hat{u}, \hat{\theta}-\bar{\theta})(t)\|_s^2+\int_0^t \|(\hat{u}, \hat{\theta}-\bar{\theta})(\tau)\|_{s+1}^2 d\tau \leq C_1^2 e^{C_2 t}\{K_1^2+C_4(M_1+N_0+N_1)^2(1+t)t\}. \tag{3.12}$$

Here $C_4=C_4(m_0, M_0, M_1, N_0, N_1)$ is a constant not depending on ε. $C_1=C_1(m_0, M_0)$ and $C_2=C_2(m_0, M_0, M_1)$ are the constants appearing in (2.8). We choose a constant $T>0$ such that

$$e^{C_2 T}\leq 2, \qquad C_4(M_1+N_0+N_1)^2(1+T)T\leq K_1^2. \tag{3.13$_1$}$$

Then (3.12) becomes

$$\|(\hat{\rho}-\bar{\rho}, \hat{u}, \hat{\theta}-\bar{\theta})(t)\|_s^2+\int_0^t \|(\hat{u}, \hat{\theta}-\bar{\theta})(\tau)\|_{s+1}^2 d\tau \leq 4C_1^2 K_1^2 \tag{3.14}$$

for $t\in[0, T]$.

Next we estimate the time derivatives $\partial_t(\hat{\rho}, \hat{u}, \hat{\theta})$. Using the equations in (3.1) and the estimates in (3.11), we have

$$\|\partial_t\hat{\rho}\|_{s-1}\leq C(\|D_x\hat{\rho}\|_{s-1}+M_1),$$
$$\|\partial_t(\hat{u}, \hat{\theta})\|_{s-1}\leq C\|D_x^2(\hat{u}, \hat{\theta})\|_{s-1}+C'(M_1+N_0+N_1),$$

where $C=C(m_0, M_0, M_1)$ and $C'=C'(m_0, M_0, M_1, N_0, N_1)$. If we combine these estimates with (3.14), we obtain

$$\int_0^t \|\partial_t(\hat{\rho}, \hat{u}, \hat{\theta})(\tau)\|_{s-1}^2 d\tau \leq C_5^2\{K_1^2+C_6(K_1+M_1+N_0+N_1)^2 t\}. \tag{3.15}$$

Here $C_5=C_5(m_0, M_0, M_1)$ and $C_6=C_6(m_0, M_0, M_1, N_0, N_1)$ are constants independent ε. We make an assumption that $T>0$ satisfies

$$C_6(K_1+M_1+N_0+N_1)^2 T\leq 3K_1^2. \tag{3.13$_2$}$$

Then (3.15) implies

$$\int_0^t \|\partial_t(\hat{\rho}, \hat{u}, \hat{\theta})(\tau)\|_{s-1}^2 d\tau \leq 4C_5^2 K_1^2 \tag{3.16}$$

for $t\in[0, T]$. Consequently,

$$|(\hat{\rho}, \hat{\theta})(t, x)-(\rho_0^\varepsilon, \theta_0^\varepsilon)(x)|\leq C_0\int_0^t \|\partial_t(\hat{\rho}, \hat{\theta})(\tau)\|_{s-1} d\tau \leq 2C_0 C_5 K_1 t^{1/2}. \tag{3.17}$$

Here C_0 is the constant in the Sobolev inequality $\sup_x |f(x)|\leq C_0\|f\|_2$. By decreasing $T>0$, we may assume the following inequality.

$$4C_0C_5K_1T^{1/2}\leq k_0=\inf_{\varepsilon}\inf_{x}\{\rho_0^\varepsilon(x),\theta_0^\varepsilon(x)\}\,. \tag{3.13$_3$}$$

Then we derive from (3.17) that

$$k_0/2\leq\hat{\rho}(t,x)\,,\quad \hat{\theta}(t,x)\leq 2K_0 \tag{3.18}$$

for $(t,x)\in[0,T]\times\boldsymbol{R}^2$. These observations show that we can take the constants m_0, M_0, M_1 and M_2, namely, the four parameters in the definition of $V^s(T)$, as follows.

$$m_0=k_0/2\,,\quad M_0=2K_0\,,\quad M_1=2C_1K_1\,,\quad M_2=2C_5K_1\,. \tag{3.19}$$

Note that $C_1=C_1(m_0,M_0)$ and $C_5=C_5(m_0,M_0,M_1)$ are the constants in (3.12) and (3.15), respectively.

We intend to prove the estimates for $(\hat{E},\hat{B})$. We shall apply the energy inequality (2.10) with $l=s$ to the equations for $(\hat{E},\hat{B}-\bar{B})$ in (3.1). From the explicit form of F_4 we have $\|F_4\|_s\leq C(M_1+N_1)$, where $C=C(m_0,M_0,M_1,N_1)$ is a constant independent of ε. Therefore, we obtain the following inequality.

$$\|(\varepsilon^{1/2}\hat{E},\hat{B}-\bar{B})(t)\|_s^2+\int_0^t\|\hat{E}(\tau)\|_s^2d\tau\leq C_3^2\{K_1^2+C_7(M_1+N_1)^2t\}\,. \tag{3.20}$$

Here $C_7=C_7(m_0,M_0,M_1,N_1)$ does not depend on ε. $C_3=C_3(m_0,M_0,M_1)$ is the constant in (2.10). By decreasing $T>0$ again, we may assume that

$$C_7(M_1+N_1)^2T\leq 3K_1^2\,. \tag{3.13$_4$}$$

Then (3.20) becomes

$$\|(\varepsilon^{1/2}\hat{E},\hat{B}-\bar{B})(t)\|_s^2+\int_0^t\|\hat{E}(\tau)\|_s^2d\tau\leq 4C_3^2K_1^2 \tag{3.21}$$

for $t\in[0,T]$. Therefore, the constant N_1, one of the parameters in the definition of $W_\varepsilon^s(T)$, can be taken as follows.

$$N_1=2C_3K_1\,. \tag{3.22$_1$}$$

Next we estimate the time derivatives $\partial_t(\hat{E},\hat{B})$. By differentiating the both members of the equations for $(\hat{E},\hat{B})$ in (3.1), we get

$$\begin{cases}\varepsilon\hat{E}_{tt}-(1/\mu_0)\operatorname{rot}\hat{B}_t+\sigma\hat{E}_t=\tilde{F}_4\,,\\ \hat{B}_{tt}+\operatorname{rot}\hat{E}_t=0\,,\end{cases} \tag{3.23}$$

where $\tilde{F}_4=-\sigma_t\hat{E}+(F_4)_t$. We wish to apply the energy inequality (2.10) with $l=s-1$ to the system (3.23) for $\partial_t(\hat{E},\hat{B})$. For this purpose, we shall derive the estimates for the initial data, and for the right hand side. By using the equations in (3.1) and the conditions $(3.8)_{1,2}$, we get the following estimate for the initial data.

$$\|\partial_t(\varepsilon^{1/2}\hat{E}, \hat{B})(0)\|_{s-1} \leq \varepsilon^{-\eta/2} C(K_2+K_3)\,, \tag{3.24}$$

where $C=C(m_0, M_0, M_1)$ is a constant independent of ε, and the exponent η is given by (3.10). On the other hand, a direct calculation gives

$$\|\tilde{F}_4\|_{s-1} \leq C(\|\partial_t(\rho, \theta)\|_{s-1}\|\hat{E}\|_{s-1} + \|\partial_t(\rho, u, \theta, B)\|_{s-1})\,. \tag{3.25}$$

Here $C=C(m_0, M_0, M_1, N_1)$ does not depend on ε. It is seen from (3.21) and $(3.22)_1$ that $\|\hat{E}\|_{s-1} \leq \|\hat{E}\|_s \leq \varepsilon^{-1/2}N_1$. Also we know that $\|\partial_t B\|_{s-1} \leq \varepsilon^{-\eta/2}N_2$ by the assumption $(E, B) \in W_\varepsilon^s(T)$. Substituting these estimates into (3.25), we have

$$\|\tilde{F}_4\|_{s-1} \leq \varepsilon^{-1/2} C(\|\partial_t(\rho, u, \theta)\|_{s-1} + N_2)\,. \tag{3.26}$$

Here we used $\eta \in [0, 1]$. $C=C(m_0, M_0, M_1, N_1)$ is a constant independent of ε. Now, applying (2.10) with $l=s-1$ to the system (3.23) and using the estimates (3.24), (3.26), we get

$$\|\partial_t(\varepsilon^{1/2}\hat{E}, \hat{B})(t)\|_{s-1}^2 + \int_0^t \|\partial_t\hat{E}(\tau)\|_{s-1}^2 d\tau \leq \varepsilon^{-1} C_8^2\{(K_2+K_3+M_2)^2 + N_2^2 t\}\,. \tag{3.27}$$

Here $C_8=C_8(m_0, M_0, M_1, N_1)$ is a constant independent of ε. We make the additional hypothesis that $T>0$ satisfies

$$N_2^2 T \leq 3(K_2+K_3+M_2)^2\,. \tag{3.13$_5$}$$

Then (3.27) implies

$$\|\partial_t(\varepsilon^{1/2}\hat{E}, \hat{B})(t)\|_{s-1}^2 + \int_0^t \|\partial_t\hat{E}(\tau)\|_{s-1}^2 d\tau \leq \varepsilon^{-1}\tilde{N}_2^2 \tag{3.28}$$

for $t \in [0, T]$, where we set $\tilde{N}_2 = 2C_8(K_2+K_3+M_2)$.

We proceed to estimate $\|E\|_{s-1}$. By using the equation for E in (3.1), we obtain

$$\begin{aligned} \|\hat{E}(t)\|_{s-1} &\leq C(\varepsilon\|\partial_t\hat{E}(t)\|_{s-1} + \|D_x\hat{B}(t)\|_{s-1} + \|F_4(t)\|_{s-1}) \\ &\leq C_9(M_1+N_1+\tilde{N}_2) \end{aligned} \tag{3.29}$$

for $t \in [0, T]$. Here we used the estimates (3.28) and (3.21) with $(3.22)_1$. The constant $C_9=C_9(m_0, M_0, M_1, N_1)$ does not depend on ε. Consequently, we can take the parameter N_0 as follows.

$$N_0 = C_9(M_1+N_1+\tilde{N}_2)\,. \tag{3.22$_2$}$$

Finally, we show the estimate (3.28) with ε^{-1} replaced by $\varepsilon^{-\eta}$. Substituting the estimates (3.29) with $(3.22)_2$ and $\|\partial_t B\|_{s-1} \leq \varepsilon^{-\eta/2}N_2$ into (3.25), we have the inequality (3.26) with $\varepsilon^{-1/2}$ replaced by $\varepsilon^{-\eta/2}$. (In this case the constant C depends on N_0.) Therefore, as the counterpart of (3.27), we obtain

(3.30)
$$\|\partial_t(\varepsilon^{1/2}\hat{E},\hat{B})(t)\|_{s-1}^2+\int_0^t\|\partial_t\hat{E}(\tau)\|_{s-1}^2d\tau$$
$$\leq\varepsilon^{-\gamma}C_{10}^2\{(K_2+K_3+M_2)^2+N_2^2t\}$$
$$\leq 4\varepsilon^{-\gamma}C_{10}^2(K_2+K_3+M_2)^2$$

for $t\in[0,T]$. Here we used $(3.13)_5$. The constant $C_{10}=C_{10}(m_0,M_0,M_1,N_0,N_1)$ is independent of ε. Consequently, we can take N_2 as follows.

$(3.22)_3$
$$N_2=2C_{10}(K_2+K_3+M_2).$$

Thus the parameters m_0, M_0, M_1, M_2, N_0, N_1, N_2 and γ are all determined by (3.19), $(3.22)_{1,2,3}$ and (3.10). The positive constant T is chosen so as to satisfy the inequalities $(3.13)_{1-5}$. This completes the proof of Proposition 3.1.

§ 4. Uniform stability of solutions of the nonelinear equations

We wish to show that a unique solution to the Cauchy problem (1.1), (1.2), (1.3) exists on $[0,T_0]\times\boldsymbol{R}^2$, where T_0 is a positive number not depending on the parameter $\varepsilon\in(0,1]$. The construction of the solution is based on the successive approximation. Let us define a sequence of approximating functions $\{(\rho^n,u^n,\theta^n,E^n,B^n)(t,x)\}_{n\geq 0}$ as follows: Let

$(4.1)_0$
$$(\rho^0,u^0,\theta^0,E^0,B^0)(t,x)=(\bar{\rho},0,\bar{\theta},0,\bar{B}).$$

For $n+1\geqq 1$, let $(\rho^{n+1},u^{n+1},\theta^{n+1},E^{n+1},B^{n+1})(t,x)$ be a unique solution of the Cauchy problem

$(4.1)_{n+1}$
$$\begin{cases}\rho_t^{n+1}+u^n\cdot\nabla\rho^{n+1}=F_1^n,\\ \rho^n u_t^{n+1}-\mu^n\Delta u^{n+1}-(\mu^n+\mu'^n)\nabla\operatorname{div}u^{n+1}=\rho^n(F_2^n+G_2^n),\\ (\rho e_\theta)^n\theta_t^{n+1}-\kappa^n\Delta\theta^{n+1}=(\rho e_\theta)^n(F_3^n+G_3^n),\\ \varepsilon E_t^{n+1}-(1/\mu_0)\operatorname{rot}B^{n+1}+\sigma^n E^{n+1}=F_4^n,\\ B_t^{n+1}+\operatorname{rot}E^{n+1}=0,\end{cases}$$

$(4.2)_{n+1}$
$$(\rho^{n+1},u^{n+1},\theta^{n+1},E^{n+1},B^{n+1})(0,x)=(\rho_0^\varepsilon,u_0^\varepsilon,\theta_0^\varepsilon,E_0^\varepsilon,B_0^\varepsilon)(x).$$

Here we use the abbreviations such as $\mu^n=\mu(\rho^n,\theta^n)$, $(\rho e_\theta)^n=\rho^n e_\theta(\rho^n,\theta^n)$, $F_1^n=F_1(\rho^n,u^n)$, $\cdots$. F_j $(j=1,\cdots,4)$ and G_j $(j=2,3)$ are functions defined by (3.3) and (3.4), respectively.

We shall show the uniform convergence of the approximating functions and obtain the solution to the Cauchy problem (1.1), (1.2), (1.3) as the limit function.

THEOREM 4.1. *Suppose* (1.4), (i) *of* (1.5) *and* (1.6). *Let* $s\geq 3$ *be an integer and let the initial data* $(\rho_0^\varepsilon,u_0^\varepsilon,\theta_0^\varepsilon,E_0^\varepsilon,B_0^\varepsilon)(x)$ *satisfy* (3.6), $(3.7)_{1,2}$ *and* $(3.8)_{1,2}$. *Then there exists a positive constant* T_0 *independent of* $\varepsilon\in(0,1]$ *such that the Cauchy problem* (1.1), (1.3) *has a unique solution*

$$(4.3) \qquad (\rho^\varepsilon, u^\varepsilon, \theta^\varepsilon, E^\varepsilon, B^\varepsilon) \in V^s(T_0; m_0, M_0, M_1, M_2) \times W_\varepsilon^s(T_0; N_0, N_1, N_2, \eta).$$

T_0 is less than T, where T is the positive constant given in Proposition 3.1. *The parameters m_0, M_j, N_j ($j=0,1,2$) and η are defined in Proposition* 3.1 *and hence are independent of ε. If, in addition, the initial data satisfy* $\operatorname{div} B_0^\varepsilon=0$ *on* $\boldsymbol{R}^2$, *the solution obtained also satisfies* (1.2) *and therefore is a unique solution to the problem* (1.1), (1.2), (1.3).

PROOF. From Proposition 3.1, we see that the approximating functions are all well-defined on $[0, T]\times\boldsymbol{R}^2$ and are uniformly bounded in the following sense:

$$(4.4) \qquad (\rho^n, u^n, \theta^n, E^n, B^n) \in V^s(T; m_0, M_0, M_1, M_2) \times W_\varepsilon^s(T; N_0, N_1, N_2, \eta)$$

for $\varepsilon\in(0, 1]$ and $n\geq 0$. In order to show the convergence of the sequence, we consider the difference $(\hat\rho^n, \hat u^n, \hat\theta^n, \hat E^n, \hat B^n)=(\rho^{n+1}-\rho^n, u^{n+1}-u^n, \theta^{n+1}-\theta^n, E^{n+1}-E^n, B^{n+1}-B^n)$ for $n\geq 0$. Subtructing $(4.1)_n$ from $(4.1)_{n+1}$, we obtain for $n\geq 1$,

$$(4.5)_n \qquad \begin{cases} \hat\rho_t^n + u^n\cdot\nabla\hat\rho^n = \hat F_1^n, \\ \rho^n\hat u_t^n - \mu^n\Delta\hat u^n - (\mu^n+\mu'^n)\nabla\operatorname{div}\hat u^n = \rho^n(\hat F_2^n + \hat G_2^n), \\ (\rho e_\theta)^n\hat\theta_t^n - \kappa^n\nabla\hat\theta^n = (\rho e_\theta)^n(\hat F_3^n + \hat G_3^n), \\ \varepsilon\hat E_t^n - (1/\mu_0)\operatorname{rot}\hat B^n + \sigma^n\hat E^n = \hat F_4^n, \\ \hat B_t^n + \operatorname{rot}\hat E^n = 0, \end{cases}$$

with the initial condition

$$(4.6)_n \qquad (\hat\rho^n, \hat u^n, \hat\theta^n, \hat E^n, \hat B^n)(0, x) = (0, 0, 0, 0, 0).$$

Here

$$(4.7) \qquad \begin{cases} \hat F_1^n = -(u^n - u^{n-1})\cdot\nabla\rho^n + F_1^n - F_1^{n-1}, \\ \hat F_2^n = F_2^n - F_2^{n-1}, \qquad \hat F_3^n = F_3^n - F_3^{n-1}, \\ \hat F_4^n = -(\sigma^n - \sigma^{n-1})E^n + F_4^n - F_4^{n-1}, \end{cases}$$

$$(4.8) \qquad \begin{cases} \hat G_2^n = (\mu^n/\rho^n - \mu^{n-1}/\rho^{n-1})\Delta u^n + \{(\mu^n+\mu'^n)/\rho^n \\ \qquad -(\mu^{n-1}+\mu'^{n-1})/\rho^{n-1}\}\nabla\operatorname{div} u^n + G_2^n - G^{n-1}, \\ \hat G_3^n = \{\kappa^n/(\rho e_\theta)^n - \kappa^{n-1}/(\rho e_\theta)^{n-1}\}\Delta\theta^n + G_3^n + G_3^{n-1}. \end{cases}$$

We want to apply the energy inequalities (2.8) and (2.10) with $l=s-1$ to the system $(4.5)_n$ for $(\hat\rho^n, \hat u^n, \hat\theta^n, \hat E^n, \hat B^n)$. To this end, we estimate the right hand side of $(4.5)_n$. By (4.4), we have

$$(4.9) \qquad \begin{cases} \|\hat F_1^n\|_{s-1} \leq C(\|\hat\rho^{n-1}\|_{s-1} + \|\hat u^{n-1}\|_s), \\ \|(\hat F_2^n, \hat F_3^n)\|_{s-1} \leq C\|(\hat\rho^{n-1}, \hat u^{n-1}, \hat\theta^{n-1}, \hat E^{n-1}, \hat B^{n-1})\|_{s-1}, \\ \|\hat F_4^n\|_{s-1} \leq C\|(\hat\rho^{n-1}, \hat u^{n-1}, \hat\theta^{n-1}, \hat B^{n-1})\|_{s-1}, \end{cases}$$

$$(4.10) \qquad \|(\hat G_2^n, \hat G_3^n)\|_{s-2} \leq C\|(\hat\rho^{n-1}, \hat u^{n-1}, \hat\theta^{n-1})\|_{s-1}.$$

Here C is a constant independent of both $\varepsilon\in(0, 1]$ and $n\geq 1$. Using the estimates

(4.9), (4.10) and noting the initial condition $(4.6)_n$, we get the following estimate from the energy inequalities: For any $\alpha\in(0, 1]$,

$$(4.11)\qquad \|(\hat\rho^n, \hat u^n, \hat\theta^n, \varepsilon^{1/2}\hat E^n, \hat B^n)(t)\|_{s-1}^2+\int_0^t \|(\hat u^n, \hat\theta^n)(\tau)\|_s^2+\|\hat E^n(\tau)\|_{s-1}^2 d\tau$$
$$\leq Ce^{\alpha^{-1}Ct}\left\{t\int_0^t \|\hat\rho^{n-1}(\tau)\|_{s-1}^2+\|\hat u^{n-1}(\tau)\|_s^2 d\tau\right.$$
$$+\alpha\int_0^t \|(\hat\rho^{n-1}, \hat u^{n-1}, \hat\theta^{n-1}, \hat E^{n-1}, \hat B^{n-1})(\tau)\|_{s-1}^2 d\tau$$
$$\left.+\int_0^t \|(\hat\rho^{n-1}, \hat u^{n-1}, \hat\theta^{n-1}, \hat B^{n-1})(\tau)\|_{s-1}^2 d\tau\right\},$$

where C is a constant independent of ε, α and $n\geq 1$. Now we fix $\alpha\in(0, 1]$ in (4.11) so as to satisfy $\alpha C\leq 1/4$. Next we choose $T_0>0$ such that

$$(4.12)\qquad T_0\leq T, \qquad e^{\alpha^{-1}CT_0}\leq 2, \qquad C(1+T_0)T_0\leq 1/8.$$

Then, setting

$$X_n(t)=\sup_{0\leq\tau\leq t}\|(\hat\rho^n, \hat u^n, \hat\theta^n, \varepsilon^{1/2}\hat E^n, \hat B^n)(\tau)\|_{s-1}^2$$
$$+\int_0^t \|(\hat u^n, \hat\theta^n)(\tau)\|_s^2+\|\hat E^n(\tau)\|_{s-1}^2 d\tau,$$

we obtain the inequality $X_n(T_0)\leq(1/2)X_{n-1}(T_0)$ for $n\geq 1$. This means that, for each $\varepsilon\in(0, 1]$, $\{(\rho^n-\bar\rho, u^n, \theta^n-\bar\theta, E^n, B^n-\bar B)\}_{n\geq 0}$ is a Cauchy sequence in $C^0(0, T_0; H^{s-1})$. Consequently, there exists uniquely a function $(\rho, u, \theta, E, B)(t, x)$ with $(\rho-\bar\rho, u, \theta-\bar\theta, E, B-\bar B)\in C^0(0, T_0; H^{s-1})$ such that

$$(\rho^n-\bar\rho, u^n, \theta^n-\bar\theta, E^n, B^n-\bar B)\to(\rho-\bar\rho, u, \theta-\bar\theta, E, B-\bar B)$$

strongly in $C^0(0, T_0; H^{s-1})$ as $n\to\infty$. On the other hand, the uniform estimate (4.4) shows that there are subsequences (still denoted by the same symbol) such that as $n\to\infty$,

$$(\rho^n-\bar\rho, u^n, \theta^n-\bar\theta, E^n, B^n-\bar B)\to(\rho-\bar\rho, u, \theta-\bar\theta, E, B-\bar B),$$
$$(u^n, \theta^n-\bar\theta)\to(u, \theta-\bar\theta) \quad\text{and}\quad E^n\to E$$

weak* in $L^\infty(0, T_0; H^s)$, weakly in $L^2(0, T_0; H^{s+1})$ and weakly in $L^2(0, T_0; H^s)$ respectively. It follows from these observations that the limit function $(\rho, u, \theta, E, B)(t, x)$ is a solution of the Cauchy problem (1.1), (1.3) satisfying

$$(\rho, u, \theta, E, B)\in\tilde V^s(T_0; m_0, M_0, M_1, M_2)\times\tilde W^s_\varepsilon(T_0; N_0, N_1, N_2, \eta).$$

Hence, by Lemmas 2.1 and 2.2, we conclude that $\partial_t^j(\rho-\bar\rho, E, B-\bar B)\in C^0(0, T_0; H^{s-j})$ and $\partial_t^j(u, \theta-\bar\theta)\in C^0(0, T_0; H^{s-2j})$ for $j=0, 1$. Therefor the solution $(\rho, u, \theta, E, B)(t, x)$ of (1.1), (1.3) obtained above satisfies (4.3). The uniqueness of the solution follows from the energy inequalities (2.8) and (2.10).

Finally we prove the last statement of the theorem. Let us assume that $\operatorname{div} B_0^\varepsilon=0$ on $\boldsymbol{R}^2$. Then, applying div to the both members of the equation for B in (1.1), we get $(\operatorname{div} B)_t=0$. Therefore the equality $\operatorname{div} B=0$ holds on $[0, T_0]\times\boldsymbol{R}^2$, and consequently the limit function $(\rho, u, \theta, E, B)(t, x)$ is a unique solution of the problem (1.1), (1.2), (1.3). Thus the proof of Theorem 4.1 is completed.

§ 5. Convergence of solutions of the nonlinear equations as $\varepsilon\to 0$

We consider the limit as $\varepsilon\to 0$ of the solution $(\rho^\varepsilon, u^\varepsilon, \theta^\varepsilon, E^\varepsilon, B^\varepsilon)(t, x)$ of the Cauchy problem (1.1), (1.3) (or (1.1), (1.2), (1.3)) constructed in the preceeding section. In addition to the hypotheses of Theorem 4.1, let us assume the following condition for the initial data $(\rho_0^\varepsilon, u_0^\varepsilon, \theta_0^\varepsilon, E_0^\varepsilon, B_0^\varepsilon)(x)$: There exist a function $(\rho_0^0, u_0^0, \theta_0^0, B_0^0)(x)$ with $(\rho_0^0-\bar\rho, u_0^0, \theta_0^0-\bar\theta, B_0^0-\bar B)\in H^s$ and a number $\gamma>0$, both independent of $\varepsilon\in(0,1]$, such that

(5.1)
$$\sup_\varepsilon \varepsilon^{-\gamma}\|(\rho_0^\varepsilon-\rho_0^0, u_0^\varepsilon-u_0^0, \theta_0^\varepsilon-\theta_0^0, B_0^\varepsilon-B_0^0)\|_{s-1}=K_4<+\infty\,.$$

It should be noted that if the initial data are independent of ε and satisfy (3.6), then (5.1) holds for any $\gamma>0$.

Under these hypotheses, we shall show that there exists a positive constant T_1 not depending on ε such that as $\varepsilon\to 0$, the solution $(\rho^\varepsilon, u^\varepsilon, \theta^\varepsilon, E^\varepsilon, B^\varepsilon)(t, x)$ of (1.1), (1.2), (1.3) converges on $[0, T_1]\times\boldsymbol{R}^2$ to a limit function $(\rho^0, u^0, \theta^0, E^0, B^0)(t, x)$. Also it will be shown that this limit function satisfies the relation (1.7) and that the function $(\rho^0, u^0, \theta^0, B^0)(t, x)$ excepting $E^0(t, x)$ is a unique solution of the magnetohydrodynamic equations (1.8), (1.9) with the initial condition

(5.2)
$$(\rho^0, u^0, \theta^0, B^0)(0, x)=(\rho_0^0, u_0^0, \theta_0^0, B_0^0)(x)\,.$$

THEOREM 5.1. *Suppose* (1.4), (i) *of* (1.5) *and* (1.6). *Let* $s\geq 3$ *be an integer and let the initial data* $(\rho_0^\varepsilon, u_0^\varepsilon, \theta_0^\varepsilon, E_0^\varepsilon, B_0^\varepsilon)(x)$ *satisfy* (3.6), $(3.7)_{1,2}$, $(3.8)_{1,2}$ *and* (5.1). *Let, furthermore,* $(\rho^\varepsilon, u^\varepsilon, \theta^\varepsilon, E^\varepsilon, B^\varepsilon)(t, x)$ *be the solution on* $[0, T_0]\times\boldsymbol{R}^2$ *to the Cauchy problem* (1.1), (1.3) *constructed in Theorem* 4.1. *Then there exists a positive constant* T_1 $(\leq T_0)$ *independent of* $\varepsilon\in(0,1]$ *such that as* $\varepsilon\to 0$, *the solution* $(\rho^\varepsilon, u^\varepsilon, \theta^\varepsilon, E^\varepsilon, B^\varepsilon)(t, x)$ *converges on* $[0, T_1]\times\boldsymbol{R}^2$ *to a limit function* $(\rho^0, u^0, \theta^0, E^0, B^0)(t, x)$ *which satisfies the equations* (1.7), (1.8) *and the initial condition* (5.2). *In particular,* $(\rho^0, u^0, \theta^0, B^0)(t, x)$ *is a solution of the Cauchy problem* (1.8), (5.2) *satisfying*

(5.3)
$$\begin{cases} \partial_t^j(\rho^0-\bar\rho)\in L^\infty(0, T_1; H^{s-j})\,, \\ \partial_t^j(u^0, \theta^0-\bar\theta)\in L^\infty(0, T_1; H^{s-2j})\cap L^2(0, T_1; H^{s+1-2j})\,, \\ \partial_t^j(B^0-\bar B)\in L^\infty(0, T_1; H^{s-2j}) \qquad \textit{for } j=0, 1\,. \end{cases}$$

If, in addition, the initial data satisfy $\operatorname{div} B_0^\varepsilon=0$ *on* $\boldsymbol{R}^2$, *then* $(\rho^0, u^0, \theta^0, B^0)(t, x)$ *becomes a unique solution of the problem* (1.8), (1.9), (5.2) *satisfying*

$$(5.4)\qquad \begin{cases} \partial_t^j(\rho^0-\bar\rho)\in C^0(0, T_1; H^{s-j}), \\ \partial_t^j(u^0, \theta^0-\bar\theta, B^0-\bar B)\in C^0(0, T_1; H^{s-2j})\cap L^2(0, T_1; H^{s+1-2j}) \\ \qquad\qquad\qquad\qquad\qquad\qquad \textit{for } j=0, 1. \end{cases}$$

Remark 5.1. (i) The solution $(\rho^\varepsilon, u^\varepsilon, \theta^\varepsilon, E^\varepsilon, B^\varepsilon)(t, x)$ converges to a limit function $(\rho^0, u^0, \theta^0, E^0, B^0)(t, x)$ in the following sense: For any $t\in[0, T_1]$,

$$(5.5)\qquad \begin{aligned} &\|(\rho^\varepsilon-\rho^0, u^\varepsilon-u^0, \theta^\varepsilon-\theta^0, B^\varepsilon-B^0)(t)\|_{s-1}^2 \\ &\quad+\int_0^t \|u^\varepsilon-u^0, \theta^\varepsilon-\theta^0)(\tau)\|_s^2+\|(E^\varepsilon-E^0)(\tau)\|_{s-1}^2 d\tau\le\varepsilon^{2\lambda}C, \end{aligned}$$

where C is a constant indesendent of ε, and

$$(5.6)\qquad \lambda=\min\{\gamma, 1-\eta/2\}>0.$$

Here $\eta\in[0, 1]$ is the number defined by (3.10).
(ii) In the simple case where the initial data are independent of ε, the exponent λ of the convergence rate in (5.5) can be taken as $\lambda=1/2$ because in this case $\gamma>0$ is arbitrary and $\eta=1$ (by Remark 3.1 (ii)). If, in addition, the initial layer for the electric field is absent, then we can take $\lambda=1$ because of $\eta=0$.

Proof of Theorem 5.1. First we prove the convergence of the sequence $\{(\rho^\varepsilon, u^\varepsilon, \theta^\varepsilon, E^\varepsilon, B^\varepsilon)(t, x)\}_{\varepsilon\in(0,1]}$ as $\varepsilon\to0$. Let $0<\delta<\varepsilon\le1$ and let $(\hat\rho, \hat u, \hat\theta, \hat E, \hat B)=(\rho^\delta-\rho^\varepsilon, u^\delta-u^\varepsilon, \theta^\delta-\theta^\varepsilon, E^\delta-E^\varepsilon, B^\delta-B^\varepsilon)$. Then we obtain

$$(5.7)\qquad \begin{cases} \hat\rho_t+u^\delta\cdot\nabla\hat\rho=\hat F_1, \\ \rho^\delta\hat u_t-\mu^\delta\Delta\hat u-(\mu^\delta+\mu'^\delta)\nabla\operatorname{div}\hat u=\rho^\delta(\hat F_2+\hat G_2), \\ (\rho e_\theta)^\delta\hat\theta_t-\kappa^\delta\Delta\hat\theta=(\rho e_\theta)^\delta(\hat F_3+\hat G_3), \\ \delta\hat E_t-(1/\mu_0)\operatorname{rot}\hat B+\sigma^\delta\hat E=\hat F_4, \\ \hat B_t+\operatorname{rot}\hat E=0, \end{cases}$$

$$(5.8)\qquad (\hat\rho, \hat u, \hat\theta, \hat E, \hat B)(0, x)=(\hat\rho_0, \hat u_0, \hat\theta_0, \hat E_0, \hat B_0)(x),$$

where $(\hat\rho_0, \hat u_0, \hat\theta_0, \hat E_0, \hat B_0)=(\rho_0^\delta-\rho_0^\varepsilon, u_0^\delta-u_0^\varepsilon, \theta_0^\delta-\theta_0^\varepsilon, E_0^\delta-E_0^\varepsilon, B_0^\delta-B_0^\varepsilon)$. The functions on the right hand side of (5.7) are given explicity as follows.

$$(5.9)\qquad \begin{cases} \hat F_1=-(u^\delta-u^\varepsilon)\cdot\nabla\rho^\varepsilon+F_1^\delta-F_1^\varepsilon, \\ \hat F_2=F_2^\delta-F_2^\varepsilon, \qquad \hat F_3=F_3^\delta-F_3^\varepsilon, \\ \hat F_4=-(\delta-\varepsilon)E_t^\varepsilon-(\sigma^\delta-\sigma^\varepsilon)E^\varepsilon+F_4^\delta-F_4^\varepsilon, \end{cases}$$

$$(5.10)\qquad \begin{cases} \hat G_2=(\mu^\delta/\rho^\delta-\mu^\varepsilon/\rho^\varepsilon)\Delta u^\varepsilon+\{(\mu^\delta+\mu'^\delta)/\rho^\delta \\ \qquad -(\mu^\varepsilon-\mu'^\varepsilon)/\rho^\varepsilon\}\nabla\operatorname{div} u^\varepsilon+G_2^\delta-G_2^\varepsilon, \\ \hat G_3=\{\kappa^\delta/(\rho e_\theta)^\delta-\kappa^\varepsilon/(\rho e_\theta)^\varepsilon\}\Delta\theta^\varepsilon+G_3^\delta-G_3^\varepsilon. \end{cases}$$

Here we use the abbreviations such as $\mu^{\varepsilon}=\mu(\rho^{\varepsilon}, \theta^{\varepsilon})$, $(\rho e_{\theta})^{\varepsilon}=\rho^{\varepsilon}e_{\theta}(\rho^{\varepsilon}, \theta^{\varepsilon})$, $F_1^{\varepsilon}=F_1(\rho^{\varepsilon}, u^{\varepsilon})$, $\cdots$. The functions F_j $(j=1, \cdots, 4)$ and G_j $(j=2, 3)$ are defined by (3.3) and (3.4), respectively.

We provide estimates for the initial data (5.8) and for the right side of (5.7) that permit us to apply the energy inequalities (2.8) and (2.10) with $l=s-1$ to the system (5.7). As for the initial data, we obtain from (3.9) and (5.1) that

$$\|(\hat{\rho}_0, \hat{u}_0, \hat{\theta}_0, \delta^{1/2}\hat{E}_0, \hat{B}_0)\|_{s-1}\leq\delta^{1/2}C(K_1+K_2)+\varepsilon^{\gamma}CK_4\,, \tag{5.11}$$

where C is a constant independent of ε and δ. Here we used $0<\delta<\varepsilon\leq 1$. Next we consider the right side of (5.7). Compare (5.9) and (5.10) with (4.7) and (4.8), respectively. The only difference lies in that we have an extra term $(\delta-\varepsilon)E_t^{\varepsilon}$ in the last equation of (5.9). Therefore, using the uniform estimate (4.3), we get the following inequalities analogous to (4.9), (4.10):

$$\begin{cases} \|\hat{F}_1\|_{s-1}\leq C(\|\hat{\rho}\|_{s-1}+\|\hat{u}\|_s)\,, \\ \|(\hat{F}_2, \hat{F}_3)\|_{s-1}\leq C\|(\hat{\rho}, \hat{u}, \hat{\theta}, \hat{E}, \hat{B})\|_{s-1}\,, \\ \|\hat{F}_4\|_{s-1}\leq C\|(\hat{\rho}, \hat{u}, \hat{\theta}, \hat{B})\|_{s-1}+\varepsilon\|\partial_t E^{\varepsilon}\|_{s-1}\,, \end{cases} \tag{5.12}$$

$$\|(\hat{G}_2, \hat{G}_3)\|_{s-2}\leq C\|(\hat{\rho}, \hat{u}, \hat{\theta})\|_{s-1}\,. \tag{5.13}$$

Here we also used $0<\delta<\varepsilon\leq 1$. C is a constant independent of ε and δ.

Now we apply the energy inequalities (2.8) and (2.10) with $l=s-1$ to the system (5.7). Substituing the estimates (5.11), (5.12) and (5.13) to the resulting inequality and using the fact that the $L^2(0, T_0; H^{s-1})$-norm of $\partial_t E^{\varepsilon}$ is bounded by $\varepsilon^{-\gamma/2}N_2$, we obtain

$$\begin{aligned} &\|(\hat{\rho}, \hat{u}, \hat{\theta}, \delta^{1/2}\hat{E}, \hat{B})(t)\|_{s-1}^2+\int_0^t \|(\hat{u}, \hat{\theta})(\tau)\|_s^2+\|\hat{E}(\tau)\|_{s-1}^2 d\tau \\ &\leq Ce^{\alpha^{-1}Ct}\Big\{\delta(K_1+K_2)^2+\varepsilon^{2\lambda}(K_4+N_2)^2+t\int_0^t \|\hat{\rho}(\tau)\|_{s-1}^2+\|\hat{u}(\tau)\|_s^2 d\tau \\ &\quad+\alpha\int_0^t \|(\hat{\rho}, \hat{u}, \hat{\theta}, \hat{E}, \hat{B})(\tau)\|_{s-1}^2 d\tau+\int_0^t \|(\hat{\rho}, \hat{u}, \hat{\theta}, \hat{B})(\tau)\|_{s-1}^2 d\tau\Big\} \end{aligned} \tag{5.14}$$

for any $\alpha\in(0, 1]$. Here λ is the number defined by (5.6), and C is a constant independent of ε, δ and α. Let us fix $\alpha\in(0, 1]$ in (5.14) so as to satisfy $\alpha C\leq 1/4$. Then we choose $T_1>0$ such that

$$T_1\leq T_0\,, \qquad e^{\alpha^{-1}CT_1}\leq 2\,, \qquad C(1+T_1)T_1\leq 1/8\,. \tag{5.15}$$

Then, setting

$$Y_{\varepsilon,\delta}(t)=\sup_{0\leq\tau\leq t}\|(\hat{\rho}, \hat{u}, \hat{\theta}, \hat{B})(\tau)\|_{s-1}^2+\int_0^t \|(\hat{u}, \hat{\theta})(\tau)\|_s^2+\|\hat{E}(\tau)\|_{s-1}^2 d\tau\,,$$

we get the inequality

$$Y_{\varepsilon,\delta}(T_1) \leq \delta C(K_1+K_2)^2 + \varepsilon^{2\lambda} C(K_4+N_2)^2 , \tag{5.16}$$

where C is a constant independent of ε and δ. Since $\lambda>0$, we see from (5.16) that $Y_{\varepsilon,\delta}(T_1) \to 0$ as $\varepsilon \to 0$ (hence $\delta \to 0$). Consequently, there exists uniquely a function $(\rho^0, u^0, \theta^0, E^0, B^0)(t, x)$ satisfying $(\rho^0-\bar{\rho}, B^0-\bar{B}) \in C^0(0, T_1; H^{s-1})$, $(u^0, \theta^0-\bar{\theta}) \in C^0(0, T_1; H^{s-1}) \cap L^2(0, T_1; H^s)$ and $E^0 \in L^2(0, T_1; H^{s-1})$ such that as $\varepsilon \to 0$,

$$(\rho^\varepsilon-\bar{\rho}, B^\varepsilon-\bar{B}) \to (\rho^0-\bar{\rho}, B^0-\bar{B}) ,$$
$$(u^\varepsilon, \theta^\varepsilon-\bar{\theta}) \to (u^0, \theta^0-\bar{\theta}) \quad \text{and} \quad E^\varepsilon \to E^0$$

strongly in $C^0(0, T_1; H^{s-1})$, $C^0(0, T_1; H^{s-1}) \cap L^2(0, T_1; H^s)$ and $L^2(0, T_1; H^{s-1})$, respectively. Combing this with the uniform estimate (4.3), we see as in the proof of Theorem 4.1 that the limit function $(\rho^0, u^0, \theta^0, B^0, E^0)(t, x)$ satisfies (5.3) for $j=0$ and $E^0 \in L^\infty(0, T_1; H^{s-1}) \cap L^2(0, T_1; H^s)$. Furthermore, we can see that the limit function satisfies the equations (1.7), (1.8) and the initial condition (5.2). Hence (5.3) holds for $j=1$. The estimate (5.5) is obtained from (5.16) by letting $\delta \to 0$. Thus the first part of the theorem has been proved.

Finally we consider the case where $\operatorname{div} B_0^\varepsilon=0$ holds on $\boldsymbol{R}^2$. Since this implies $\operatorname{div} B^\varepsilon=0$ by Theorem 4.1, we conclude by letting $\varepsilon \to 0$ that $\operatorname{div} B^0=0$ on $[0, T_1] \times \boldsymbol{R}^2$. Hence the limit function $(\rho^0, u^0, \theta^0, B^0)(t, x)$ is a solution of the Cauchy problem (1.8), (1.9), (5.2) satisfying (5.3) for $j=0, 1$. In order to prove the uniqueness and the regularity (5.4), we recall the well known formula in vector analysis:

$$-\operatorname{rot}\{(1/\sigma\mu_0) \operatorname{rot} B\} = (1/\sigma\mu_0)(\Delta B - \nabla \operatorname{div} B) - \nabla(1/\sigma\mu_0) \times \operatorname{rot} B .$$

By this formula and (1.9), the equation of B in (1.8) can be regarded as a symmetric system of strongly parabolic type. Therefore, by the argument analogous to that employed in the proof of Lemmas 2.1 and 2.2, we can prove the regularity (5.4) and the uniqueness of the solution to the problem (1.8), (1.9), (5.2). Thus the proof of Theorem 5.1 is completed.

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Department of Mathematics
Nara Women's University
Nara, 630 Japan