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ON JOINT NUMERICAL RANGES AND JOINT NORMALOIDS IN A C*-ALGEBLA

by

K. R. UNNI and C. PUTTAMADAIAH

The notion of the joint numerical range of a finite system of elements in a unital complex Banach algebra was introducəd by Bonsall and Duncan (p. 23, [2]), and also proved that it is a convex compact subset of C^n. Later Mocanu [5] extended this definition to a C*-algebra and obtained several interesting results in this set up. The result (Lemma 5, p. 43, [3]) that if a and b are single elements in unitial Banach algebras A and B respectively, then the numerical range V((a. b)) of $(a, b)\in A+B$ is equal to the convex hull of $V(a)\cup V(b)$, is also valid in case of a C*-algebra. The purpose of this paper is to generalize this result to an n-tuple of elements in a C*-algebra. It is also proved, on contrary to the expentation that the generalization of a well known result that a single element a in a C*-algebra is normaloid if and only if $\|a^k\|=\|a\|^k$ for all positive integers k, is not true for a finite system of elements in a C*-algebra.

1. Joint numerical range

If A and B are unital C*-algebra with unit elements e_1 and e_2 respectively, then

$$A+B=\{(a, b): a\in A, b\in B\}$$

with componentwise addition, multiplication, scalar-multiplication, and conjugation together with the norm

$$\|(a, b)\|=\max\{\|a\|, \|b\|\}$$

is a unital C*-algebra with the unit element (e_1, e_2)

If $a=(a_1, a_2, \cdots, a_n)$ and $b=(b_1, b_2, \cdots, b_n)$ are n-tuples of elements of A and B respectively, then $a+b$ is given by $a+b=((a_1, b_1), (a_2, b_2), \cdots, (a_n, b_n))$. where $(a_i,b_i)\in a+b$, $1\leq i\leq n$. Throughout we shall consider complex C*-alsebras only.

A linear functional f on a unital C*-algebra is positive if $f(a^*a)\geq 0$ for all

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$a \in A$. It is known that f is positive if and only if f is bounded and $\|f\|=f(e)$ ([1], p. 40 [4], prop. 3. 3. p. 24 [9] and cor. 4. 5. 3, p. 215, Th. 4. 8. 16 [7]). A positive functional f such the $f(e)=1$ is called a state on A.

DEFINITION 1. 1. For an n-tuple $a=(a_1, \cdots, an)$ of elements in a unital C*-algebra, the joint numerical range $V(a)$ is defined by

$$V(a)=\{(s(a_1), \cdots, s(a_n)) \in C^n, s \in S_A\},$$

where S_A is the set of all states on A. We note that $V(a)$ is a compact convex subset of C^n.

LEMMA 1. 2. If P and Q are convex sets in a vector space, then

$$\mathrm{Co}(P \cup Q)=\bigcup_{0\le\lambda\le1} \lambda P+(1-\lambda)Q$$

This is (Theorem 1.25, p. 16 [8]).

LEMMA 1. 3. Let A and B be unital C*-algebras. Then a functional $F \in S_{A+B}$ if and only if it can be represented in the form

$$F(a, b)=\lambda f(a)+\mu g(b)$$

for all $(a, b) \in A+B$, where $f \in S_A$, $g \in S_B$ and $\lambda, \mu \geq 0$ with $\lambda+\mu=1$.

PROOF. Suppose $f \in S_A$, $g \in S_B$. If λ and μ are such that $\lambda, \mu \geq 0$, $\lambda+\mu=1$, then set

$$F(a, b)=\lambda f(a)+\mu g(b),$$

Clearly F is a linear functional on $A+B$ such that $F(e_1, e_2)=1$. Since $f(a^*a) \geq 0$, and $g(b^*b) \geq 0$,

$$\begin{aligned} F((a, b)^* (a, b)) &= F((a^*, b^*), (a, b)) \\ &= F(a^*a, b^*b) \\ &= \lambda f(a^*a)+\mu g(b^*b) \geq 0. \end{aligned}$$

Thus $F \in S_{A+B}$.

To prove the converse, first we shall observe that if D is a unital C*-algebra with unit element e and P is a linear functional on D such that $P(x^*, x) \geq 0$, $x \in D$, then

$$|P(x)| \leq P(e)\|x\|, \ x \in D \quad \text{(p. 40 [4] and Prop. 3. 3, p. 29 [9]).}$$

Now let $F \in S_{A+B}$. setting $h(a)=F(a, 0)$, $a \in A$, it follows that h is a linear functional on A such that $h(a^*a) \geq 0$, $a \in A$. Since e_1 is a unit element in A,

$$(1) \qquad |h(a)| \leq h(e_1)\|a\|, \ a \in A.$$

Analogously, if $K(b)=F(0, b)$, then K is a linear fuctional on B with $K(b^*b) \geq 0$, and

(2) $$|K(b)| \leq K(e_2) \|\|b\|, \quad b \in B.$$

Clearly

(3) $$F(a, b)=h(a)+K(b), \ (a, b) \in A+B$$

and

(4) $$1=F(e_1, e_2)=h(e_1)+K(e_2)$$

(i) If $h(e_1)=0$, then the inequality(l) implies that $h(a)=0$ for $a \in A$. From (4) it follows that $K(e_2)=1$ showing $K \in S_B$. Then from (3) $F(a, b)=K(b)$.

(ii) Similarly, if $K(e_2)=0$, $F(a, b)=h(a)$ with $h \in S_A$.

(iii) If $h(e_1)=\lambda \neq 0$ and $K(e_2)=\mu \neq 0$, then $\lambda+\mu=1$. By Setting $f(a)=(1/\lambda)h(a)$ and $g(b)=(1/\mu)K(b)$, we get $f \in S_A$, $g \in S_B$ and $F(a, b)=\lambda f(a)+\mu g(b)$ by (3). This completes the proof.

We now prove our main ɹesult.

THEOREM 1. 4. Let A and B be unital C*-algebras. If $a=(a_1, \cdots, a_n)$ and $b=(b_1, \cdots, b_n)$ are n-tuples of elements of A and B respectively, then

$$\begin{aligned} V(a+b) &= V((a_1, b_1), \cdots, (a_n, b_n)) \\ &= \{(s(a_1, b_1), \cdots, s(a_n{}^* b_n)) \in C^n : s \in S_{A+B}\} \\ &= \mathrm{Co}(V(a) \cup V(b)). \end{aligned}$$

PROOF: Suppose $\lambda \in \mathrm{Co}(V(a) \cup V(b))$. Then $\lambda=t\mu+(1-t)\nu$, $\mu=(\mu_1, \cdots, \mu_n) \in (V(a)$ and $\nu=(\nu_1, \cdots, \nu_n) \in V(b)$ and $0 \leq t \leq 1$ using Lemma 1. 2. Then $\mu_1=f(a_i)$, and $\nu_i=g(b_i)$ for some $f \in S_A$ and $g \in S_B$, $1 \leq i \leq n$. Since $f \in S_A$ and $g \in S_B$, by Lemma 1. 3, there exists $F \in S_{A+B}$ such that

$$F(a_i, b_i)=tf(a_i)+(1-t)g(b_i) \text{ for all } (a_i, b_i) \in A+B.$$

Now

$$\begin{aligned} t\mu+(1+t)\nu &= t(\mu_1, \cdots, \mu_n)+(1-t)(\nu_1, \cdots, \nu_n) \\ &= ((t\mu+(1-t)\nu_1), \cdots, ((t\mu_n+(1-t)\nu_n)) \\ &= (Fa_1, b_1), \cdots, F(a_n, b_n)) \in V(a+b) \end{aligned}$$

Hence

$$\mathrm{Co}(V(a) \cup V(b)) \subset V(a+b).$$

Conversely, suppose $\eta \in V(a+b)$. Then $\eta=\eta_1, \cdots, \eta_n$ with $\eta_i=F(a_i, b_i)$ for some $F \in S_{A+B}$, $1 \leq i \leq n$. Since $F \in S_{A+B}$, by Lemma 1. 3, we can find $f \in S_A$ and $g \in S_B$ and

λ, $\mu\geq 0$ with $\lambda+\mu=1$ such that

$$F(x, y)=\lambda f(x)+(1-\lambda)g(y)$$

for all $(x, y)\in A+B$. Therefore, in particular

$$F(a_i, b_i)=\lambda f(a_i)+(1-\lambda)g(b_i),\ (a_i, b_i)\in A+B.$$

$$\begin{aligned}\eta&=(\eta_1, \cdots, \eta_n)=(F(a_1, b_1), \cdots, F(a_n, b_n))\\ &=((\lambda f(a_1)+(1-\lambda)g(b_1), \cdots, (\lambda f(a_n)+(1-\lambda)g(b_n)))\\ &=\lambda(f(a_1), \cdots, f(a_n))+(1-\lambda)(g(b_1), \cdots, g(b_n))\in \mathrm{Co}(Va)\cup V(b)).\end{aligned}$$

Thus
$$V(a+b)=\mathrm{Co}(V(a)\cup V(b)).$$

2. Joint Normaloids

Definition 2.1. Let A be a C*-algebra with unit element e. Then for an n-tuple $a=(a_1, \cdots, a_n)$ of elements in A, the joint spectrum $\sigma(a)$ of a is defined by

$$\sigma(a)=\{(\lambda_1, \cdots, \lambda_n)\in C^n : \sum_{i=1}^{n}(a_i-\lambda_i)A\neq A \quad\text{or}\quad \sum_{i=1}^{n}A(a_i-\lambda_i)\neq A\}$$

$\sigma(a)\subset V(a)$ (Theorem 12, p. 24, [2], also see [5]).

The Cartesian product $A^n=A\times A\times\cdots\times$ (n times) becomes an algebra with involution if we define all the operations componentswise. In particular, if $a=(a_1, \cdots, a_n)$ and $b=(b_1, \cdots, b_n)$ are elements of A^n, we have

$$a^*=(a_1^*, \cdots, a_n^*),$$
$$ab=(a_1b_1, \cdots, a_nb_n)$$

and a norm is defined by

$$||a||=(\sum_{i=1}^{n}||a_i||^2)^{1/2}.$$

If $z=(z_1, \cdots, z_n)$ C^n, we set $|z|=(\sum_{i=1}^{m}|z_i|^2)^{1/2}$.

Definition 2.2. The joint numerical radius and joint spectral radius of $a\in A^n$ defined by

$$V(a)=\sup\{|\lambda| : \lambda\in V(a)\}$$

and

$$r(a)=\sup\{|\eta| : \eta\in\sigma(a)\{$$

respectively. It is easy to see that $r(a)\leq v(a)\leq||a||$.

Definition 2.3. The joint approximate spectrum $\pi(a)$ of $a=(a_1, \cdots, a_n)\in A^n$ is defined to be the set of all n-tuples of complex numbers $\lambda=(\lambda_1, \cdots, \lambda_n)$ such that there exists a sequence U_k of unit vectors in A satisfying

$$||(a_i-\lambda_i)U_k||\rightarrow 0 \text{ as } K\rightarrow\infty, \text{ for } i=1, 2, \cdots, n.$$

DEFINITION 2. 4. We say that $a=(a_1, \cdots, a_n)$ A^n is jointly normaloid if $r(a)=||a||$.

THEOREM 2. 5. The joint numerical radius has the following properties:

(i) $v(a)<\infty$, $v(a)\geq 0$ and $v(a)=0$ if and only if $a=0\in A^n$.

(ii) $v(\alpha a)=|\alpha|\, v(a)$ for all scalars α

(iii) $v(a+b)\leq v(a)+v(b)$ for all $a, b\in A^n$

(iv) $v(a)=v(a^*)$ for all $a\in A$.

Proof is easy, and hence omitted.

LEMMA 2. 6. Let $a=(a_1, \cdots, a_n)$ be n-tuple of elements in A. If $\lambda=(\lambda_1, \cdots, \lambda_n)\in V(a)$ with $|\lambda_i|=||a_i||$, $1\leq i\leq n$, then $\lambda\in\pi(a)$.

This is Theorem 3 of Mocanu [5].

In the following we prove the invalidity of the generalization of a well known characterisation that a single element $a\in A$ is normaloid if and only if $||a^k||=||a||^k$ for all positive integers k. For simplicity of exposition we shall consider the case $n=2$ and the general result follows on the similar lines.

THEOREM 2. 7. suppose $a=(a_1, a_2)\in A\times A$. if a is jointly normaloid, then $a_1^2=(a_2^2, a_2^2)$ is also jointly normaloid. If in addition a_1 and a_2 are non-zero, then $r(a^2)\neq r(a)^2$, that is $||a^2||\neq||a||^2$.

PROOF: Since a is jointly normaloid, we have $r(a)=||a||$. There exists $\lambda=(\lambda_1, \lambda_2)\in\sigma(a)$ such that $|\lambda|=r(a)$. Thus

$$|\lambda_1|^2+|\lambda_2|^2=||a_1||^2+||a_2||^2.$$

This shows that

$$||a_i||=|\lambda_i| \quad \text{for} \quad i=1, 2 \tag{5}$$

and hence $\lambda\in\pi(a)$ by Lemma 2. 6. Since $\lambda=(\lambda_1, \cdots, \lambda_n)\in\pi(a)$, there is a sequence $\{U_k\}$ of unit vectors in A such that

$$||(a_i-\lambda_i)U_k||\rightarrow 0 \quad \text{as } k\rightarrow\infty, \ i=1, 2.$$

From which it follows that

$$||(a_i^2-\lambda_1^2)\, U_k||\rightarrow 0 \quad \text{as } k\rightarrow\infty, \ i=1, 2.$$

Hence

$$\mu=(\lambda_1^2, \lambda_2^2)\in\pi(a^2).$$

Using (5) and the fact that a_i is a normaloid, we have

$$|\lambda_i^2|=|\lambda_i|^2=||a_i||^2=||a_i^2||$$

for each $i=1$, 2 and therefore

$$|\mu|=(|\lambda_1^2|^2+|\lambda_2^2|^2)^{1/2}$$
$$=(||a_1^2||^2+||a_2^2||^2)^{1/2}=||a^2||.$$

Hence $r(a^2)=||a^2||$ and a^2 is jointly normaloid. This proves the first part.

Now suppose $a=(a_1, a_2)$ is jointly normaloid and a_1, a_2 are both non-zero. If possible, suppose $r(a^2)=r(a)^2$, By the first part of the theorem a^2 is jointly normaloid, and hence $||a^2||=||a||^2$. This gives

$$||a_1^2||^2+||a_2^2||^2=(||a_1||^2+||a_2||^2)^2$$

That is,

$$(||a_1||^4-||a_1^2||^2)+(||a_2||^4-||a_2^2||^2)+2||a_1||^2||a_2||^2=0.$$

Since the left side of this equation is the sum of three nonnegative terms, we conclude that each term must be zero, in particular either $a_1=0$ or $a_2=0$. This is a contradiction.

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Institute of Mathematical Sciences
MADRAS-600 113, India.
Department of Mathematics
University of Mysore
MYSORE-570 006, India.