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ASYMPTOTIC POWERS OF ONE- AND TWO-SAMPLE RANK TESTS AGAINST LOCATION-ALTERNATIVES INCLUDING CONTAMINATION

By

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1. Introduction

For the nonparametric hypotheses of symmetry about zero and equality of distribution functions in one- and two-sample problems, we often consider only location alternatives. But after treatments are received, we cannot predict enough that many observations give rise to only variation of location of a distribution. In fact, Lehmann, in [2], considered three sorts of alternatives which are not the location alternatives and, in Section 7 A of Chapter 2 of [3], pointed out that the location alternative may be an oversimplication. So in this paper, we consider the alternative distribution of the form $(1-\varepsilon)F(x-\theta)+\varepsilon H(x-\theta)$ for the null distribution of the form $F(x)$ and discuss asymptotic powers of one- and two-sample rank tests under contiguous sequences of the above alternatives. When $F(x)$ and $H(x)$ are symmetric distributions about zero, θ is the mean and the median of $(1-\varepsilon)F(x-\theta)+\varepsilon H(x-\theta)$. Then it follows that we test whether the mean equals zero or not in the one-sample case and the difference of the two means of the two-sample case.

In Section 2, we shall state the one-sample case and will show that asymptotic relative efficiencies (ARE's) of signed rank tests with respect to the t-test are equivalent to the classical ARE-results against shift alternatives for the distribution $H(x)$ that is symmetric about zero and that ARE of the signed rank test based on normal scores with respect to the t-test is one for $F(x)=$normal irrespective of $H(x)$ and that the Wilcoxon signed rank test is asymptotically most powerful for $F(x)=$logistic and $H(x)=\{F(x)\}^2$. In Section 3, we shall give results of the two-sample case similar to some results obtained in Section 2 and will discuss asymptotic powers of k-sample rank tests additionally.

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2. One-sampe problem.

Let $X_1, \cdots, X_n$ be independent and identically distributed with continuous distribution function $F_1(x)$. We consider the testing problem of the null hypothesis $H_0: F_1(x)=F(x)$ versus the alternative $K: F_1(x)=(1-\varepsilon)F(x-\theta)+\varepsilon H(x-\theta)$, where $F(x)$ has density $f(x)$ that is symmetric about zero and $H(x)$ is absolutely continuous and $\varepsilon>0$ and $\theta\neq 0$. Further assume that there exists a distribution function $G(u)$ on [0, 1] such that $H(x-\theta)=G(F(x-\theta))$ fer all x. In order to derive asymptotic powers of signed rank tests, the student t-test and the most powerful test, we consider the following sequence of alternatives included in K which is converging to the null hypothesis H_0.

(2. 1) $$K_n: F_1(x)=(1-\lambda/\sqrt{n})F(x-\Delta/\sqrt{n})+(\lambda/\sqrt{n})G(F(x-\Delta/\sqrt{n}))$$

where $\lambda>0$ and $\Delta\neq 0$. The joint density functions of $(X_1, \cdots, X_n)$ under H_0 and under K_n are respectively given by

(2. 2) $$p_n(x)=\prod_{i=1}^{n} f(x_i) \text{ and}$$

(3. 3) $$q_n(x) = \prod_{i=1}^{n} \{(1-\lambda/\sqrt{n})f(x_i-\Delta/\sqrt{n})+(\lambda/\sqrt{n})g(F(x_i-\Delta/\sqrt{n}))f(x_i-\Delta/\sqrt{n})\}.$$

where $g(u)=G'(u)$.
We shall set Assumptions (1) and will prove the contiguity introduced by VI. 1 of Hájek and Šidák [1].

ASSUMPTIONS (1)

(i) $f(x)$ has positive and finite Fisher's information $I(f)=\int_{-\infty}^{\infty}\{f'(x)/f(x)\}^2 f(x)dx$.
(ii) $g(u)$ is bounded.
(iii) $\lim_{\theta\to 0}\int_{-\infty}^{\infty}[\{g(F(x-\theta))-g(F(x))\}/\theta]f(x)\,dx=-\int_{-\infty}^{\infty}g'(F(x))\{f(x)\}^2dx$.
(iv) $\int_{-\infty}^{\infty}g'(F(x))\{f(x)\}^2dx=-\int_{-\infty}^{\infty}g(F(x))f'(x)\,dx$.

Then lemma 2. 1 and the equation (2. 10) of Shiraishi [4] show the following Lemma 2. 1.

LEMMA 2. 1. Suppose that Assumptions (1) are satisfied. Then we get, as n tends to infinity, under H_0,

(2. 4) $$|\log\{q_n(X)/p_n(X)\}-(1/\sqrt{n})\sum_{i=1}^{n}[\lambda\{g(F(X_i))-1\}-\Delta f'(X_i)/f(X_i)] + (1/2)[\lambda^2 \operatorname{Var}\{g(F(X_1))\}+\Delta^2 I(f)]+\lambda\Delta E\{g'(F(X_1)f(X_1)\}| \xrightarrow[P]{} 0$$

where $p_n(x)$ and $q_n(x)$ are respectively defined by (2. 2) and (2. 3) and $\xrightarrow[P]{}$ denotes convergence in probability. Moreover the family of densities $\{q_n(x)\}$ is contiguous to $\{p_n(x)\}$.

Taking the absolute values of observations, Let R_i^+ be the rank of $|X_i|$ among the values $\{|X_i| : i=1, \cdots, n\}$ and let $a_n(\cdot)$ be a mapping from $\{1, \cdots, n\}$ to real values. Further define sign$(X)=1$ for $X>0$, 0 for $X=0$ and -1 otherwise. Then we can describe the signed rank test statistic as

$$S=\sum_{i=1}^{n}\{\text{sign}(X_i)\}\,a_n(R_i^+)/\sqrt{\sum_{k=1}^{n}\{a_n(k)\}^2}.$$

THEOREM 2. 2. Suppose that there exists square integrable function $\phi(u)$ such that $\int_0^1\{\varphi(u)\}^2du>0$ and

$$\lim_{n\to\infty}\int_0^1\{a_n(1+[un])-\varphi(u)\}^2\,du=0 \tag{2. 5}$$

with $[un]$ being the largest integer not exceeding un and that Assumptions (1) are satisfied. Then the signed rank test statistic S has asymptotically a normal distribution with mean ν and variance 1 under K_n as n tends to infinity, where

$$\nu=\int_{-\infty}^{\infty}\{\lambda g(F(t))f(t)-\Delta f'(t)\}\{\text{sign}(t)\}\varphi(2F(|t|)-1)\,dt.$$

PROOF. Set $T=(1/\sqrt{n})\sum_{i=1}^{n}\{\text{sign}(X_i)\}\varphi(2F(|X_i|)-1)$. Then from the central limit theorem, $((1/\sqrt{n})\sum_{i=1}^{n}[\lambda\{g(F(X_i))-1\}-\Delta f'(X_i)/f(X_i)],\ T)'$, under $\{p_n(x)\}$ has asymptotically a bivariate normal distribution with mean zero vector and convariance matrix

$$\Sigma=\begin{pmatrix}\sigma_0^2, & \sigma_{12}\\ \sigma_{12}, & \sigma_2^2\end{pmatrix} \tag{2. 6}$$

where

$$\sigma_0^2=\lambda^2\,\text{Var}\{g(F(X_1))\}+\Delta^2I(f)-2\lambda\Delta E\{g'(F(X_1))f(X_1)\}, \tag{2. 7}$$

$$\sigma_{12}=\int_{-\infty}^{\infty}\{\lambda g(F(x))f(x)-\Delta f'(x)\}\{\text{sign}(x)\}\,\varphi(2F(|x|)-1)\,dx \tag{2. 8}$$

and $\sigma_2^2=\int_{-\infty}^{\infty}\{\varphi(2F(|x|)-1)\}^2f(x)\,dx$. Also if we set $S_1=\sqrt{\sum_{k=1}^{n}\{a_n(k)\}^2/n}\cdot S$, by the proof of V. 1. 7 theorem of Hájek and Šidák [1], we get

$$S_1-T\xrightarrow[P]{}0 \tag{2. 9}$$

Hence combining (2. 9) with (2. 4), we get, under $\{p_n(x)\}$,

$$(\log\{q_n(X)/p_n(X)\},\ S_1)'\xrightarrow[L]{}N((-\sigma_0^2/2,\ 0)',\ \Sigma),$$

where $\xrightarrow[L]{}$ denotes convergence in law and σ_0^2 and Σ are respectively defind by

(2. 7) and (2. 6). From LeCam's third lemma stated in VI. 1. 4 lemma of Hájek and Šidák [1], $S_1 \xrightarrow[L]{} N(\sigma_{12}, \sigma_2^2)$. As V. 1. 7 theorem of Hájek and Šidák [1] shows $\lim_{n\to\infty}[n\int_0^1\{\varphi(u)\}^2 du/\sum_{k=1}^n \{a_n(k)\}^2=1$, we get the result.

Further we get the following theorem for the parametric t-test based on

$$U=\sqrt{n(n-1)}\bar{X}/\sqrt{\sum_{i=1}^n (X_i-\bar{X})^2}, \quad \text{where } \bar{X}=\sum_{i=1}^n X_i/n.$$

THEOREM 2. 3. Suppose that Assumptions (1) are satisfied. Then U has asymptotically a normal distribution with mean ν' and variance 1 under $\{q_n(x)\}$, where

$$\sigma^2=\int_{-\infty}^{\infty} x^2 f(x)\,dx \text{ and } \nu'=\int_{-\infty}^{\infty}\{\lambda g(F(x))f(x)-\Delta f'(x)\}\,x dx/\sigma.$$

PROOF. From the way similar to the proof of Theorem 2. 4, we find that $(\log\{q_n(X)/p_n(X)\},\ U)'$ has asymptotically a bivariate normal distribution with mean $(-\sigma_0^2/2,\ 0)'$ and covariance matrix $\begin{pmatrix}\sigma_0^2, & \nu' \\ \nu', & 1\end{pmatrix}$ under $\{p_n(x)\}$, where σ_0^2 is defined by (2. 7).
Hence LeCam's third lemma shows the result.

The square of the ratio of the asymptotic mean stated in Theorem 2. 2 to the one stated in Theorem 2. 3 gives the asymptotic relative efficiency of S with respect to U.

COROLLARY 2. 4. If the assumptions of Theorem 2. 2 are satisfied, the asymptotic relative efficiency of the signed rank test based on S with respect to the t-test based on U for H_0 versus K_n is given by

$$\text{ARE}(S, U)=\sigma^2\left[\int_{-\infty}^{\infty}\{\lambda g(F(x))f(x)-\Delta f'(x)\}\{\text{sign}(x)\}\varphi(2F(|x|)-1)\,dx\right]^2 \Big/\left(\left[\int_{-\infty}^{\infty}\{\lambda g(F(x))f(x)-\Delta f'(x)\}\,x dx\right]^2\int_0^1\{\varphi(u)\}^2 du\right).$$

Further if $g(F(x))f(x)$ is a symmetric density function about 0, we get

$$\text{ARE}(S, U)=\sigma^2\left[\int_{-\infty}^{\infty} f'(x)\{\text{sign}(x)\}\varphi(2F(|x|)-1)\,dx\right]^2 \Big/\left[\left\{\int_{-\infty}^{\infty} f'(x)\,x dx\right\}^2\int_0^1\{\varphi(u)\}^2 du\right].$$

$\varphi(u)$'s satisfying (2. 5) for $a_n(k)=k/(n+1)$ (Wilcoxon-type) and $E|Z_n|^{(k)}$ (normal scores) are respectively equal to u and $\Phi^{-1}(u/2+1/2)$, where $\Phi(x)$ is the standard normal distribution function.

If $F(x)$ is nomal and $a_n(k)$ is the normal scores function, we get ARE(S, U)

$=1$ without assuming that $g(F(x))f(x)$ is a symmetric function about 0. The values of ARE (S, U) are equal to the classical ARE-results against shift alternatives for density $g(F(x))f(x)$ which is symmetric about 0, for instance, ARE$(S, U)=3/\pi$ for $F(x)=$normal and $a_n(k)=k/(n+1)$.

COROLLARY 2. 5. Assume that $G(u)=u^2$ and $\lambda+\Delta>0$ and $F(x)$ is a logistic distribution function. Then the one-sided Wilcoxon signed rank test is asymptotically most powerful for H_0 versus K_n as n tends to infinity.

PROOF. The assumptions and (2. 4) show that

$$\log\{q_n(X)/p_n(X)\} \xrightarrow[L]{} N(-(\lambda+\Delta)^2/6,\ (\lambda+\Delta)^2/3) \text{ under } \{p_n(x)\}.$$

Using LeCam's third lemma, the asymptotic power of the most powerful test of level α based on $\log\{q_n(X)/p_n(X)\}$ is given by

$$1-\Phi(s_\alpha-(\lambda+\Delta)/\sqrt{3}), \tag{2. 10}$$

where s_α is the upper 100α percentage point of the standard normal distribution. Further from Theorem 2. 2, the asymptotic power of the Wilcoxon signed rank test is given by (2. 10). Hence the result follows.

Corollary 2. 5 shows that the Wilcoxon signed rank test is asymptotically admissible for logistic distribution $F(x)$ and unknown distributon $G(u)$.

3. Two-sample problem.

In this section, we shall give some results similar to the one-sample problem. Throughout this section, we shall use the same notations as in section 2. Let $X_{11}, \cdots, X_{1n_1}, X_{21}, \cdots, X_{2n_2}$ be independent random variables and suppose that the distribution of X_{ij} is given by a cotinuous distribution function $F_i(x)$ for $i=1, 2$ and $j=1, \cdots, n_i$. Let $N=n_1+n_2$ be the size of the pooled sample and consider the testing problem of the null hypothesis $H_0: F_1(x)=F_2(x)=F(x)$ versus the alternative $K: F_i(x)=(1-\varepsilon_i)F(x-\theta_i)+\varepsilon_i G(F(x-\theta_i))$ $(i=1, 2)$, where $\varepsilon_1, \varepsilon_2\geqq 0$ and $\theta_1\neq\theta_2$. In order to compare asymptotic powers of two-sample rank tests and a t-test, we consider the following sequence of alternatives included in K which is converging to the null hypothesis H_0.

$$K_N: F_i(x)=(1-\lambda_i/\sqrt{N})F(x-\Delta_i/\sqrt{N})+(\lambda_i/\sqrt{N})G(F(x-\Delta_i/\sqrt{N}))\ (i=1, 2), \tag{3. 1}$$

where $\lambda_1, \lambda_2\geqq 0$ and $\Delta_1\neq\Delta_2$.
The joint density functions of $(X_{11}, \cdots, X_{1n_1}, X_{21}, \cdots, X_{2n_2})$ under H_0 and under

K_N are respectively given by

$$p_N(x)=\prod_{i=1}^{2}\prod_{j=1}^{n_i}f(x_{ij}) \quad \text{and} \tag{3. 2}$$

$$\begin{aligned} q_N(x)=\prod_{i=1}^{2}\prod_{j=1}^{n_i}\{&(1-\lambda_i/\sqrt{N})f(x_{ik}-\Delta_i/\sqrt{N}) \\ &+(\lambda_i/\sqrt{N})\,g(F(x_{ij}-\Delta_i/\sqrt{N}))f(x_{ij}-\Delta_i/\sqrt{N})\}, \end{aligned} \tag{3. 3}$$

where $f(x)=F'(x)$ and $g(u)=G'(u)$.

We get Lemma 3. 1 from the argument similar to that of Lemma 2. 1.

LEMMA 3. 1. Assume that Assumptions (1) stated in Section 2 and $\lim_{N\to\infty}(n_1/N)=\alpha$ $(0<\alpha<1)$ and $E\{f'(X)/f(X)\}=0$ are satisfied, where random variable X has the distribution function $F(x)$. Then as N tends to infinity, we get

$$\begin{aligned} &|\log\{q_N(X)/p_N(X)\}-(1/\sqrt{N})\sum_{i=1}^{2}\sum_{j=1}^{n_i}[\lambda_i\{g(F(X_{ij}))-1\}-\Delta_i f'(X_{ij})/f(X_{ij})] \\ &+(1/2)\{\alpha\lambda_1^2+(1-\alpha)\lambda_2^2\}\operatorname{Var}\{g(F(X))\}+\{\alpha\lambda_1\Delta_1+(1-\alpha)\lambda_2\Delta_2\}E\{g'(F(X))f(X)\} \\ &+(1/2)\{\alpha\Delta_1^2+(1-\alpha)\Delta_2^2\}I(f)| \xrightarrow[P]{} 0, \end{aligned} \tag{3. 4}$$

where $p_N(x)$ and $q_N(x)$ are respectively defined by (3. 2) and (3. 3). Moreover the family of densities $\{q_N(x)\}$ is contiguous to $\{p_N(x)\}$.

Let R_{2j} be the rank of X_{2j} among the pooled sample $\{X_{ij}: i=1, 2, j=1, \cdots, n_i\}$ and let $b_N(\cdot)$ be a mapping from $\{1, \cdots, N\}$ to real values. Then the two-sample rank test statistic is given by

$$S=\{\sum_{j=1}^{n_2}b_N(R_{2j})-n_2\bar{b}_N\}\Big/\sqrt{[n_1n_2/\{N(N-1)\}]\sum_{k=1}^{N}\{b_N(k)-\bar{b}_N\}^2} \tag{3. 5}$$

where $\bar{b}_N=\sum_{k=1}^{N}b_N(k)/N$.

THEOREM 3. 2. Assume that, for some square integrable function $\phi(u)$ such that $\int_0^1\{\phi(u)-\int_0^1\phi(v)\,dv\}^2\,du>0$,

$$\lim_{N\to\infty}\int_0^1\{b_N(1+[uN])-\phi(u)\}^2\,du=0 \tag{3. 6}$$

holds. Further suppose that the assumptions of Lemma 3. 1 are satisfied. Then the two-sample rank statistic S has asymptotically a normal distribution with mean ν and variance 1 under $\{q_N(x)\}$, where randam variable X has the distribution function $F(x)$ and

$$\begin{aligned} \nu=\sqrt{\alpha(1-\alpha)}\operatorname{Cov}\{&\phi(F(X)), \\ &(\lambda_2-\lambda_1)\,g(F(X))-(\Delta_2-\Delta_1)f'(X)/f(X)\}/\sqrt{\operatorname{Var}\{\phi(F(X))\}}. \end{aligned} \tag{3. 7}$$

PROOF. If we set

$$T=\sqrt{N/(n_1n_2)}\,(-(n_2/N)\sum_{j=1}^{n_1}[\psi(F(X_{1j}))-E\{\psi(F(X))\}]$$
$$+(1-n_2/N)\sum_{j=1}^{n_2}[\psi(F(X_{2j}))-E\{\psi(F(X)\}])/\sqrt{\mathrm{Var}\{\psi(F(X))\}},$$

from V. 1. 5 theorem a and V. 1. 6 theorem a of Hájek and Šidák [1], we get $S\text{-}T \xrightarrow[P]{} 0$ under $\{p_N(x)\}$. Further it follows that $(\log\{q_N(X)/p_N(X)\},\ T)'$ has asymptotically a bivariate normal distribution with mean $(-\sigma_0^2,\ 0)'$ and covariance matrix $\begin{pmatrix}\sigma_0^2, & \nu\\ \nu, & 1\end{pmatrix}$ where,

$$\sigma_0^2=\{\alpha\lambda_1^2+(1-\alpha)\lambda_2^2\}\,\mathrm{Var}\{g(F(X))\}+\{\alpha\Delta_1^2+(-\alpha)\Delta_2^2\}\,I(f)$$
$$+2\{\alpha\lambda_1\Delta_1+(1-\alpha)\lambda_2\Delta_2\}\,E\{g'(F(X))f(X)\}. \tag{3. 8}$$

Hence LeCam's third lemma shows the result.

Next we derive the asymptotic distribution of the two-sample t-test based on

$$U=\sqrt{n_1n_2(N-1)/N}\,(\hat{X}_{2\cdot}-\hat{X}_{1\cdot})\Big/\sqrt{\sum_{i=1}^{2}\sum_{j=1}^{n_i}(X_{ij}-\hat{X}_{i\cdot})^2},\quad \text{where } \hat{X}_{i\cdot}=\sum_{j=1}^{n_i}X_{ij}/n_i\ (i=1,\ 2).$$

THEOREM 3. 3. If the assumptions of Lemma 3. 1 are satisfied, the two-sample t-test statistic U has asymptotically a normal distribution with mean ν' and variance 1 under $\{q_N(x)\}$, where

$$\nu'=\sqrt{\alpha(1-\alpha)/\mathrm{Var}(X)}\ \mathrm{Cov}\{X,\ (\lambda_2-\lambda_1)\,g(F(X))-(\Delta_2-\Delta_1)f'(X)/f(X)\}. \tag{3. 9}$$

PROOF. $(\log\{q_N(X)/p_N(X)\},\ U)'$ has asymptotically a bivariate normal distribution with mean $(-\sigma_0^2/2,\ 0)'$ and covariance matrix $\begin{pmatrix}\sigma_0^2, & \nu'\\ \nu', & 1\end{pmatrix}$ under $\{p_N(x)\}$. LeCam's third lemma shows the result.

The square of the ratio of the asymptotic mean (3. 7) stated in Theorem 3. 2 to (3. 9) in Theorem 3. 3 gives the asymptotic relative efficiency of the rank test with respect to the t-test.

COROLLARY 3. 4. If the assumptions of Theorem 3. 2 are satisfied, the asymptotic relative efficiency of the rank test based on S with respect to the two-sample t-test based on U for H_0 versus K_N is given by

$$\mathrm{ARE}(S,\ U)=\mathrm{Var}(X)[\mathrm{Cov}\{\psi(F(X)),\ (\lambda_2-\lambda_1)\,g(F(X))$$
$$-(\Delta_2-\Delta_1)f'(X)/f(X)\}]^2/(\mathrm{Var}\{\psi(F(X))\}$$
$$\times\mathrm{Cov}\{X,(\lambda_2-\lambda_1)\,g(F(X))-(\Delta_2-\Delta_1)f'(X)/f(X)\}]^2).$$

Further assume that

$$\mathrm{Cov}\{\psi(F(X)),\ g(F(X))\}=\mathrm{Cov}\{X,\ g\,(F(X))\}=0, \tag{3. 10}$$

then values of ARE(S, U) are equal to the classical ARE-results against shift alternatives.

$\psi(u)$'s satisfying (3. 6) for $b_N(k)=2k/(N+1)-1$ (Wilcoxon type), $E(Z^{(k)})$ (normal score) and sign$\{2k/(N+1)-1\}$ (sign score) are respectively equal to $2u-1$, $\Phi^{-1}(u)$ and sign$(2u-1)$. If $f(x)$ and $g(F(x))f(x)$ are symmetric about $E(X)$, it follows that (3. 9) is satisfied for these $\psi(u)$'s. Also we find that ARE$(S, U)=1$ for $b_N(k)=E(Z^{(k)})$ and $F(x)=$normal without assuming (3. 10). Though Corollary 2. 5 shows that the Wilcoxon signed rank test is asymptotically most powerful against a specified alternative, there exists no rank test that is asymptotically most powerful against a specified alternative for $\lambda_1 \neq \lambda_2$ in the two-sample problem.

Additionally we shall investigate the case of the k-sample problem. Let $\{X_{ij} : i=1, \cdots, k,\ j=1, \cdots, n_i\}$ be independent random variables and suppose that the distribution of X_{ij} is given by continuous distribution function $F_i(x)$ for $i=1, \cdots, k$ and $j=1, \cdots, n_i$. Let $N=\sum_{i=1}^{k} n_i$ and consider the testing problem H_0': $F_i(x)=F(x)$ $(i=1, \cdots, k)$. Further let a sequence of alternatives be defined by

$$K_N' : F_i(x)=(1-\lambda_i/\sqrt{N})F(x-\Delta_i/\sqrt{N})+(\lambda_i/\sqrt{N})G(F(x-\Delta_i/\sqrt{N})) \quad (i=1, \cdots, k)$$

where $\lambda_i \geqq 0$ and there exist i and j satisfying $\Delta_i \neq \Delta_j$. Setting $R_{ij}=$rank of X_{ij}, the Kruskal-Wallis type rank test statistic is expressed as

$$S'=\{(N-1)/\sum_{k=1}^{N}(b_N(k)-\bar{b})^2\}\sum_{i=1}^{k} n_i\{\bar{b}_N(R_{i\cdot})-\bar{b}\}^2,$$

where $$\bar{b}_N(R_{i\cdot})=\sum_{j=1}^{n_i} b_N(R_{ij})/n_i \text{ and } \bar{b}=\sum_{k=1}^{N} b_N(k)/N.$$

We give some results corresponding to Theorem 3. 2 and Corollary 3. 4 of the two-sample case.

THEOREM 3. 5. Suppose that $\lim_{N\to\infty}(n_i/N)=\alpha_i$ and $\alpha_i>0$ for $i=1, \cdots, k$, the assumption of Lemma 3. 1 and the assumption of Theorem 3. 2 for $b_N(\cdot)$ are satified. Then the Kruskal-Wallis type rank test statistic S' has asymptotically a noncentral χ-square distribution with $k-1$ degrees of freedom and noncentrality parameter

$$\sum_{i=1}^{k} \alpha_i[\mathrm{Cov}\{\psi(F(X)), (\lambda_i-\sum_{l=1}^{k}\alpha_l\lambda_l)g(F(X))-(\Delta_i-\sum_{l=1}^{k}\alpha_l\Delta_l)f'(X)/f(X)\}]^2/\mathrm{Var}\{\psi(F(X))\}$$

under K_N' as N tends to infinity, where the random variable X has the distribution function $F(x)$, and the asymptotic relative efficiency of the test based on S' with respect to the F-test is given by

$$\mathrm{ARE}(S', F\text{-test}) = \mathrm{Var}(X) \sum_{i=1}^{k} [\mathrm{Cov}\{\psi(F(X)),\ (\lambda_i - \sum_{l=1}^{k} \alpha_l \Delta_l)\, g(F(X))$$
$$-(\Delta_i - \sum_{l=1}^{k} \alpha_l \Delta_l) f'(X)/f(X)\}]^2 / (\mathrm{Var}\{\psi(F(X))\} \sum_{i=1}^{k} \alpha_i [\mathrm{Cov}\{X,\ (\lambda_i - \sum_{l=1}^{k} \alpha_l \lambda_l)\, g(F(X))$$
$$-(\Delta_i - \sum_{l=1}^{k} \alpha_l \Delta_l) f'(X)/f(X)\}]^2).$$

Further assume that $\mathrm{Cov}\{\psi(F(X)),\ g(F(X))\} = \mathrm{Cov}\{X,\ g(F(X))\} = 0$, that the values of $\mathrm{ARE}(S', F\text{-test})$ are equal to the classical ARE-results against shift alternatives.

If $b_N(k)$ is normal score and $F(x)$ is normal, $\mathrm{ARE}(S', F\text{-test}) = 1$ irrespective of $G(u)$.

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