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A REMARK ON HIGMAN'S RESULT ABOUT SEPARABLE ALGEBRAS

In memory of Professor Akira HATTORI

By

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1. Introduction.

Let k be a field with arbitarary characteristics and A a finite dimentional k-algebra. Then A is said to be a separable algebra if for any extension field E of k, $A\otimes_k E$ is a semisimple E-algebra. Let $a_1, \cdots, a_n$ be a k-basis of A and $T_{A|k}=T$ a trace form of A over k, that is, for $x\in A$, if $a_i x=\sum_j \alpha_{ij}a_j$ $(\alpha_{ij}\in k)$ then $T(x)=\Sigma\alpha_{ii}$.

It is well known that if A is a finite extension field of k, then A is a separable extension of k iff A is a separable algebra and iff T is nondegenerate (the bilinear form $A\times A\rightarrow k$ is defined by $(x, y)\longmapsto T(xy)$ as usual). But in case where an algebra A is not necessarily a field, the situation is more complicated.

Before stating Higman's result, let us fix the algebra A with several assumptions. Throughout this paper unless otherwise specified let k be a field with arbitrary characteristic and A a finite dimensional Frobenius k-algebra with nondegenerate associative bilinear from $\phi: A\times A\rightarrow k$ with fixed dual bases $\{a_i\}$, $\{b_i\}$ (i.e. $\phi(a_i, b_j)=\delta_{ij}$) $(i, j=1, \cdots, n)$.

Now Higman's result is as follows.

THEOREM 1 (Higman). *The following are equivalent.*

(i) *A is a separable k-algebra.*

(ii) *There exists a nondegenerate associative bilinear from $\phi: A\times A\rightarrow k$ with dual bases $\{a_i'\}$, $\{b_i'\}$ such that $\Sigma b_i' x a_i'=1$ for some $x\in A$.*

It should be noted that under the same situation of the theorem $\Sigma b_i' x a_i'$ is always in the center of A for any $x\in A$ but $\Sigma a_i' x b_i'$ is not (we shall give a simple example later). It seems natural to ask that what $\Sigma a_i' x b_i'=1$ does imply. To this question we obtained the following result.

THEOREM 2. *Let $\{a_i\}$, $\{b_i\}$ be dual bases with respect to ϕ. Then the following are equivalent.*

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(i) *A is a separable k-algebra.*
(ii) $\Sigma b_i c a_i = 1$ *for some unit* $c \in A$.
(iii) $\Sigma b_i c a_i$ *is a unit for some unit* $c \in A$.
(iv) $\Sigma b_i x a_i$ *is a unit for some* $x \in A$.
(v) $\Sigma a_i c b_i$ *is a unit for some unit* $c \in A$.
(iv) $\Sigma a_i x b_i$ *is a unit for some* $x \in A$.

Each condition in the above theorem is not equivalent to the condition "$\Sigma a_i x b_i = 1$ for some $x \in A$ (or a unit $x \in A$)" (see Example). Conditions (ii) to (iv) are known, but conditions (v) and (vi) are new, which we want to remark in this paper.

For the next theorem the equivalence of (i) and (iii) are well known.

THEOREM 3. *The following are equivalent.*
(i) *The trace form* $T_{A/k}$ *is nondegenerate,*
(ii) $\Sigma a_i b_i$ *is a unit.*
(iii) *There exists an extention field* E *of* k *such that* $A \otimes_k E \equiv \oplus_i M_{n_i}(E)$ *such that* $n_i \neq 0$ *in* k *for each* i, *where* $M_{n_i}(E)$ *is a full matrix algebra of degree* n_i *over* E.

2. Proof of Theorems.

To prove Theorem 2 we need several lemmas. The methods of proofs of next two lemmas are the same as in [1, section 71], so the proofs are omitted.

LEMMA 1. *Let* U_A *and* ${}_A V$ *be right and left A-modules with finite dimentions over* k, *respectively. Let* $\psi : U \times V \rightarrow k$ *be a nondegenerate associative bilinear from and* $\psi' : U \times V \rightarrow k$ *any map. Then* ψ' *is a nondegenerate asociative bilinear form if and only if there exist automorphisms* F *of* U_A *and* G *of* ${}_A V$ *such that the diagram*

$$\begin{tikzcd} U \times V \arrow[rr, "F \times G"] \arrow[dr, "\psi'"'] & & U \times V \arrow[dl, "\psi"] \\ & k & \end{tikzcd}$$

is commutative.

COROLLARY. *Let* $\psi : A \times A \rightarrow k$ *be a nondegenerate associative bilinear form. Then there exist units* $c_1, c_2 \in A$ *such that* $\{c_1 a_i\}$, $\{b_i\}$ *and* $\{a_i\}$, $\{b_i c_2\}$ *are dual bases with respect to* ψ.

LEMMA 2. *Let* U, V *and* $\psi : U \times V \rightarrow k$ *be as in Lemma 1. Let* $\{u_i\}$, $\{v_i\}$ *and*

$\{u_i'\}$, $\{v_i'\}$ be two pairs of dual bases. Then for any $a\in A$, $\Sigma u_i a\otimes v_i=\Sigma u_i' a\otimes v_i'$ in $U\otimes_k V$.

COROLLARY. *Let e and f be idempotents in A. Suppose there exists a nondegenerate associative bilinear form $\phi: eA\times Af\to k$ with dual bases $\{c_i\}$, $\{d_i\}$ and $\{c_i'\}$, $\{d_i'\}$. Then for any $a\in A$, $\Sigma c_i a d_i=\Sigma c_i' a d_i'$ holds.*

LEMMA 3. *Let B be a finite dimensional simple k-algebra. If B is separable then there exists a nondegenerate associative bilinear form $\phi: B\times B\to k$ with dual bases $\{c_i\}$, $\{d_i\}$ such that $\Sigma d_i c_i=1$.*

PROOF. We may assume $B=M_n(D)$ with D a division algebra. For any extension field E of k, $M_n(D)\otimes_k E\cong M_n(D\otimes_k E)$ is a semisimple E-algebra by assumption. Hence $D\otimes_k E$ is also a semisimple E-algebra by Morita theorem. This prove that D is a separable k-algebra. Then by Theorem 1 and Lemma 1, there exists a nondegenerate associative bilinear form $\phi_0: D\times D\to k$ with dual bases $\{g_i\}$, $\{h_i\}$ such that $\Sigma h_i g_i=1$. Let us define

$$\phi_1: M_n(D)\times M_n(D)\to k \text{ via } \phi_1(\alpha\beta)=\sum_{i,j}\phi_0(\alpha_{ij},\ \beta_{ji})$$
$$\text{for } \alpha=(\alpha_{ij}) \text{ and } \beta=(\beta_{ij})$$

Then by a routine verification we know that ϕ_1 is a nondegenerate associative bilinear form. Let $e_{ij}^k=(\alpha_{lm})$ and $f_{ji}^k=(\beta_{lm})$ be such that $\alpha_{ij}=g_k$, $\beta_{ji}=h_k$, $\alpha_{lm}=0$ for $(l,\ m)\neq(i,\ j)$ and $\beta_{lm}=0$ for $(l,\ m)\neq(j,\ i)$. Then

$$\begin{aligned}\phi_1(e_{ij}^k,\ f_{rs}^t)&=\sum_{lm}\phi_0(\alpha_{lm},\ \beta_{ml})\\&=\phi_0(\alpha_{ij},\ \beta_{ji})+\phi_0(\alpha_{sr},\ \beta_{rs})\\&=\delta_{is}\delta_{jr}\phi_0(g_k,\ h_t)\\&=\delta_{is}\delta_{jr}\delta_{kt}.\end{aligned}$$

Thus $\{e_{ij}^k\}$, $\{f_{ji}^k\}$ are dual bases. Moreover

$$\begin{aligned}\sum_{i,j,k}f_{ij}^k e_{ji}^k&=\sum_{i,j,k}(h_k e_{ij})(g_k e_{ji})\quad (e_{ij}=\text{matrix unit})\\&=\sum_{i,j}(\sum_k h_k g_k)e_{ii}\\&=\sum_{i,j}e_{ii}\\&=nI\quad (I=\text{identity}).\end{aligned}$$

Let $c=(c_{ij})\in M_n(D)$ be nonsingular such that $c_{11}=1$ and $e_{ii}=0$ for $i\geq 2$. Thus by Lemma 1, $\{e_{ij}^k\}$, $\{f_{ji}^k c\}$ are dual bases of some nondegenerate associative bilinear form $\phi: M_n(D)\times M_n(D)\to k$. In this case $\sum_{i,j,k}f_{ji}^k c e_{ij}^k=I$ holds.

LEMMA 4. *If A is local but not a division algebra, then $\Sigma a_i x b_i\in N$ for all*

$x \in A$, where N is the Jacobson radical of A.

PROOF. Let $\theta: A_A \to \hat{A}_A$ ($\hat{A} = \mathrm{Hom}_k(A, k)$) be the isomorphism induced from ϕ. We can obtain a k-basis $\{b_i'\}$ of A as a continuation of a k-basis of N. Let $\{a_i'\}$ be the dual basis of $\{b_i'\}$ with respct to ϕ. Then $\theta(a_i') = \eta_{b_i'}$. where $\eta_{b_i'}(b_j') = \delta_{ij}$. By the corollary of Lemma 2, $\Sigma a_i x b_i = \Sigma a_i' x b_i'$ for all $x \in A$. We show that if a_i' is a unit then $b_i' \in N$. Suppose b_i' is a unit. Since $\theta(a_i') = \theta(1) a_i' = \eta_{b_i'}$, $\theta(1) = \eta_{b_i'} a_i'^{-1}$. If $0 \neq \alpha \in N$ then $\theta(\alpha) = \theta(1)\alpha = \eta_{b_i'} a_i'^{-1} \alpha$. Hence for all $a \in A$, $(\eta_{b_i'} a_i'^{-1} \alpha)(a) = \eta_{b_i'}(a_i'^{-1} \alpha a) = 0$ since $a_i'^{-1} \alpha a \in N$ and b_i' is a unit (hence $a_i'^{-1} \alpha a$ is expressed as a linear combination of $b_j' \in N$). Therefore $\eta_{b_i'} a_i'^{-1} \alpha = 0$. This prove $\theta(\alpha) = 0$. But θ is an isomorphism, which is a contradiction.

Now we can prove our result.

PROOF of THEOREM 2.

As we mentioned before, it is known that conditions (i) to (iv) are equivalent each other. (i)$\Rightarrow$(v) is clear since A is a symmetric algebra. (v)$\Longrightarrow$(vi) is also clear. So we only need to prove (vi)$\Longrightarrow$(i). Let $1 = e_1 + \cdots + e_n$ be a decomposition of 1 to orthogonal primitive idempotents. Then $A = e_1 A \oplus \cdots \oplus e_n A \cong \widehat{Ae_1} \oplus \cdots \oplus \widehat{Ae_n}$. Hence there is a permutation σ of $\{1, 2, \cdots, n\}$ such that $e_1 A \cong \widehat{Ae_{\sigma(i)}}$. Note that $e_i A \cong e_j A$ iff $Ae_i \cong Ae_j$. Suppose $eA \cong_{\sigma(i)} A$. Then $Ae_{\sigma^2(i)} \cong \widehat{e_{\sigma(i)} A} \cong \widehat{e_i A} \cong Ae_{\sigma(i)}$. Hence $e_{\sigma(i)} A \cong e_{\sigma^2(i)} A$. By the same argument we obtain $e_i A \cong e_{\sigma^r(i)} A$ for all integers r. Let s be the smallest positive integer such that $\sigma^s(i) = i$. Let $\tau = (i\, \sigma(i) \cdots \sigma^{s-1}(i))$ be a cyclic permutation. Then $\sigma\tau^{-1}$ fixes $\sigma^r(i)$ for $r = 0, 1, \cdots, s-1$. It is clear that $\sigma\tau^{-1}$ plays the same role as σ. So we may assume that $e_i A \simeq e_{\sigma(i)} A$ iff $\sigma(i) = i$. Let $\phi_i: e_i A \times Ae_{\sigma(i)} \to k$ be the nondegenerate associative bilinear form induced from $e_i A \cong \widehat{Ae_{\sigma(i)}}$. Let $\psi: A \times A \to k$ be such that $\psi(a, b) = \sum_i \phi_i(e_i a, be_{\sigma(i)})$ for $a, b \in A$. Then it is easily verified that ψ is a nondegenerate associative bilinear form. Let $\{a_{ij} | j\}$, $\{b_{j\sigma(i)} | j\}$ be dual bases with respect to ϕ_i. Then $\{a_{ij} | i, j\}$, $\{b_{j\sigma(i)} | i, j\}$ form dual bases with respect to ψ. By assumption $\sum_i a_i x b_i$ is a unit in A. There exists a unit $c \in A$ such that $\{a_i\}$, $\{b_i c\}$ are dual bases with respect to ψ. Now $u = \sum_i a_i x b_i c = \sum_{j,k} a_{jk} x b_{k\sigma(j)}$ is a unit. Note that if $\sigma(i) = i$ then $\sum_j a_{ij} x b_{j\sigma(i)} = u_i$ may be a unit in $e_i A e_i$ and if $\sigma(i) \neq i$ then $\sum_j a_{ij} x b_{j\sigma(i)} = r_i \in N$ since $e_i A \not\cong e_{\sigma(i)} A$ implies $a_{ij} x b_{j\sigma(i)} \in e_i A e_{\sigma(i)} = e_i N e_{\sigma(i)}$ After renumbering we may assume $u = u_1 + \cdots + u_t + r_{t+1} + \cdots + r_n$. Then $u_1 + \cdots + u_t = u - (r_{t+1} + \cdots$ i. $+ r_n)$ is a unit in A since $r_{t+1} + \cdots + r_n \in N$. Thus $t = n$ and this implies $e_i A \cong \widehat{Ae_i}$ for all i. Then it is well known that $e_i A$ had the same top and socle. Let ϕ_{ij}: $e_i A e_j \times e_j A e_i \to k$ be the restriction of ϕ_i. Then ϕ_{ij} is a nondegenerate associative bilinear form over the algebra e_jAe_j. Now let $\{a^k_{ij}|k\}$, $\{b^k_{ji}|k\}$ be dual bases with respect to ϕ_{ij}. Then $\{a^k_{ij}|j, k\}$, $\{b^k_{ji}|j, k\}$ are dual bases with respect to ϕ_i since $\phi_i(a^k_{ij}, b^m_{li})=\delta_{jl}\phi_{ij}(a^k_{ij}, b^m_{li})=\delta_{jl}\delta_{km}$. Since $\sum_{j,k} a^k_{ij}xb^k_{ji}$ is a unit in e_iAe_i, there is an index j such that $\sum_k a^k_{ij}xb^k_{ij}$ is a unit in e_iAe_i. Let $f=e_jfe_i$, $g=e_ige_j$ be such that $fg=e_j$, $gf=e_i$. Let us define $\psi_{ij}: e_iAe_i\times e_iAe_i\to k$ as $\psi_{ij}(a, b)=\phi_{ij}(ag, fb)$. Then it is easily verified that $\{a^k_{ij}f\}$, $\{gb^k_{ji}\}$ form dual bases with respect to ψ_{ij}. Then $\sum_k (a^k_{ij}f)(gxf)(gb^k_{ji})=\sum_k a^k_{ij}xb^k_{ji}$, which is a unit in e_iAe_i. Hence by Lemma 4, e_iAe_i is a division algebra. Since the top and the socle of e_iA concide, e_iA has to be simple. This proves that A is semisimple. For any extension field E of k, $A\otimes_k E$ satisfies the condition (vi). This proves that A is a separable algebra. This completes the proof of the theorem.

Next let us give a proof of Theorem 3.

PROOF of THEOREM 3.

Since the equivalence between (i) and (iii) is well known, we only show the equivalence of (i) and (ii). If T is nondegenerate then A is a separable algebra. Hence in particular A is a direct sum of simple algebras. Thus the proof is complete by the next lemma.

LEMMA 5. *The following are equivalent if A is simple.*

(i) $T_{A/k}$ *is nondegenerate.*

(ii) $\Sigma a_i b_i$ *is a unit.*

(iii) $\Sigma a_i b_i \neq 0$.

PROOF. (i)$\Longrightarrow$(ii). Let $\{a'_i\}$, $\{b'_i\}$ be dual bases with respect to T. Let $b'_i a'_j =\sum_l \lambda_{il} b'_l$ $(\lambda_{il}\in k)$. For a fixed a'_j,

$$\begin{aligned}(1-\textstyle\sum a'_i b'_i)a'_j &= a'_j-\sum_i a'_i(b'_i a'_j)\\ &= a'_j-\sum_i a'_i(\sum_l \lambda_{il} b'_l)\\ &= a'_j-\sum_{i,l}\lambda_{il} a'_i b'_l.\end{aligned}$$

Thus

$$\begin{aligned}T((1-\textstyle\sum a'_i b'_i)a'_j) &= T(a'_j)-\sum_{i,l}\lambda_{il} T(a'_i b'_l)\\ &= T(a'_j)-\sum_i \lambda_{ii}\\ &= T(a'_j)-T(a'_j)\\ &=0.\end{aligned}$$

This holds for all a'_j. Therefore $\sum a'_i b'_i=1$. Now by the corollary of Lemma 1, $\sum a'_i b'_i=1=\sum a_i b_i c$ for some unit $c\in A$. This proves $\sum a_i b_i$ is a unit.

(ii)$\Longrightarrow$(iii). Clear.

(iii)$\Longrightarrow$(i). It always holds that $\phi(\sum a_i b_i, a_j) = T(a_j)$ for all j. Hence there exists j such that $T(a_j) \neq 0$. If $0 \neq a \in A$ and $T(aA)=0$, then $0=T(aA)=T(AaA)=T(A)$ since A is simple. This is a contradiction. Therefore T is nondegenerate.

Finally we give an example as we mentioned in the introduction.

EXAMPLE. Let char $k=2$ and $A=M_2(k)$. Let $P=\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$. Then $P^{-1}=\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$. Let ϕ and ψ be bilinear forms such that $\phi(e_{ij}, e_{kl})=\delta_{il}\delta_{jk}$ and $\psi(a, b)=\phi(a, bP^{-1})$ for $a, b \in A$, where e_{ij}, e_{kl} are matrix units in A. Then $\{e_{ij}\}$, $\{e_{ji}P\}$ are dual bases with respect to ψ. For $x \in A$, if $x=\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ then $\sum_{i,j} e_{ij} x e_{jj} P=\begin{pmatrix} a+d & a+d \\ a+d & 0 \end{pmatrix}$. Therefore there is no possibility for $\sum_{i,j} e_{ij} x e_{ji} P$ to become a central unit.

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