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CARDINAL FUNCTIONS OF SPACES WITH ORTHO-BASES

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§ 1. Introduction.

Throughout this paper, "space" will mean T_1-space. Let $\mathcal{B}$ be a base of a space X. $\mathcal{B}$ is said to be an ortho-base if for every $\mathcal{B}'\subset\mathcal{B}$, $\cap\mathcal{B}'$ is open or $\mathcal{B}'$ is a neighborhood base of some point. $\mathcal{B}$ is said to have subinfinite rank if for every $\mathcal{B}'\subset\mathcal{B}$ such that $\cap\mathcal{B}'\neq\phi$ and $\mathcal{B}'$ is infinite, at least two elements of $\mathcal{B}'$ are related by set inclusion. Spaces having an ortho-base, and spaces having a base of subinfinite rank were introduced by Nyikos as natural generalizations of non-archimedean spaces [4] [5].

Concerning cardinal functions of spaces with special bases, Gruenhage showed that for each regular space X having a base of subinfinite rank, $d(X)=hd(X)\geq hl(X)=s(X)$ holds [3]. $d(X)$ is the density of X, $hd(X)$ is the hereditary density, $hl(X)$ is the hereditary Lindelöf degree, and $s(X)$ is the spread (i. e., the supremum of the discrete subspaces of X). In this paper we investigate cardinal functions of spaces having ortho-bases. We shall show that $hd(X)\geq hl(X)=s(X)$ holds for each space X having an ortho-base.

§ 2. Main result.

We need two lemmas. For convenience, for a cardinal τ, we say a space X to be τ-developable if there exist τ open covers $\{\mathcal{H}_\alpha\}_{\alpha<\tau}$ such that for each $x\in X$ $\{\mathrm{St}(x,\mathcal{H}_\alpha)\}_{\alpha<\tau}$ is a neighborhood base of x.

LEMMA 1. *Let X be a space having an ortho-base $\mathcal{B}$ and D be the set of isolated points of X. If D is dence in X, then X is $|D|$-developable.*

PROOF. Set $D=\{d_\alpha|\alpha<\tau\}$, where τ is a cardinal. For each $x\in X-D$ and $\alpha<\tau$, we take $B_\alpha(x)\in\mathcal{B}$ such that $x\in B_\alpha(x)$ and $d_\alpha\notin B_\alpha(x)$. Put $\mathcal{H}_\alpha=\{\{d_\alpha\}|\alpha<\tau\}\cup\{B_\alpha(x)|x\in X-D\}$. $\mathcal{H}_\alpha$ is obviously an open cover of X. Let x be a point of X and W be a neighborhood of x. If $x\in D$, then $\mathrm{St}(x,\mathcal{H}_\alpha)=\{x\}\subset W$ for some α. So, we assume $x\in X-D$. Suppose that $\mathrm{St}(x,\mathcal{H}_\alpha)\not\subset W$ for any $\alpha<\tau$. Then for each α, we can take $H_\alpha\in\mathcal{H}_\alpha$ such that $x\in H_\alpha$ and $H_\alpha\not\subset W$. Since $\{H_\alpha\}_{\alpha<\tau}$ can not be a neigh-

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borhood base of x, $H=\bigcap_{\alpha<\tau} H_\alpha$ must be open. But $H\cap D=\phi$, because $H_\alpha \not\ni d_\alpha$. Since D is dense in X, this is a contradiction.

The following lemma is well known in the countable case and can be easily carried over to the general case. So we omit the proof.

LEMMA 2. *Let X be τ-developable. If the cardinality of each closed discrete subspace is at most τ, then X is τ-Lindelöf (i.e., every open cover has a subcover of the cardinality τ).*

THEOREM 3. *Let X be a space having an ortho-base $\mathcal{B}$. Then $hd(X)\geq s(X)=hl(X)$ holds.*

PROOF. Since $hd(X)\geq s(X)$ and $hl(X)\geq s(X)$ are obvious, we show $s(X)\geq hl(X)$. Let $s(X)=\tau$. Since for each subspace Y of X, $s(Y)\leq\tau$ and Y has an ortho-base, the proof is complete if we show that X is τ-Lindelöf. Suppose that there exists an open cover $\mathcal{U}$ of X which has not a subcover of the cardinality τ. Firstly we take $x_0\in X$ and $U_0\in\mathcal{U}$ such that $x_0\in U_0$. Put $V_0=U_0$. Let $\gamma<\tau^+$. We assume that for each $\beta<\gamma$ we could take $x_\beta\in X$ and an open set V_β such that the following (*) is satisfied.

$$(*)\qquad \begin{cases} V_\beta\cap\{x_\alpha|\alpha<\gamma\}=\{x_\beta\} & \text{for each } \beta<\gamma. \\ \text{There exists } U_\beta\in\mathcal{U} \text{ such that } V_\beta\subset U_\beta \text{ for each } \beta<\gamma. \end{cases}$$

Then, if we set $A=\{x_\alpha|\alpha<\gamma\}$, since $|A|\leq\tau$, $\mathrm{Cl}\,A$ is τ-Lindelöf by Lemma 1 and Lemma 2. Thus $\mathrm{Cl}\,A\cup(\bigcup_{\beta<\gamma}V_\beta)$ is covered by τ elements of $\mathcal{U}$. So we can take $x_\gamma\in X-\mathrm{Cl}\,A\cup(\bigcup_{\beta<\gamma}V_\beta)$. We take $U_\gamma\in\mathcal{U}$ and an open set V_γ such that $x_\gamma\in V_\gamma\subset U_\gamma$ and $V_\gamma\cap A=\phi$. Now by the induction we get the discrete space $\{x_\alpha|\alpha<\tau^+\}$. This is a contradiction to $s(X)=\tau$.

There exists a space having an ortho-base such that $hd(X)\neq d(X)$. In fact the space in [6, 3.6.I] is such a space.

Concerning SH (Souslin's hypothesis), we note the following theorem.

THEOREM 4. *The following (a), (b) and (c) are equivalent.*

(a) SH is false.

(b) There exists a non-metrizable non-archimedean space such that $s(X)$ is countable.

(c) There exists a non-metrizable regular space having an ortho-base such that $s(X)$ is countable.

PROOF. The equivalence of (a) and (b) is due to [1]. Also, refer [5, Theorem 1.7]. (b)$\rightarrow$(c) is trivial. We show (c)$\rightarrow$(b). Let X be a space of (c). Since by Theorem 3 X is regular Lindelöf, it is paracompact. Therefore X is a proto-metrizable space (i. e., paracompact space having an ortho-base). It follows from Fuller's result [2, Theorem 6] that X is the perfect irreducible image of a non-archimedean space Y. Since metrizability is an invariant of perfect maps, Y is not metrizable. Since the spread of a non-archimedean space is equal to the cellularity, by the irreducibility of the map, $s(Y)$ must be countable. Thus Y is the desired space.

COROLLARY 5. *The following (a) and (b) are equivalent.*

(*a*) *SH.*

(*b*) *Each regular space having an ortho-base is metrizable if the spread is countable.*

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