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SCALENE METRIC SPACES

By

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Abstract. In this paper, we introduce the notion of scalene metric and study it. In particular, we prove that a compactum with scalene metric is an AR and a locally compact space with locally scalene metric is an ANR. Also, we show that scalene metric subsets of a metric space play important roles as convex subsets of a Banach space in some selection theorems, and the notion of scalene metric gives another aspect which differs from that of E. Michael with respect to the constructions of selections ([6], [7], [8] and [9]).

0. Introduction.

A compactum is a compact metric space and a connected compactum is a continuum. It is well-known that a continuum is locally connected if and only if it has a convex metric. Natually, the following problem is raised: Is there a metric characterization of an absolute retract, i.e., AR or absolute neighborhood retract, i.e., ANR? In this paper, we consider the following problem: Which metric implies AR or ANR?

A metric ρ on a space X is said to be a *scalene metric* provided that if a, b are different points of X, then there is a point c of X such that for each $x\in X$, either $\rho(x, a)>\rho(x, c)$ or $\rho(x, b)>\rho(x, c)$ holds. Scalene metric spaces are generalization of convex subsets in the Hilbert space l_2. In fact, take two points a, b of a convex subset X in l_2. Choose a point $c\in\{x|x=(1-t)a+tb, 0<t<1\}\subset X$. Clearly, c satisfies the desired property. A metric ρ on a space X is said to be a *locally scalene metric* provided that for each point x of X there is a neighborhood U of x in X such that the restriction ρ_U of ρ to U is a scalene metric.

We study some properties of scalene metrics and locally scalene metrics. In particular, we prove that if a compactum has a scalene metric, then it is an AR. Moreover, if a locally compact space has a locally scalene metric, then it is an ANR. But the converse assertions are not true. Also, by using the notion of scalene metric, we investigate some selection theorems from another aspect which differs from that of E. Michael ([6], [7], [8] and [9]).

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In [8, (8.1)], Michael proved the following theorem.

THEOREM (E. Michael). *Let X be a topological space, Y a Banach space, and $F(Y)$ the family of non-empty, closed, convex subsets of Y. If $\varphi: X \to F(Y)$ is continuous, then it admits a continuous selection.*

In the statement of the above theorem, we show that the family $F(Y)$ of non-empty, closed convex subsets of a Banach space Y is replaced by the family $S(Y)$ of non-empty, compact scalene metric subsets of a metric space Y.

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1. Notations and preliminaries.

Let X be a matric space with metric ρ. For any subsets A and B of X, let $\rho(A, B)=\inf\{\rho(a, b) | a \in A, b \in B\}$. Also, let $H_\rho(A, B)=\max\{\sup_{a\in A} \rho(a, B), \sup_{b\in B} \rho(b, A)\}$. H_ρ is called *the Hausdorff metric*. The hyperspace $2^X=\{A \subset X | A$ is non-empty and compact$\}$ is metrized with H_ρ. It is well-known that X is a locally connected continuum if and only if 2^X is an AR [10]. Moreover, it was proved that 2^X is homeomorphic to the Hilbert cube $Q=[-1, 1]^\infty$ [2]. Let $a, b \in X$. We define the set $I_\rho(a, b)$ as follows: If $a=b$, $I_\rho(a, b)=\{a\}$ and if $a \neq b$, $I_\rho(a, b)=\{c \in X | \max\{\rho(x, a), \rho(x, b)\} > \rho(x, c)$ for each $x \in X\}$. Note that ρ is a scalene metric if and only if for any $a, b \in X$ $I_\rho(a, b) \neq \phi$. A function $\varphi: Y \to 2^X$ is *lower semi-continuous* if $\{y \in Y | \varphi(y) \cap V \neq \phi\}$ is open in Y for each open subset V of X. A function $\varphi: Y \to 2^X$ is *upper semi-continuous* if $\{y \in Y | \varphi(y) \subset V\}$ is open in Y for each open subset V of X. A function $\varphi: Y \to 2^X$ is *continuous* if φ is lower semi-continuous and upper semi-continuous. A *continuous selection* for $\varphi: Y \to 2^X$ is a continuous function $s: Y \to X$ such that $s(y) \in \varphi(y)$ for each $y \in Y$.

2. Examples of scalene metric spaces.

In this section, we give several examples in order to clarify the definition of scalene metric.

(2.1) EXAMPLE. Let X be a dendrite, i.e., 1-dimensional compact AR, and let $p \in X$. For $y, z \in X$, let $y \leqq_p z$ mean y lies on the unique arc in X from p to z. Then $(X, \leqq_p)$ is a partially ordered space, and hence X has a metric d which is radially convex with respect to $\leqq_p$. Define a metric ρ on X as follows [3, (2.16)]: Let $y, z \in X$ and let $y \wedge z$ denote the last point with respect to $\leqq_p$ where the arc from p to y intersects the arc from p to z. Set

$$\rho(y, z)=d(y, y\wedge z)+d(y\wedge z, z).$$

Then ρ is a scalene metric.

(2.2) Example. Let P be a 1-dimensional locally finite polyhedron with triangulation T. For any points x, y of P which belong to 1-simplex $\langle v_0, v_1\rangle \in T$, define $d(x, y)=|t-t'|$, where $x=tv_0+(1-t)v_1$, $y=t'v_0+(1-t')v_1$. If points y, z belong to a component of P, we define

$$\rho(y, z)=\inf\left\{\sum_{i=0}^{m} d(x_i, x_{i+1})\,\middle|\, x_0=y,\ x_{m+1}=z,\ x_i\ (1\leqq i\leqq m) \text{ is a vertex of } T \text{ and each successive points } x_i,\ x_{i+1} \text{ belongs to 1-simplex of } T\right\}.$$

Otherwise we define $\rho(y, z)=1$. Then for each $x\in P$, ρ_B is a scalene metric, where $B=\{y\in P|\rho(x, y)\leqq 1/2\}$. Hence P has a locally scalene metric ρ.

(2.3) Example. A scalene metric is not always convex. Recall that a metric d on X is *convex* if for any two points x and y of X there is a point z of X such that $d(x, z)=d(z, y)=1/2\cdot d(x, y)$. In the plane E^2 with Euclidean metric ρ, consider the set $S=\{(x, y)\in E^2|x^2+y^2=1, y\geqq 0\}$. Clearly ρ_S is scalene but convex.

3. Compact scalene metric spaces are ARs.

In this section, we study scalene or locally scalene metric spaces. In particular, we prove that a compactum with scalene metric is an AR and a locally compact space with locally scalene metric is an ANR.

(3.1) Lemma. *Suppose that X has a scalene metric ρ. Let $a, b\in X$ and $a\neq b$. If $c\in I_\rho(a, b)$, then $\rho(a, b)>\rho(a, c)>0$ and $\rho(a, b)>\rho(b, c)>0$.*

This follows immediately from the definition of scalene metric.

(3.2) Lemma. *If X has a scalene metric ρ, then for each $x\in X$ and $t>0$ ρ_B is a scalene metric, where $B=\{y\in X|\rho(x, y)\leqq t\}$. Furthermore, for any points $a, b\in X$ ρ_A is scalene, where $A=\overline{I_\rho(a, b)}$.*

Proof. We shall prove that ρ_A is a scalene metric. Note that if $c\in A$, either $\rho(x, a)\geqq\rho(x, c)$ or $\rho(x, b)\geqq\rho(x, c)$ holds for each $x\in X$. Let $x_1, x_2\in A$ and $x_1\neq x_2$. Take a point $c\in I_\rho(x_1, x_2)$. Then we can easily see that $c\in I_\rho(a, b)\subset A$, which implies that ρ_A is scalene. Similarly, ρ_B is scalene.

(3.3) Proposition. *If a compactum X has a scalene metric ρ, then X is connected and locally connected.*

Proof. Suppose, on the contrary, that X is not connected. There exist two disjoint nonempty closed subsets A and B of X such that $X=A\cup B$. Since X is compact, we can choose two points $a\in A$ and $b\in B$ such that $\rho(A,B)=\rho(a,b)>0$. Since ρ is scalene, there is a point $c\in I_\rho(a,b)$. Assume that $c\in A$. By (3.1), we have $\rho(A,B)=\rho(a,b)>\rho(b,c)$, which is a contradiction. Hence X is connedted. Also, by (3.2) we can see that X is locally connected.

(3.4) PROPOSITION. *Suppose that a compactum X has a scalene metric. Let $a,b\in X$ and $a\neq b$. Then $\overline{I_\rho(a,b)}$ is a locally connected continuum containing a and b and* diam $\overline{I_\rho(a,b)}=\rho(a,b)$.

PROOF. By (3.2) and (3.3), $\overline{I_\rho(a,b)}$ is a locally connected continuum. We shall prove that $a,b\in\overline{I_\rho(a,b)}$. Suppose, on the contrary, that $a\notin\overline{I_\rho(a,b)}$. Since $I_\rho(a,b)$ is compact, there is a point $z\in\overline{I_\rho(a,b)}$ such that $\rho(a,\overline{I_\rho(a,b)})=\rho(a,z)$. Note that for each $x\in X$, either $\rho(x,a)\geqq\rho(x,z)$ or $\rho(x,b)\geqq\rho(x,z)$ holds. Choose a point $c\in I_\rho(a,z)$. Since $I_\rho(a,z)\subset I_\rho(a,b)$, by (3.1) we have $\rho(a,\overline{I_\rho(a,b)})=\rho(a,z)>\rho(a,c)$ and $c\in I_\rho(a,b)$. This is a contradiction. Similarly, $b\in\overline{I_\rho(a,b)}$. If $c_1,c_2\in I_\rho(a,b)$, then $\rho(c_2,a)>\rho(c_2,c_1)$ or $\rho(c_2,b)>\rho(c_2,c_1)$. By (3.1), $\rho(a,b)>\rho(c_2,c_1)$. Hence diam $\overline{I_\rho(a,b)}=\rho(a,b)$.

(3.5) PROPOSITION. *If a locally compact space X has a locally scalene metric, then X is locally connected.*

This follows from (3.2) and (3.3).

(3.6) EXAMPLE. In the statement of (3.3), we cannot omit the condition that X is compact. Consider the following set in the real line E:

$$X=\{x\in E\mid 0\leqq x\leqq 1\}-\{1/2\}.$$

Let ρ be the metric defined by $\rho(x,y)=|x-y|$ for $x,y\in X$. Then ρ is scalene and X is locally compact but not connected.

(3.7) EXAMPLE. In the statement of (3.5), we cannot omit the condition that X is locally compact. Consider the following set in E^n with Euclidean metric ρ:

$$X=\{(x_1,x_2,\cdots,x_n)\in E^n\mid \text{ each } x_i \text{ is a rational number}\}.$$

Then ρ_X is scalene but X is totally disconnected.

(3.8) EXAMPLE. The Euclidean metric ρ on E^n is scalene. Then if $a,b\in E^n$ and $a\neq b$,

$$I_\rho(a,b)=\{x\in E^n\mid x=ta+(1-t)b \text{ and } 0<t<1\}.$$

(3.9) EXAMPLE. Let S^2 be the unit sphere in E^3. Define the metric ρ on S^2 by

$$\rho(x, y)=\arccos\left(\sum_{i=1}^{3} x_i y_i\right) \qquad \text{for } x=(x_1, x_2, x_3),\ y=(y_1, y_2, y_3)\in S^2 .$$

Consider the following sets:

$$A=\{x\in S^2 \mid \rho(a, x)+\rho(x, b)=\rho(a, b)\},$$

where $a=\left(\frac{\sqrt{2}}{2}, 0, -\frac{\sqrt{2}}{2}\right)$ and $b=\left(0, \frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$.

$$X=\{x\in S^2 \mid \rho(z_0, x)+\rho(x, a')=\rho(z_0, a') \text{ for some } a'\in A\},$$

where $z_0=(0, 0, 1)$ (see Figure 1).

Then ρ_X is scalene and $\overline{I_\rho(a, b)}\cap A=\{a, b\}$. Note that $\overline{I_\rho(a, b)}$ is not an arc (see (4.1)).

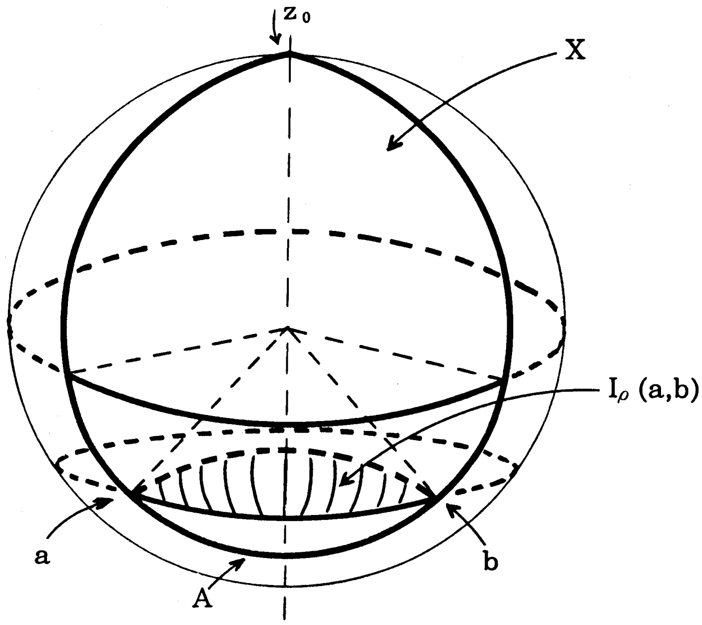

Fig. 1.

The main result of this section is the following theorem.

(3.10) THEOREM. *If a compactum X has a sclene metric ρ, then X is an AR.*

PROOF. By (3.3), X is a locally connected continuum. Hence by [10], 2^X is an AR. Let $A\in 2^X$. Define a function $f_A : X\to[0, \infty)$ by

(1) $f_A(x)=\sup\{\rho(x, a) \mid a\in A\}=H_\rho(\{x\}, A)$.

Clearly f_A is continuous. Consider the following:

(2) $m(A)=\inf\{f_A(x) \mid x\in X\}$ and

(3) $R(A)=\{x\in X \mid f_A(x)=m(A)\}$.

We shall prove that $R(A)$ consists of only one point. If $x_1, x_2 \in R(A)$ and $x_1 \neq x_2$, then there is a point $c \in I_\rho(x_1, x_2)$. Since A is compact, there is a point $a_0 \in A$ such that $f_A(c)=\rho(a_0, c)$. Then either $f_A(c)=\rho(a_0, c)<\rho(a_0, x_1) \leqq f_A(x_1)=m(A)$ or $f_A(c)=\rho(a_0, c)<\rho(a_0, x_2) \leqq f_A(x_2)=m(A)$ holds, which implies that $f_A(c)<m(A)$. This is a contradiction. Let us define a function $r: 2^X \to X$ by $\{r(A)\}=R(A)$ for each $A \in 2^X$. We must prove that r is continuous. Suppose, on the contrary, that there is a sequence $A, A_1, A_2, \cdots$, of points in 2^X such that

(4) $H_\rho(A_n, A)<1/n$ for each $n=1, 2, \cdots$, and

(5) $\lim_{n\to\infty} r(A_n) \neq r(A)$.

Let $a=\lim_{n\to\infty} r(A_n)$. Since $R(A)$ is a one point set, we see that $f_A(a)-f_A(r(A))=\varepsilon>0$. Since $\lim_{n\to\infty} f_{A_n}(r(A_n))=f_A(a)$, there is a natural number n_0 such that for each $n \geqq n_0$,

(6) $f_{A_n}(r(A_n))-f_A(r(A))>2\varepsilon/3$.

By (4), we can choose a natural number $n_1 \geqq n_0$ such that

(7) $H_\rho(A, A_n)<\varepsilon/3$ for each $n \geqq n_1$.

Then by (6) and (7), for each $n \geqq n_1$,

(8) $$\begin{aligned} f_{A_n}(r(A)) &\leqq f_A(r(A))+H_\rho(A, A_n)<f_{A_n}(r(A_n))-2\varepsilon/3+\varepsilon/3 \\ &=f_{A_n}(r(A_n))-\varepsilon/3<m(A_n). \end{aligned}$$

This is a contradiction. Hence $r: 2^X \to X$ is continuous. Note that $r(\{x\})=x$ for each $x \in X$. This implies that X is an AR. This completes the proof.

(3.11) THEOREM. *If a locally compact space X has a locally scalene metric, then X is an ANR. Moreover, each point of X has a compact neighborhood which is an AR.*

PROOF. It follows from (3.2) and (3.10) that for each $x \in X$ there is a compact AR V such that x is an interior point of V in X. Hence X is an ANR (e.g., see [1, p. 102]).

(3.12) REMARK. In the statement of (3.10), we cannot replace "compactum" by "locally compact space" (see (3.6)). Also, in the statement of (3.11), we cannot omit the condition that X is locally compact (see (3.7)).

(3.13) REMARK. There is a compact 2-dimensional AR X not admitting a locally scalene metric. In fact, the space X in [1, p. 155, (4.17)] is one of such compacta. It cannot be decomposed into a finite or countable number of compact ARs distinct from X. By (3.11), X does not admit a locally scalene metric.

4. Some properties of scalene metric spaces.

In this section, we study further properties of scalene metric spaces.

(4.1) PROPOSITION. *Let X be a compactum with metric ρ. If $a, b \in X$ and $a \neq b$, then $I_\rho(a, b)$ is open in X.*

PROOF. Let $c \in I_\rho(a, b)$. Since X is compact, there is a positive number ε such that $\max\{\rho(x, a), \rho(x, b)\} \geqq \rho(x, c)+\varepsilon$ for each $x \in X$. Set $U_\varepsilon(c)=\{x \in X \mid \rho(x, c)<\varepsilon\}$. If $y \in U_\varepsilon(c)$, then

$$\begin{aligned}\rho(x, y) \leqq \rho(x, c)+\rho(c, y) &\leqq \max\{\rho(x, a), \rho(x, b)\}-\varepsilon+\rho(c, y) \\ &< \max\{\rho(x, a), \rho(x, b)\}.\end{aligned}$$

Hence $U_\varepsilon(c) \subset I_\rho(a, b)$, which implies that $I_\rho(a, b)$ is open in X.

(4.2) LEMMA. *Let X be a compactum with scalene metric ρ. If $a, b \in X$, $a \neq b$ and ε is a positive number, then there is a point $c \in I_\rho(a, b)$ such that $\rho(a, c)<\rho(a, b)$ and $\rho(c, b)<\varepsilon$.*

PROOF. By (3.4), $b \in \overline{I_\rho(a, b)}$. Choose a point $c \in I_\rho(a, b)$ with $\rho(b, c)<\varepsilon$. Since $c \in I_\rho(a, b)$, $\rho(a, c)<\rho(a, b)$.

(4.3) LEMMA. *Let X be a compactum with scalene metric ρ. If $a, b \in X$, then*

$$\overline{I_\rho(a, b)}=\{c \in X \mid \max\{\rho(x, a), \rho(x, b)\} \geqq \rho(x, c) \text{ for each } x \in X\}.$$

PROOF. We may assume that $a \neq b$. It is easily seen that if $c \in \overline{I_\rho(a, b)}$, $\max\{\rho(x, a), \rho(x, b)\} \geqq \rho(x, c)$ for each $x \in X$. Conversely, let $c \in X$ such that $\max\{\rho(x, a), \rho(x, b)\} \geqq \rho(x, c)$ for each $x \in X$. Then $I_\rho(a, c) \subset I_\rho(a, b)$, and hence $c \in \overline{I_\rho(a, c)} \subset \overline{I_\rho(a, b)}$.

(4.4) PROPOSITION. *Let X be a compactum with scalene metric ρ. If $K_\rho: 2^X \times [0, \infty) \rightarrow 2^X$ is the function defined by*

$$K_\rho(A, t)=\{x \in X \mid \rho(x, A) \leqq t\} \quad \text{for } A \in 2^X \text{ and } t \in [0, \infty),$$

then K_ρ is continuous.

PROOF. Suppose that $A_1, A_2, \cdots$, is a sequence of closed subsets of X and $t_1, t_2, \cdots$, is a sequence of positive numbers such that $\lim A_n=A$ and $\lim t_n=t$. Then it is easily seen that

(1) $\limsup K_\rho(A_n, t_n) \subset K_\rho(A, t)$.

Let $y \in K_\rho(A, t)$ and $\varepsilon > 0$. Choose a point $a \in A$ such that $\rho(a, y) \leqq t$. By (4.2), there is a point $c \in X$ such that $\rho(y, c) < \varepsilon$ and $\rho(a, c) < \rho(a, y) \leqq t$. Choose a natural number n_0 such that if $n \geqq n_0$, there is a point $a_n \in A_n$ such that $\rho(a_n, a) < t_n - \rho(a, c)$. Then we have

(2) $\rho(a_n, c) \leqq \rho(a_n, a) + \rho(a, c) < t_n$.

Hence $c \in K_\rho(A_n, t_n)$ for each $n \geqq n_0$. This implies that

(3) $\liminf K_\rho(A_n, t_n) \supset K_\rho(A, t)$.

By (1) and (3), $\lim K_\rho(A_n, t_n) = K_\rho(A, t)$. Therefore K_ρ is continuous.

A metric ρ defined on a space X is *strongly convex* provided that for each $a, b \in X$, there is only one point $c \in X$ such that $\rho(a, c) = \rho(c, b) = 1/2 \cdot \rho(a, b)$.

(4.5) PROPOSITION. *If a scalene metric is convex, then it is strongly convex.*

PROOF. Let X be a space with scalene metric ρ, and let $a, b \in X$ and $a \neq b$. Consider the set $C = \{c \in X \mid \rho(a, c) = \rho(c, b) = 1/2 \cdot \rho(a, b)\} \neq \phi$. We must show that C is a one point set. Suppose, on the contrary, that there exist $c_1, c_2 \in C$ and $c_1 \neq c_2$. Since ρ is a scalene metric, there is $c_0 \in I_\rho(c_1, c_2)$. Then

$$\rho(a, b) \leqq \rho(a, c_0) + \rho(b, c_0) < \frac{1}{2} \cdot \rho(a, b) + \frac{1}{2} \cdot \rho(a, b) = \rho(a, b) .$$

This is a contradiction. Hence ρ is strongly convex.

(4.6) REMARK. Suppose that ρ is a scalene metric on a compactum X. If ρ is convex (and hence strongly convex), the retraction $r: 2^X \cong Q \to X$ in the proof of (3.10) is a cell-like map. Moreover, for each $x \in X$, $r^{-1}(x)$ is contractible. For let $t_0 = \sup\{f_A(x) \mid A \in r^{-1}(x)\} \geqq 0$. If $t_0 = 0$, $r^{-1}(x)$ is one point set. Assume that $t_0 > 0$. Let us defined a function $H: r^{-1}(x) \times [0, t_0] \to 2^X$ by

$$H(A, t) = \begin{cases} V(A, t), & 0 \leqq t \leqq f_A(x), \\ B(x, t), & f_A(x) \leqq t \leqq t_0, \end{cases}$$

where $A \in r^{-1}(x)$, $V(A, t) = \{y \in X \mid \rho(y, A) \leqq 2t$ and $\rho(x, y) \leqq f_A(x)\}$ and $B(x, t) = \{y \in X \mid \rho(x, y) \leqq t\}$. By (3.1), (3.2), (3.4) and the same argument as (4.4), H is continuous. Next, we shall show that $H(r^{-1}(x) \times [0, t_0]) \subset r^{-1}(x)$. Let $A \in r^{-1}(x)$ and $0 \leqq t \leqq f_A(x)$. By the definitions of r and $f_A(x)$, $H(A, t) \in r^{-1}(x)$. Let $A \in r^{-1}(x)$ and $f_A(x) < t \leqq t_0$. Note that $r(H(A, t_0)) = r(B(x, t_0)) = x$. We may assume that $f_A(x) < t < t_0$. Suppose, on the contrary, that for some t ($f_A(x) < t < t_0$), $r(H(A, t)) = r(B(x, t)) = x' \neq x$. Then $f_{B(x,t)}(x') < f_{B(x,t)}(x) = t$. Let $y \in B(x, t_0)$. If $y \in B(x, t)$, $\rho(x', y) \leqq f_{B(x,t)}(x') < f_{B(x,t)}(x) = t < t_0$. If $y \in B(x, t_0) - B(x, t)$, there is a point $y' \in B(x, t)$ such that $\rho(x, y) = \rho(x, y')$

$+\rho(y', y)$ and $\rho(x, y')=t$, because ρ is convex and X is compact. Then

$$\rho(x', y)\leqq\rho(x', y')+\rho(y', y)<t+\rho(y', y)=\rho(x, y)\leqq t_0 .$$

Since $B(x, t_0)$ is compact, $f_{B(x,t_0)}(x')<t_0$, hence $r(B(x, t_0))\neq x$. This is a contradiction. Thus $r: 2^X\cong Q\rightarrow X$ is a cell-like map.

But we cannot omit the condition that ρ is convex, as shown in the next example.

(4.7) EXAMPLE. There is a scalene metric ρ on a compactum X such that the retraction $r: 2^X\cong Q\rightarrow X$ in the proof of (3.10) is not monotone. Consider the following sets in the plane E^2:

$$M=\{x\in E^2|ta+(1-t)b,\ 0\leqq t\leqq 1\},$$

$$N=\{x\in E^2|x=t_1b+t_2c+t_3d,\ t_i\geqq 0\ (i=1, 2, 3) \text{ and } t_1+t_2+t_3=1\},$$

where $a=(-1, 0)$, $b=(0, 1)$, $c=(1, 0)$, $d=(3, 0)$, and

$$X=M\cup N.$$

Let ρ be the Euclidean metric on E^2. Then we can easily check that ρ_X is a scalene metric. Let r: $2^X\rightarrow X$ be the retraction in the proof of (3.10). Note that $r^{-1}(c)\ni\{a, d\}$. Then the point c is an isolated point of $r^{-1}(c)$. In fact, if $A\in r^{-1}(c)$ and $0<f_A(c)\leqq 1$, then $r(B(c, f_A(c))\neq c$. Let $c'=\frac{1}{2}e+\frac{1}{2}f$, where $e=\left(1-\frac{f_A(c)}{\sqrt{2}}, \frac{f_A(c)}{\sqrt{2}}\right)$, $f=(1+f_A(c), 0)$. Then $f_A(c')\leqq f_{B(c,f_A(c))}(c')<f_A(c)$. Hence c is an isolated point of $r^{-1}(c)$.

Let A be a subset of a metric space X with metric ρ. Then A is *scalene convex in* X provided that if $a, b\in A$ and $a\neq b$, there is a point $c\in A$ such that for each $x\in X$, $\max\{\rho(x, a), \rho(x, b)\}>\rho(x, c)$ holds. Note that every convex subset of l_2 is scalene convex in l_2, and if A is scalene convex in X, the restriction ρ_A is a scalene metric on A.

(4.8) PROPOSITION. *If A is a scalene convex, compact subset of a metric space X with metric ρ, then there is the unique continuous retraction $r: X\rightarrow A$ such that $\rho(x, r(x))=\rho(x, A)$ for each $x\in X$.*

PROOF. For $x\in X$, let $r(x)$ be a point of A with $\rho(x, r(x))=\rho(x, A)$. We have to prove that such $r(x)$ is unique. Suppose that there are points $x_0\in X$, $y_0\in A$ such that $y_0\neq r(x_0)$ and $\rho(x_0, y_0)=\rho(x_0, A)$. Since A is scalene convex, we can find a point $c\in A$ such that $\rho(x_0, c)<\max\{\rho(x_0, y_0), \rho(x_0, r(x_0))\}=\rho(x_0, A)$. This is a contradiction. Thus we have a function $r: X\rightarrow A$. We shall show that r is continuous. Let $x_0\in X$ and x_1, x_2, $\cdots$ be a sequence of points of X with $\lim x_n=x_0$.

Since A is compact, we may assume that the sequence $r(x_1)$, $r(x_2)$, $\cdots$, converges. Then

$$\begin{aligned}\rho(x_0, r(x_0)) = \rho(x_0, A) &= \rho(\lim x_n, A)\\ &= \lim \rho(x_n, A) = \lim \rho(x_n, r(x_n))\\ &= \rho(\lim x_n, \lim r(x_n))\\ &= \rho(x_0, \lim r(x_n)).\end{aligned}$$

Hence $r(x_0) = \lim r(x_n)$. This implies that r is continuous. Obviously, r is a retraction from X to A.

(4.9) EXAMPLE. There is a smooth arc A in the plane E^2 with Euclidean metric ρ such that ρ_A is scalene but there is no neighborhood U of A in E^2 in which A is scalene convex. In fact, let A_n $(n=1, 2, \cdots)$ be the set of points (x, y) $\in E^2$ such that $(-1)^n y \geqq 0$ and

$$\left(x - \frac{2n+1}{2n(n+1)}\right)^2 + \left(y + \frac{(-1)^n}{n(n+1)}\right)^2 = \left(\frac{\sqrt{2}}{n(n+1)}\right)^2.$$

Set $A = \{(0, 0)\} \cup \bigcup_{n=1}^{\infty} A_n$ (see Figure 2). Then A is a smooth arc and ρ_A is scalene, but there is no neighborhood U of A in E^2 such that A is scalene convex in U.

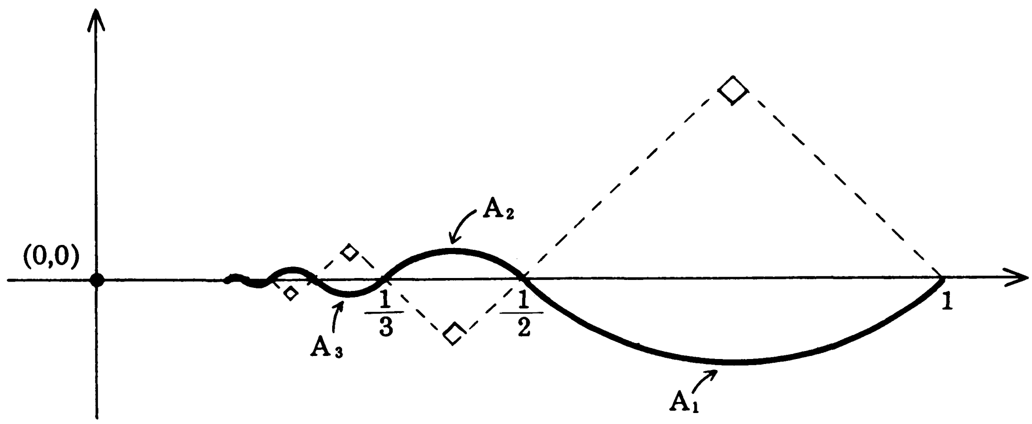

Fig. 2.

(4.10) PROPOSITION. *Let X be a compactum with scalene metric ρ. Then for any maps $f, g: Y \to X$, there is a homotopy $H: Y \times I \to X$ such that $H(y, 0) = f(y)$, $H(y, 1) = g(y)$ and* diam $H(\{y\} \times I) = \rho(f(y), g(y))$ *for each $y \in Y$.*

This follows from (5.2) which will be proved in the next section.

(4.11) PROPOSITION. *Let Y be a compactum with metric ρ and let X_1, X_2, $\cdots$ be an increasing sequence of subcompacta in Y. If each X_n is scalene convex in Y, then $X = \overline{\bigcup_{n=1}^{\infty} X_n}$ is an AR.*

PROOF. Let $r_n: X \to X_n$ be the retraction such that

(1) $\rho(x, r_n(x)) = \rho(x, X_n)$ for $x \in X$ (see (4.8)).

By (3.10) and Dowker's result [4, p. 105], it is sufficient to show that for every $\varepsilon > 0$ there is a homotopy $H: X \times I \to X$ such that $H(x, 0) = x$, $H(x, 1) = r_{n_0}(x)$ and $\rho(H(x, t), x) < \varepsilon$ for each $x \in X$, $t \in I$ and some positive integers n_0. Choose a positive integer n_0 such that $X \subset U_{\varepsilon/3}(X_{n_0})$. Without loss of generality, we may assume that $n_0 = 1$. By (4.10), for each n there is a map $\phi_n: X_n \times X_n \times I \to X_n$ such that

(2) $\phi_n(x, y, 0) = x$, $\phi_n(x, y, 1) = y$ and

(3) $\rho(\phi_n(x, y, t), x) \leqq \rho(x, y)$ for each $x, y \in X_n$ and $t \in I$.

Consider the set

(4) $\widetilde{X \times I} = \{(x, t) \in X \times I \mid 0 \leqq t \leqq \rho(x, X_1)\}$.

Define a map $p: X \times I \to \widetilde{X \times I}$ by

(5) $p(x, t) = (x, t \cdot \rho(x, X_1))$ for $(x, t) \in X \times I$.

Also, define a function $G: \widetilde{X \times I} \to X$ by

$$(6) \quad G(x, t) = \begin{cases} x, & \text{if } t = 0, \\ \phi_{n+1}(r_{n+1}(x), r_n(x), \alpha_n(x, t)), & \text{if } \rho(x, X_{n+1}) \leqq t \leqq \rho(x, X_n), \end{cases}$$

$$\text{where } \alpha_n(x, t) = \begin{cases} 0, & \text{if } \rho(x, X_{n+1}) = t, \\ \dfrac{(t - \rho(x, X_{n+1}))}{(\rho(x, X_n) - \rho(x, X_{n+1}))}, & \text{if } \rho(x, X_{n+1}) < t \leqq \rho(x, X_n). \end{cases}$$

By (1), (2), (3), (4) and (6), we can prove that G is continuous. Finally, define $H = G \circ p: X \times I \to \widetilde{X \times I} \to X$. Then $H(x, 0) = x$, $H(x, 1) = r_1(x)$ and $\rho(H(x, t), x) \leqq 3 \cdot \rho(x, X_1) \leqq \varepsilon$ for each $x \in X$. This completes the proof.

(4.12) EXAMPLE. In the statement of (4.11), we cannot conclude that $X = \overline{\bigcup_{n=1}^{\infty} X_n}$ has a scalene metric ρ_X. Consider the following sets in the 3-dimensional space E^3 with Euclidean metric ρ:

$$Y = \{(x, y, z) \in E^3 \mid |y| \leqq x,\ -1 \leqq z \leqq 1,\ x^2 + y^2 + z^2 \geqq 1 \text{ and } (x+1)^2 + y^2 + z^2 \leqq 3^2\},$$

and for each $n = 2, 3, \cdots$,

$$X_n = Y \cap \left\{(x, y, z) \in E^3 \,\middle|\, \left(x + \frac{1}{n}\right)^2 + y^2 + z^2 \geqq \left(\frac{n+2}{n}\right)^2\right\}.$$

Then $Y = X = \overline{\bigcup_{n=1}^{\infty} X_n}$. It can be checked that each X_n is scalene convex in Y, but ρ_Y is not a scalene metric. In fact, if $a = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right)$ and $b = \left(\frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}, 0\right)$,

$I_{\rho_Y}(a, b)=\phi$.

In general, the limit of scalene metric subspaces in a metric space with metric ρ is not a scalene metric space with respect to ρ. We need the following definition. Let Y be a metric space with metric ρ. A family $\{X_\alpha\}$ of subsets of Y is *uniformly scalene* provided that for any $\varepsilon>0$ and if $a_\alpha, b_\alpha \in X_\alpha$ with $\rho(a_\alpha, b_\alpha)\geqq\varepsilon$, then there are points $c_\alpha \in X_\alpha$ such that if $\delta(a_\alpha, b_\alpha)$ denotes $\inf_{x\in X_\alpha}\{\max\{\rho(x, a_\alpha), \rho(x, b_\alpha)\}-\rho(x, c_\alpha)\}$, then $\inf_\alpha \delta(a_\alpha, b_\alpha)>0$

(4.13) PROPOSITION. *Let Y be a compactum with metric ρ. If a sequence $\{X_n\}$ of subcompacta of Y is uniformly scalene and* $\lim X_n=X$, *then ρ_X is a scalene metric of X. In particular, X is an AR.*

PROOF. Let $a, b\in X$ and $a\neq b$. Put $\varepsilon=\rho(a, b)$. Choose sequences $\{a_n\}$, $\{b_n\}$ of points such that $a_n, b_n\in X_n$, $\lim a_n=a$ and $\lim b_n=b$. We may assume that $\rho(a_n, b_n)\geqq 2\varepsilon/3$ for each n. Since $\{X_n\}$ is uniformly scalene, there are points $c_n\in X_n$ and some positive number δ such that $\max\{\rho(x, a_n), \rho(x, b_n)\}-\rho(x, c_n)\geqq\delta$ for each $x\in X_n$. We may assume that $\lim c_n=c\in X$. Then if $x\in X$, for every $x_n\in X_n$ we have

$$\begin{aligned}\rho(x, c)&\leqq\rho(x, x_n)+\rho(x_n, c_n)+\rho(c_n, c)\\&\leqq\rho(x, x_n)+\max\{\rho(x_n, a_n), \rho(x_n, b_n)\}+\rho(c_n, c)-\delta\,.\end{aligned}$$

Choose a sequence $x_1, x_2, \cdots$ of points such that $x_n\in X_n$ and $\lim x_n=x$. Since $\lim\rho(x_n, a_n)=\rho(x, a)$ and $\lim\rho(x_n, b_n)=\rho(x, b)$, we have $\rho(x, c)<\max\{\rho(x, a), \rho(x, b)\}$ for $x\in X$. Hence $c\in I_\rho(a, b)$.

(4.14) QUESTION. Let Y be a compactum with metric ρ and let $\{X_n\}$ be a sequence of subcompacta of Y. If ρ_{X_n} is scalene $(n=1, 2, \cdots)$ and $\lim X_n=X$, is X an AR?

(4.15) QUESTION. If X_i $(i=1, 2, 3)$ is a compactum and X_1, X_2 and $X_3=X_1\cap X_2$ admit scalene metrics, does $X_1\cup X_2$ admit a scalene metric? Does $X_1\times X_2$ admit a scalene metric?

In relation to (4.15), the following is clear.

(4.16) PROPOSITION. *If X_1 and X_2 admit scalene metrics and have only one common point, then $X_1\cup X_2$ admits a scalene metric.*

5. Scalene metrics and some selection theorems.

In this section, we shall prove some selection theorems in an aspect which differs from that of E. Michael.

(5.1) THEOREM. *Suppose that Y is any topological space and X is a metric space with metric ρ. If $\varphi: Y\to 2^X$ is continuous and $\rho_{\varphi(y)}$ is a scalene metric for each $\varphi(y)$, then there is a continuous selection $s: Y\to X$ for φ.*

PROOF. For every $y\in Y$, let $s: Y\to X$ be the function defined by $s(y)=r_y(\varphi(y))$, where $r_y: 2^{\varphi(y)}\to\varphi(y)$ is defined as in the proof of (3.10). We have to prove that s is continuous. Let $y_0\in Y$ and $\varepsilon_1>0$. Note that

(1) $\inf\{f_{\varphi(y_0)}(x)\,|\,x\in\varphi(y_0)-U_{\varepsilon_1/2}(s(y_0))\}-m(\varphi(y_0))=\varepsilon_2>0$.

(for the definitions of $f_{\varphi(y_0)}$ and $m(\varphi(y_0))$, see the proof of (3.10)). Choose a neighborhood V of y_0 in Y such that if $y\in V$, $H_\rho(\varphi(y_0),\varphi(y))<\delta=1/4\cdot\min\{\varepsilon_1,\varepsilon_2\}$. We show $s(V)\subset U_{\varepsilon_1}(s(y_0))$. Suppose, on the contrary, that $s(y)\notin U_{\varepsilon_1}(s(y_0))$ for some $y\in V$. Choose a point $y_1\in\varphi(y)$ such that $\rho(y_1, s(y_0))<\delta$. Note that $y_1\neq s(y)$. Then we have

(2) $f_{\varphi(y)}(y_1)\leqq\rho(y_1, s(y_0))+m(\varphi(y_0))+H_\rho(\varphi(y_0),\varphi(y))<m(\varphi(y_0))+2\delta$.

On the other hand, choose a point $y_2\in\varphi(y_0)$ such that $\rho(s(y), y_2)<\delta$. Note that $y_2\in\varphi(y_0)-U_{\varepsilon_1/2}(s(y_0))$. Then by (1) and (2), we have

$$
\begin{aligned}
(3)\quad m(\varphi(y))=f_{\varphi(y)}(s(y))&\geqq f_{\varphi(y_0)}(y_2)-\rho(s(y), y_2)-H_\rho(\varphi(y),\varphi(y_0))\\
&\geqq m(\varphi(y_0))+\varepsilon_2-2\delta\\
&\geqq m(\varphi(y_0))+2\delta>f_{\varphi(y)}(y_1).
\end{aligned}
$$

This is a contradiction. Hence $s: Y\to X$ is continuous. This completes the proof.

(5.2) THEOREM. *Let X, Y, ρ and φ be as in the preceding theorem. If $f, g: Y\to X$ are continuous selections for φ, then there is a homotopy $h: Y\times I\to X$ such that $h(y,0)=f(y)$, $h(y,1)=g(y)$, $h(\{y\}\times I)\subset\varphi(y)$ and* $\operatorname{diam}(h(\{y\}\times I))=\rho(f(y), g(y))$ *for each $y\in Y$.*

PROOF. Define a function $\Psi: Y\to 2^X$ by

(1) $\Psi(y)=\overline{I_{\rho_{\varphi(y)}}(f(y), g(y))}\subset\varphi(y)$ for each $y\in Y$.

Then Ψ is continuous.

To prove this, we first show that Ψ is upper semi-continuous. Let $y\in Y$ and $\varepsilon>0$. Note $\Psi(y)=\{c\in\varphi(y)\,|\,\max\{\rho(x, f(y)),\rho(x, g(y))\}\geqq\rho(x, c)$ for $x\in\varphi(y)\}$. For each $c\in\varphi(y)-U_{\varepsilon/2}(\Psi(y))$, let $\alpha(c)=\sup\{\rho(x,c)-\max\{\rho(x, f(y)),\rho(x, g(y))\}\,|\,x\in\varphi(y)\}$. Then we have

(2) $\inf\{\alpha(c)\,|\,c\in\varphi(y)-U_{\varepsilon/2}(\Psi(y))\}=\varepsilon_1>0$.

Put $\delta=1/4\cdot\min\{\varepsilon,\varepsilon_1\}>0$. Since $\varphi: Y\to 2^X$ is continuous, there is a neighborhood V of y in Y such that for $z\in V$, $\rho(f(z), f(y))<\delta$, $\rho(g(z), g(y))<\delta$ and $H_\rho(\varphi(z),\varphi(y))<\delta$.

We show that if $z\in V$, $\Psi(z)\subset U_\varepsilon(\Psi(y))$. Suppose, on the contrary, that for some $z\in V$, $\Psi(z)\not\subset U_\varepsilon(\Psi(y))$. Choose a point $c\in\Psi(z)$ such that $c\notin U_\varepsilon(\Psi(y))$ and a point $c_0\in\varphi(y)$ such that $\rho(c, c_0)<\delta$. Since $c_0\in\varphi(y)-U_{\varepsilon/2}(\Psi(y))$, there is a point $x_0\in\varphi(y)$ such that $\alpha(c_0)=\rho(x_0, c_0)-\max\{\rho(x_0, f(y)), \rho(x_0, g(y))\}\geqq\varepsilon_1>0$. Let $x_1\in\varphi(z)$ with $\rho(x_0, x_1)<\delta$. Then by (2) we have

$$
\begin{aligned}
(3)\quad \rho(c, x_1) &\geqq \rho(c_0, x_0)-\rho(c, c_0)-\rho(x_0, x_1)\\
&> \max\{\rho(x_0, f(y)), \rho(x_0, g(y))\}+\varepsilon_1-2\delta\\
&\geqq \max\{\rho(x_1, f(z)), \rho(x_1, g(z))\}-2\delta+\varepsilon_1-2\delta\\
&\geqq \max\{\rho(x_1, f(z)), \rho(x_1, g(z))\}\,.
\end{aligned}
$$

(3) implies that $c\notin\Psi(z)$. This is a contradiction. Hence Ψ is upper semi-continuous.

Next, we show that Ψ is lower semi-continuous. Let $c_0\in\Psi(y)$ and $\varepsilon>0$. By (4.2), there is a point $c_1\in I_{\rho_{\varphi(y)}}(f(y), g(y))$ such that $\rho(c_0, c_1)<\varepsilon/2$. Put $\varepsilon_1=\inf\{\max\{\rho(x, f(y)), \rho(x, g(y))\}-\rho(x, c_1)|x\in\varphi(y)\}>0$. Since φ is continuous, there is a neighborhood V of y in Y such that if $z\in V$, $H_\rho(\varphi(z), \varphi(y))<1/4\cdot\min\{\varepsilon, \varepsilon_1\}=\delta>0$ and $\rho(f(z), f(y))<\delta$, $\rho(g(z), g(y))<\delta$. Let $z\in V$. Take a point $c\in\varphi(z)$ with $\rho(c, c_1)<\delta$. For each $x\in\varphi(z)$, choose a point $x'\in\varphi(y)$ with $\rho(x, x')<\delta$. Then we have

$$
\begin{aligned}
(4)\quad \rho(c, x) &\leqq \rho(c, c_1)+\rho(c_1, x')+\rho(x', x)\\
&\leqq \max\{\rho(x', f(y)), \rho(x', g(y))\}-\varepsilon_1+2\delta\\
&< \max\{\rho(x, f(z)), \rho(x, g(z))\}-\varepsilon_1+4\delta\\
&\leqq \max\{\rho(x, f(z)), \rho(x, g(z))\}\,.
\end{aligned}
$$

Hence $c\in I_{\rho_{\varphi(z)}}(f(z), g(z))\subset\Psi(z)$. Note that

$$(5)\quad \rho(c, c_0)\leqq\rho(c, c_1)+\rho(c_1, c_0)<\varepsilon\,.$$

This implies that Ψ is lower semi-continuous. Hence Ψ is continuous.

Next, we define a homotopy $h: Y\times I\to X$ as follows. Since $\rho_{\Psi(y)}$ is a scalene metric, there is a retraction $r_y: 2^{\Psi(y)}\to\Psi(y)$. Define a homotopy $F: Y\times I\to X$ by

$$F(y, t)=r_y(K_\rho(f(y), t\cdot\rho(f(y), g(y)))\cap\Psi(y)) \qquad \text{for } y\in Y,\ t\in I.$$

By (4.4) and the proof of (3.10), F is continuous. Then

$$(6)\quad F(\{y\}\times I)\subset\Psi(y),\ F(y, 0)=f(y) \text{ and } F(y, 1)=r_y(\Psi(y)) \text{ for } y\in Y.$$

Similarly, we have a homotopy $G: Y\times I\to X$ such that

$$(7)\quad G(\{y\}\times I)\subset\Psi(y),\ G(y, 0)=g(y) \text{ and } G(y, 1)=r_y(\Psi(y)) \text{ for } y\in Y.$$

By (6) and (7), define a homotopy $h: Y\times I\to X$ by

$$h(y, t)=\begin{cases} H(y, 2t)\,, & \text{if } 0\leqq t\leqq 1/2\,,\\ G(y, 2-2t)\,, & \text{if } 1/2\leqq t\leqq 1\,.\end{cases}$$

Since diam $\Psi(y)=\rho(f(y), g(y))$, we have diam $h(\{y\}\times I)=\rho(f(y), g(y))$. This completes the proof.

(5.3) COROLLARY. *Let X and Y be metric spaces and ρ be a metric on X. If $f: X\to Y$ is a proper open map and $\rho_{f^{-1}(y)}$ is scalene for each $y\in Y$, then f is a fiber homotopy equivalence. In particular, if X is an ANR, then Y is an ANR.*

(5.4) COROLLARY. *Let Y be a complete metric space and X be a compactum with metric ρ. If $\varphi: Y\to 2^X$ is upper (or lower) semi-continuous and $\rho_{\varphi(y)}$ is a scalene metric for each $y\in Y$, then there is a dense G_δ-subset Y' of Y and a continuous selection $s: Y'\to X$ for $\varphi|Y': Y'\to 2^X$.*

(5.4) follows from (5.1) and [5, Corollary 1, p. 71].

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