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ON REPRESENTATION-FINITE ALGEBRAS WHOSE AUSLANDER-REITEN QUIVER CONTAINS A STABLE COMPLETE SLICE

By

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0. Introduction

Tilting modules and associated tilted algebras, introduced by Brenner and Butler in [7] and generalized by Happel and Ringel [12, 13] has been shown in [1, 8, 12, 13, 14, 16, 18, 19] to be of interest in representation theory. Recall [12] that a module T_A over a finite-dimensional algebra A is called a *tilting module* provided it satisfies the following three properties:

(1) proj $\dim_A(T_A)\leqslant 1$

(2) $\mathrm{Ext}_A^1(T_A, T_A)=0$

(3) There is an exact sequence $0\longrightarrow A_A\longrightarrow T'_A\longrightarrow T''_A\longrightarrow 0$ with T', T'' being direct sums of summands of T.

An algebra B is called a *tilted algebra* if there is an hereditary algebra A and a tilting module T_A such that $B=\mathrm{End}(T_A)$. Tilted algebras together with recently developed covering techniques provide a rather general setting for dealing with arbitrary representation-finite algebras, that is, algebras with finitely many non-isomorphic finitely generated indecomposable modules. Happel and Ringel showed in [12] (see also [6, 15]) that representation-finite tilted algebra have the following nice characterization in the term of the associated Auslander-Reiten quiver: A connected representation-finite algebra B is a tilted algebra if and only if the Auslander-Reiten quiver of B contains a *complete slice*, that is, a set $\mathcal{S}$ of indecomposable modules with the following properties

(i) Given any indecomposable module X, $\mathcal{S}$ contains precisely one module from the orbit $\{\tau^r X; r\in Z\}$ of X, where $\tau=DTr$ and $\tau^{-1}=TrD$ and $\tau^{-1}=TrD$ are the Auslander-Reiten operators [3].

(ii) If $X_0\longrightarrow X_1\longrightarrow X_2\longrightarrow\dots\longrightarrow X_r$ is a chain of non-zero maps and indecomposable modules, and X_0, X_r belong to $\mathcal{S}$, then all X_i belong to $\mathcal{S}$.

(iii) There is no oriented cycle of irreducible maps $U_0\longrightarrow U_1\longrightarrow\dots\longrightarrow U_r\longrightarrow U_0$ with all U_i in $\mathcal{S}$.

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Recently two interesting classes of representation-finite algebras, *PHI* algebras considered by Simson-Skowroński [18, 19] and trivial extension algebras investigated by Hughes-Waschbüsch [16] (see also [14]), have been completely classified by invariants involving only tilted algebras. In general the Auslander-Reiten quiver of such algebras contains no complete slice but the Auslander-Reiten quiver modulo projective-injectives has a complete slice of a Dynkin class.

In this paper we shall give a rather simple description of all algebras having this property. We use many ideas and extend results from [12, 16, 19].

We use the term algebra to mean finite-dimensional algebra over a fixed commutative field K and the term module to mean a finitely generated right module. Algebras, as is usual in representation theory, are assumed to be basic and connected. For any algebra A and an A-module M we shall denote by $E_A(M)$ the A-injective envelope of M, by $P_A(M)$ the A-projective cover of M, by $\mathrm{top}_A(M)$ the top of M, by $\mathrm{soc}_A(M)$ the socle of M, by $\mathrm{rad}(M)$ the radical of M. For any indecomposable projective-injective A-module Q, define $\sigma_A(\mathrm{soc}_A(Q))=\mathrm{top}_A(Q)$. Further, we will denote by mod A the category of (finite dimensional) A-modules and by ind A the full subcategory of mod A formed by the chosen representatives of the isomorphism classes of indecomposable modules. We will frequently ignore the distinction between the isomorphism class of a module and the module itself. Left modules will usually be regarded as right modules over the opposite algebra. We shall denote by D: mod $A \longrightarrow$ mod A^{op} the usual duality $\mathrm{Hom}_K(-, K)$. We will use freely the properties of irreducible maps, almost split sequences, almost split morphisms, and the Auslander-Reiten operators $\tau=DTr$ and $\tau^{-1}=TrD$. For any algebra A, we will denote by Γ_A the Auslander-Reiten quiver of A [10]. For definitions and further details we refer to [2, 3, 4, 5, 10]. Finally, for the definition of valued quivers and of the Cartan class of a valued quiver we refer to [11, 17].

1. Main result

In this section we formulate the main result of the paper. Let A be a connected basic algebra over a field K and let $\mathfrak{C}$ be a connected component of Γ_A. Then a subquiver $\mathcal{S}$ of $\mathfrak{C}$ is said to be *path-complete* if, whenever M and N are vertices of $\mathcal{S}$ and there is a path $M \longrightarrow \dots \longrightarrow L \longrightarrow \dots \longrightarrow N$ in $\mathfrak{C}$, L is a vertex of $\mathcal{S}$. We say that a full subquiver $\mathcal{S}$ of $\mathfrak{C}$ is a *stable complete slice* of $\mathfrak{C}$ if the following conditions are satisfies:

(1) $\mathcal{S}$ is path-complete.

(2) There is no oriented cycles $X_0 \longrightarrow X_1 \longrightarrow \dots \longrightarrow X_r \longrightarrow X_0$ with all X_i in $\mathcal{S}$.

(3) $\mathcal{S}$ has no projective-injective modules.

(4) Given any non-projective-injective module X in $\mathfrak{C}$, $\mathcal{S}$ contains precisely one module from the orbit $\{\tau^r X; r \in Z\}$ of X.

It is easy to see that $\mathcal{S}$ is a stable complete slice in $\mathfrak{C}$ if and only if $\mathcal{S}$ is a complete slice of the full subquiver ${}_S\mathfrak{C}$ of $\mathfrak{C}$ obtained by suppressing the vertices corresponding to projective-injective indecomposable modules.

A complete slice $\mathcal{S}$ of $\mathfrak{C}$ is of Dynkin class Δ provided $\mathcal{S}$, considered as a nonoriented graph, is a Dynkin graph Δ. It follows from [9] that if A is a connected representation-finite hereditary algebra, then the vertices of Γ_A corresponding to the indecomposable projective A-modules form in Γ_A a complete slice of Dynkin class. If A is a hereditary representation-finite algebra, and T_A a tilting module, then the Cartan class of the tilted algebra $B=\mathrm{End}(T_A)$ is defined to be that of A (see [16]).

For any algebra A, we will denote by $F(A)$, the set of isomorphism classes of simple A-modules.

A *system C of Dynkin class* Δ is defined to be $C=(B, n, m, F_*, F'_*)$, where B is a tilted algebra of Dynkin class Δ, n and m are nonnegative integers, and F_*, F'_* are chains

$$F_* : F(B)=F_0 \supset F_1 \supset \dots \supset F_n$$
$$F'_* : F(B^{op})=F'_0 \supset F'_1 \supset \dots \supset F'_m$$

of nonempty subsets of $F(B)$ and $F(B^{op})$.

Then the algebra $\mathcal{R}(C)$, for a given system $C=(B, n, m, F_*, F'_*)$, is defined to be $\mathcal{R}(C)=R(-m)$, where the sequence of algebras

$$B=R(0),\ R(1),\ \cdots,\ R(n),\ R(-1),\ \cdots,\ R(-m)$$

is obtained as follows:

$$R(1)=\begin{pmatrix} E(1), & I(1) \\ 0\ , & R(0) \end{pmatrix}$$

where $I(1)=\bigoplus_{S\in F_1} E_B(S)$, $E(1)=\mathrm{End}_B(I(1))$, and $I(1)$ has the canonical structure of $E(1)-R(0)-$bimodule. Let $i\geqslant 1$ and write $\sigma_{R(i)}=\sigma_i$; similarly as in [19] one shows that the set $F(R(i))$ of $R(i)$-simples has a natural identification with the union of $F(R(i-1))$ and a new set of simples $\bar{F}_i=\{\sigma_i\sigma_{i-1}\cdots\sigma_1(S); S\in F_i\}$. Then $R(i+1)$, for $i=1, \cdots, n-1$, is the triangular matrix algebra

$$R(i+1)=\begin{pmatrix} E(i+1), & I(i+1) \\ 0\ \ , & R(i) \end{pmatrix}$$

where $I(i+1)=\bigoplus_{S\in F_{i+1}} F_{R(i)}(\sigma_i\cdots\sigma_1(S))$ and $E(i+1)=\mathrm{End}_{R(i)}(I(i+1))$. Further, $R(-1)$ is the triangular matrix algebra

$$R(-1)=\begin{pmatrix} R(n), & I(-1) \\ 0\ , & E(-1) \end{pmatrix}$$

where $I(-1)=\bigoplus_{S\in F_1'} E_{R(n)^{op}}(S)$ and $E(-1)=\mathrm{End}_{R(n)^{op}}(I(-1))$. Finally, for $-m\leqslant i\leqslant -2$, $R(i)$ is the triangular matrix algebra

$$R(i)=\begin{pmatrix} R(i+1), & I(i) \\ 0 \quad, & E(i) \end{pmatrix}$$

where $I(i)=\bigoplus_{S\in F_i'} E_{R(i+1)^{op}}(\sigma_{i+1}\cdots\sigma_{-1}(S))$ and $E(i)=\mathrm{End}_{R(i+1)^{op}}(I(i))$.

We can now formulate the main result of this paper.

THEOREM. *Let A be a connected finite-dimensional basic algebra. Then A is representation-finite and Γ_A contains a stable complete slice of a Dynkin class if and only if A is isomorphic to an algebra $\mathcal{R}(C)$ for some system C of a Dynkin class.*

2. Proof of the theorem

First we shall show that for any system $C=(B, n, m, F_*, F_*')$ of a Dynkin class, the algebra $\mathcal{R}(C)$ is representation-finite and $\Gamma_{\mathcal{R}(C)}$ contains a stable complete slice of a Dynkin class. We will apply results from [16].

Let $C=(B, n, m, F_*, F_*')$ be a system of a Dynkin class Δ and consider the doubly infinite matrix algebra without identity

$$\hat{B}=\begin{pmatrix} \ddots & \ddots & & & & \\ & B_{n-1} & M_{n-1} & & & \\ & & B_n & M_n & & \\ & & & B_{n+1} & M_{n+1} & \\ & & & & \ddots & \ddots \\ & & & & & \ddots \end{pmatrix}$$

in which matrices are assumed to have only finitely many entries different from zero, $B_n=B$ and $M_n={}_BD(B)_B$ for all integers n, all remaining entries are zero, and the multiplication is induced from the canonical maps $B\otimes_B D(B)\longrightarrow D(B)$, $D(B)\otimes_B B\longrightarrow D(B)$, and zero maps $D(B)\otimes_B D(B)\longrightarrow 0$. Hughes and Waschbüsch proved in [16] that mod B has almost split sequences, the stable Auslander-Reiten quiver ${}_s\Gamma_B$ is isomorphic to $Z\Delta$ and that for any indecomposable projective B-module P, $\mathrm{Hom}_B(P, X)\neq 0$ only for a finite number of nonisomorphic indecomposable B-modules X. It is not hard to see that all algebras $R(0)=B$, $R(1)$, $\cdots$, $R(n)$, $R(-1)$, $\cdots$, $R(-m)$ occuring in the definition of $\mathcal{R}(C)$ are full finite subcategories of $\hat{B}$. Then from [2, §3], the algebras $R(i)$ are representation-finite and consequently $\mathcal{R}(C)=R(-m)$ is so. Now it suffices to prove that $\Gamma_{\mathcal{R}(C)}$ contains a stable complete slice of a Dynkin class. By assumption Γ_B contains a complete slice $\mathcal{M}$. We shall prove that the modules from $\mathcal{M}$ being no projective-injective $\mathcal{R}(C)$-modules form a stable complete slice in $\Gamma_{\mathcal{R}(C)}$. First observe that the set $\mathcal{M}'$ of all modules from $\mathcal{M}$

being no projective-injective B-modules form a stable complete slice in Γ_B. Let X be the set of all (isomorphism classes of) indecomposable projective $R(n)$-modules $Q \in X$ such that $\mathrm{top}(Q) \in F(R(n)) \setminus F(B)$. From [20] we know that all modules from X are also injective $R(n)$-modules. Choose a module Q_1 from X such that $\mathrm{rad}(Q_1)$ is not successor in Γ_B of any module $\mathrm{rad}(Q')$ for $Q' \in X$. Note that $\mathrm{rad}(Q_1)$ is a B-module. Consequently, $\mathrm{rad}(Q_1)$ is an injective B-module isomorphic to $\mathrm{Hom}_{R(n)}(B, Q_1)$ and the algebras $T_1 = \mathrm{End}_{R(n)}(B \oplus Q_1)$ and

$$\begin{pmatrix} C_1, & \mathrm{rad}(O_1) \\ 0, & B \end{pmatrix}$$

where $C_1 = \mathrm{End}_B(\mathrm{rad}(Q_1))$, are isomorphic. Algebra T_1 is representation-finite as a full subcategory of B. Moreover, just as in [16, 3.5, 3.6], we see that $\mathcal{M}'$ is a stable complete slice in Γ_{T_1} provided $\mathrm{rad}(Q_1)$ is not projective B-module. On the other hand, if $\mathrm{rad}(Q_1)$ is projective-injective (as B-module), then $\mathrm{rad}(Q_1)$ belongs to $\mathcal{M}$ and $\mathcal{M}' \cup \{\mathrm{rad}(Q_1)\}$ forms a stable complete slice in Γ_{T_1}. Moreover, since all modules from X are projective-injective $R(n)$-modules, if $Y = \mathrm{rad}R(n)(Q)$, for $Q \in X$, is a projective-injective T_1-module, then Y is a projective-injective B-module and so belongs to $\mathcal{M}$. Then we can repeat this procedure taking T_1 instead of B. Consequently, after a finite number of steps, we obtain $R(n)$ and the modules Z from $\mathcal{M}$ being no projective-injective $R(n)$-modules form a stable complete slice in $\Gamma_{R(n)}$. Then the corresponding $R(n)^{op}$-modules $D(Z)$ form a stable complete slice in $\Gamma_{R(n)^{op}}$. Considering $\mathcal{R}(C)^{op}$-modules Q whose tops belong to $F'(\mathcal{R}(C)) \setminus F'(R(n))$, and applying above arguments, we conclude that the modules $D(Z)$, where Z ranges over all modules Z from M being no projective-injective $\mathcal{R}(C)$-modules, form a stable complete slice in $\Gamma_{\mathcal{R}(C)^{op}}$. Consequently, $\Gamma_{\mathcal{R}(C)}$ contains a stable complete slice $\mathcal{S}$ being a connected subgraph of the complete slice $\mathcal{M}$ of Γ_B. Since $\mathcal{M}$ is of Dynkin class, $\mathcal{S}$ is so and we are done.

At the end of this paper we shall give an example showing that the graphs $\mathcal{M}$ and $\mathcal{S}$ can be different.

Now let A be a representation-finite algebra and let Γ_A contains a stable complete slice $\mathcal{M} = \{M_1, \cdots, M_t\}$ of Dynkin class Δ. We shall show that A is isomorphic to an algebra $\mathcal{R}(C)$ for some system $C = (B, n, m, F_*, F'_*)$ of Dynkin class Δ.

We start with the following lemma.

LEMMA 1. *Under the above assumption, Γ_A has no oriented cycle.*

PROOF. Assume that Γ_A has an oriented cycle

$$X_0 \longrightarrow X_1 \longrightarrow \dots \longrightarrow X_r \longrightarrow X_0 .$$

Since Γ_A has a stable complete slice, one of the modules $X_0, X_1, \cdots X_r$ is projective-injective. Indeed, in the opposite case, similarly as in [12, Prop. 8.1] one proves that there is an oriented cycle $Y_0 \longrightarrow Y_1 \longrightarrow \cdots \longrightarrow Y_s \longrightarrow Y_0$ with all modules Y_j from $\mathcal{M}$, but this is a contradiction to the stable slice condition (2). Denote by $\mathfrak{D}$ the full subcategory of ind A formed by all non-projective-injective modules. From the stable slice condition (4), for each module X of $\mathfrak{D}$, there is exactly one module M_i from $\mathcal{M}$ and one integer z such that $X=\tau^z(M_i)$, and put $z=z(X)$. Suppose that there is an irreducible map $X=\tau^{z(X)}(M_i) \longrightarrow Y=\tau^{z(X)}(M_j)$ between two objects X and Y from $\mathfrak{D}$. Then $z(X)=z(Y)$ and there is an irreducible map $M_i \longrightarrow M_j$ or $z(X)=z(Y)+1$ and there is an irreducible map $M_j \longrightarrow M_i$. Indeed, if $z(X)=z(Y)$ then obviously there is an irreducible map $M_i \longrightarrow M_j$. If $z(X) \leqslant 0$ and $z(Y) \geqslant 0$, then there is a chain of irreducible maps $M_i \longrightarrow \cdots \longrightarrow \tau^{z(X)}(M_i) \longrightarrow \tau^{z(Y)}(M_j) \longrightarrow \cdots \longrightarrow M_j$ and by the stable slice condition (4), $z(X)=z(Y)=0$. Consider the case $z(X)>z(Y)>0$. Then there is an irreducible map $\tau^{z(X)-z(Y)}(M_i) \longrightarrow M_j$, hence a chain of irreducible maps $M_j \longrightarrow \tau^{z(X)-z(Y)-1}(M_i) \longrightarrow \cdots \longrightarrow M_i$ and, by the stable slice condition (4), $z(X)=z(Y)+1$. Similarly, if $z(Y)>z(X)>0$, there is a chain of irreducible maps $M_i \longrightarrow \tau^{z(Y)-z(X)} M_j \longrightarrow \cdots \longrightarrow M_j$ and $z(Y)-z(X)>0$, contrary to the stable slice conditions (1) and (4). Analogically one proves that $z(X)=z(Y)+1$ if $z(X) \neq z(Y)$, $z(X)<0$, $z(Y)<0$. Finally, if $z(X)>0$, $z(Y) \leqslant 0$, then $z(X)=1$, $z(Y)=0$; and $z(X)=0$, $z(Y)=-1$ in case $z(X) \geqslant 0$ and $z(Y)<0$.

Consequently one of the modules in the cycle $X_0 \longrightarrow X_1 \longrightarrow \cdots \longrightarrow X_r \longrightarrow X_0$ is projective-injective. Without loss of generality we can assume that this is X_1. If X_i is projective-injective, then $X_{i-1}=\operatorname{rad}(X_i)$, $X_{i+1}=X_i/\operatorname{soc}(X_i)$, $X_{i-1}=\tau(X_{i+1})$, and $z(X_{i-1})=z(X_{i+1})+1$. Thus, from the above remarks, $z(X_0)>z(X_2) \geqslant \cdots \geqslant z(X_0)$ and we get a contradiction. Therefore Γ_A has no oriented cycles and the lemma is proved.

Denote by $\mathfrak{P}_A$ (resp. $\mathfrak{J}_A$) the set of projective (resp. injective) modules in ind A and by Σ_A the sum $\mathfrak{P}_A \cup \mathfrak{J}_A$. Let us denote by $\nu: \Sigma_A \longrightarrow \Sigma_A$ and $\nu^{-1}: \Sigma_A \longrightarrow \Sigma_A$ two partial functions defined as follows: For each $X \in \Sigma_A$, $\nu(X)$ is defined iff $X \in \mathfrak{P}_A$, and then $\nu(X)=E(\operatorname{top}(X))$; $\nu^{-1}(X)$ is defined iff $X \in \mathfrak{J}_A$, and then $\nu^{-1}(X)=P(\operatorname{soc}(X))$. Then the set $\{\nu^z(X);\ z \in Z,\ \nu^z(X) \text{ is defined}\}$ is said to be the ν-orbit of $X \in \Sigma_A$.

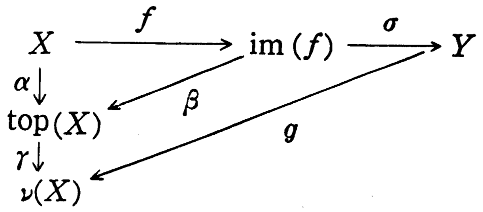
Let us denote by $\mathcal{S}=\{S_1, \cdots, S_r\}$ the set of all composition factors of modules $M_1, \cdots, M_t$, and by B the algebra $\operatorname{End}_A(P_A(S_1) \oplus \cdots \oplus P_A(S_r))$. As in [16, Lemmas 3.2, 3.3] one proves that any ν-orbit in Σ_A contains exactly one module from the set $\{P_A(S_1), \cdots, P_A(S_r)\}$ and that the set $\mathcal{M}$ considered as a set of B-modules is a complete slice of Γ_B of Dynkin class Δ. In particular, B is a tilted algebra of Dynkin class Δ. Moreover, any ν-orbit in Σ_A is the ν-orbit of some module $P_A(S_j)$, $j=1, \cdots, r$, and we can define the function $s: \Sigma_A \longrightarrow Z$ such that, for $X \in \Sigma_A$, $s(X)=i$ iff $X=\nu^i(P_A(S_j))$ for some $j=1, \cdots, r$. Thus, for $X \in \Sigma_A$, $s(X) \leqslant 0$ implies $X \in \mathfrak{P}_A$, and $s(X)>0$ implies $X\in\mathfrak{J}_A$.

Let $A_A=Q_1\oplus\cdots\oplus Q_m$ be some decomposition as a direct sum of indecomposable projective A-modules, $n=\max\{s(X);\ X\in\Sigma_A\}$, $m=-\min\{s(X);\ X\in\Sigma_A\}$, and we let A_p, $-m\leqslant p\leqslant n$, be the direct sum of all modules Q_k such that $s(Q_k)=p$. Then $A_A=\bigoplus_{p=-m}^{n} A_p$ and put for $-m\leqslant p\leqslant q\leqslant n$, $E_{p,q}=\operatorname{End}_A(\bigoplus_{k=p}^{q} A_k)$. We will write simply E_p instead of $E_{p,p}$, B_p, $0\leqslant p\leqslant n$, instead of $E_{0,p}$, and B_q, $-m\leqslant q\leqslant -1$, instead of $E_{q,n}$. Obviously the algebras B and E_0 are isomorphic.

In our proof an important role is played by the following lemma.

LEMMA 2. *In the above notation*, $\operatorname{Hom}_A(A_p, A_q)=0$ *for* $p>q$ *and* $q>p+1$.

PROOF. Suppose that $\operatorname{Hom}_A(A_p, A_q)\neq 0$ for some $p>q$. Then there are two indecomposable summands X of A_p and Y of A_q with $\operatorname{Hom}_A(X, Y)\neq 0$. First assume $p>0$, $q\leqslant 0$. In this case X is projective-injective, there is a sequence of non-zero maps $\bigoplus_{i=1}^{t} M_i\longrightarrow\nu^{1-p}(X)\longrightarrow\cdots\longrightarrow X\longrightarrow Y\longrightarrow\cdots\longrightarrow\nu^{-q}(Y)\longrightarrow\bigoplus_{i=1}^{t} M_i$, implying the corresponding sequence of irreducible maps, and we get a contradiction to the stable slice condition (3). If $p=0$ and $f\colon X\longrightarrow Y$ is a non-zero map, then since $\nu(X)$ is injective, there is a commutative diagram



where α, β are canonical epimorphisms, γ, σ canonical monomorphisms, and obviously $g\neq 0$. Similarly, there is a non-zero map $h\colon \nu(X)\longrightarrow\nu(Y)$. But $s(\nu(Y))=q+1\leqslant 0$, $\nu(Y)$ is projective-injective, there is a sequence of irreducible maps $\bigoplus_{i=1}^{t} M_i\longrightarrow\cdots\longrightarrow\nu(X)\longrightarrow\cdots\longrightarrow\nu(Y)\longrightarrow\cdots\longrightarrow\bigoplus_{i=1}^{t} M_i$, and we get a contradiction to the stable slice conditions (1) and (3). If $0>p>q$, then as above we conclude that $\operatorname{Hom}_A(\nu^{-p}(X), \nu^{-p}(Y))\neq 0$, but this is impossible since $s(\nu^{-p}(X))=0$ and $s(\nu^{-p}(Y))=q-p<0$. Finally, in the case $p>q>0$, similarly, as in [19, Lemma p. 60], we prove that $\operatorname{Hom}_A(\nu^{-q}(X), \nu^{-q}(Y))\neq 0$. Since, $s(\nu^{-q}(X))=p-q>0$, $s(\nu^{-q}(Y))=0$, from the first part of our proof, it is impossible. Consequently, $\operatorname{Hom}_A(A_p, A_q)=0$ for $p>q$.

Now assume that $\operatorname{Hom}_A(X, Y)\neq 0$ for $p<q-1$ and indecomposable direct summands X of A_p and Y of A_q. If $p\geqslant 0$, as in [19, Lemma p. 60], $\operatorname{Hom}_A(\nu^{-1}(Y), X)\neq 0$, and since $s(\nu^{-1}(Y))=q-1>p$ we get a contradiction to the fact that $\operatorname{Hom}_A(A_{q-1}, A_p)=0$. If $p<0$, $\nu(X)$ is projective-injective, and, as in the first part of the proof, we conclude that $\operatorname{Hom}_A(Y, \nu(X))\neq 0$. This is a contradiction since

$s(\nu(X))=p+1<q=s(Y)$ and $\mathrm{Hom}_A(A_q, A_{p+1})=0$. Therefore, $\mathrm{Hom}_A(A_p, A_q)=0$ for $p+1<q$ and the lemma is proved.

In our proof we shall need the following fact.

LEMMA 3. *For* $p<0$, $\mathrm{Hom}_A(A_p, A)$ *is a projective-injective* A^{op}*-module.*

PROOF. Let X be an indecomposable direct summand of $\mathrm{Hom}_A(A_p, A)$. Then $D(X)\cong E(D(\mathrm{top}_{A^{op}}(X)))\cong E(\mathrm{top}_A(Y))=\nu(Y)$ for $Y=\mathrm{Hom}_{A^{op}}(X, A)$. Since Y is a direct summand of A_p, and $s(\nu(Y))=p+1\leqslant 0$, $\nu(Y)$ is a projective-injective A-module. Hence $X\cong D(\nu(Y))$ is a projective-injective A^{op}-module and we are done.

Now we shall define a system $C=(B, n, m, F_*, F'_*)$ where $B=\mathrm{End}_A(A_0)$, $n=\max\{s(X)\,;\, X\in\Sigma_A\}$, $m=-\min\{s(X)\,;\, X\in\Sigma_A\}$. The canonical action of B on $\mathrm{top}_A(A_0)$ (resp. $\mathrm{top}_{A^{op}}(\mathrm{Hom}_A(A_0, A))$) enabling us to identify the set $F(B)$ (resp. $F(B^{op})$) with the set F_0(resp. F'_0) of simple A-module (resp. A^{op}-module) components of $\mathrm{top}_A(A_0)$ (resp. of to $\mathrm{top}_{A^{op}}(\mathrm{Hom}_A(A_0, A))$). Then F_1 consists of the simple components of of $\mathrm{soc}_A(A_1)$ (a summand of $F_0=\mathrm{top}(A_0)$); for $1\leqslant i<n$, F_{i+1} consists of the simples S in F_0 such that $\sigma^i_A(S)$ is a component of $\mathrm{soc}_A(A_{i+1})$. Similarly, F'_1 consists of the simple components of $\mathrm{soc}_{A^{op}}(\mathrm{Hom}_A(A_{-1}, A))$ (a summand of F'_0); for $1\leqslant j<m$, F'_{j+1} consists of the simples S in F'_0 such that $\sigma^j_{A^{op}}(S)$ is a component of $\mathrm{soc}_{A^{op}}(\mathrm{Hom}_A(A_{-j-1}, A))$.

From Lemma 2 it follows that $A=\mathrm{End}_A(A_A)$ is isomorphic to the matrix algebra

$$\left|\begin{array}{cccccccccc}
E_n & {}_nM_{n-1} & 0 & & & & & & & \\
0 & E_{n-1} & {}_{n-1}M_{n-2} & 0 & & & & & 0 & \\
 & 0 & \ddots & \ddots & & & & & & \\
 & & & E_1 & {}_1M_0 & 0 & & & & \\
 & & & 0 & E_0 & {}_0M_{-1} & 0 & & & \\
 & 0 & & & 0 & E_{-1} & {}_{-1}M_{-2} & 0 & & \\
 & & & & & 0 & \ddots & \ddots & 0 & \\
 & & & & & & & E_{-m+1} & {}_{-m+1}M_{-m} & \\
 & & & & & & & 0 & E_{-m} &
\end{array}\right|$$

where ${}_{i+1}M_i$ is the E_{i+1}-E_i-bimodule $\mathrm{Hom}_A(A_i, A_{i+1})$. First we shall prove that the algebras B_i and $R(i)$, $i=0, \cdots, n$, are isomorphic. We shall proceed by induction, using [19, Proposition 2] and Lemma 2. For $i=0$, $B_0=R(0)$ by definition. Assume that for some $i\geqslant 0$ there is an isomorphism $h\colon B_i \longrightarrow R(i)$. Observe that there is a canonical isomorphism of algebras

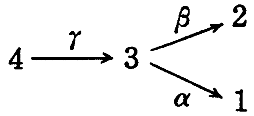
$$B_{i+1}\cong\begin{pmatrix} E_{i+1}\ , & {}_{i+1}M_i \\ 0\ , & B_i \end{pmatrix}$$

Then $A_{i+1}\cong\mathrm{Hom}_A(\bigoplus_{k=0}^{i+1} A_k\ ,\ A_{i+1})$ is an injective B_{i+1}-module and ${}_{i+1}M_i=\mathrm{Hom}(\bigoplus_{k=0}^{i} A_k,$ $A_{i+1})$ is an injective B_i-module as the greatest B_i-submodule of A_{i+1}. Similarly as in [19, Proposition 2] we conclude that the algebras E_{i+1} and $\mathrm{End}_{B_i}({}_{i+1}M_i)$ are isomorphic. By definition of A_{i+1} and $I(i+1)$ it is not hard to see that $I(i+1)\cong H({}_{i+1}M_i)$ where H: mod $B_i\longrightarrow$mod $R(i)$ is the functor induced by h. Hence $B_{i+1}\cong R(i+1)$ and consequently $B_n\cong R(n)$. Further, using Lemma 3 and repeating the above arguments for A^{op}-modules, we get isomorphisms of algebras B_j and $R(j)$, $j=-1,\cdots,-m$. Then $A\cong B_{-m}\cong R(-m)=\mathscr{R}(C)$ and this completes the proof of the theorem.

We end the paper with an example illustrating previously considered questions.

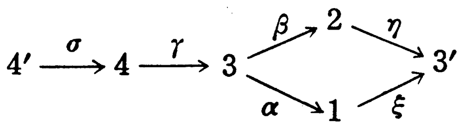
Let B be the tilted algebra of Dynkin class D_4 given by the bounden quiver algebra (see [10]) KQ/I, where

Q:

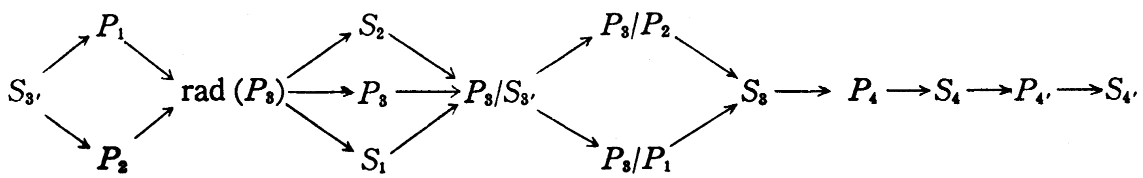


and I is generated by the composed arrows $\alpha\gamma$ and $\beta\gamma$. Consider the system $C=(B, 1, 1, F_*, F'_*)$ where F_1 consists of one simple B-module given by the vertex 4 and F'_1 consists of one simple B^{op}-module given by the vertex 3. Then it is easy to see that $\mathscr{R}(C)$ is the bounden quiver algebra KQ'/I' where

Q':



and I' is generated by $\alpha\gamma$, $\beta\gamma$, $\gamma\sigma$ and $\xi\alpha$-$\eta\beta$. Then a straightforward calculation shows that $\Gamma_{\mathscr{R}(C)}$ is of the form



where $P_i=P(S_i)$ and S_i denotes the simple module given by the vertex i. Here, P_3, P_4 and $P_{4'}$ are projective-injective and the modules S_1, S_2 and $P_3/S_{3'}$ form a stable complete slice of class A_3, so different from the Dynkin class of B. On the other hand, $\mathscr{R}(C)$ is isomorphic to the algebra $\mathscr{R}(\bar{C})$ where $\bar{C}$ is the system $(\bar{B}, 2, 1, \bar{F}_*, \bar{F}'_*)$ and $\bar{B}$ is the path algebra of $1\longleftarrow 3\longrightarrow 2$, $\bar{F}_1=\bar{F}_2$ (resp. $\bar{F}'_1$) consists of the simple $\bar{B}$-module (resp. $\bar{B}^{op}$-module) given by the vertex 3.

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