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BASES FOR ARONSZAJN TREES

By

James E. BAUMGARTNER[1]

1. Introduction.

In [3], Hanazawa defined and studied the notion of a non-Souslin base for certain Aronszajn trees. We extend his definition here by defining a base for an arbitrary tree of height ω_1 to be a collection of subtrees with the property that every subtree contains an element of the base. We show that if there is a Kurepa tree with κ branches then there is a special Aronszajn tree for which every base must have cardinality $\geqslant\kappa$. This slightly improves a result of Hanazawa [3] which draws a similar conclusion from $\diamondsuit^+$. Then we show that it is consistent, relative to the existence of an inaccessible cardinal, that every Aronszajn tree has a base of cardinality $\aleph_1$, and that this may be obtained even with the continuum large. This answers a question of Hanazawa [3]. Finally, we observe that if T is a tree of height ω_1 which is essentially non-Aronszajn in the sense that every element has exactly $\aleph_1$ immediate successors, then every base for T must have $2^{\aleph_1}$ elements.

A precise statement of the results follows a brief discussion of terminology. The remainder of the paper is then devoted to the proofs.

A *tree* is a partially ordered set $(T, \leqslant_T)$ with the property that for every $t\in T$, $\{s\in T: s<_T t\}$ is well ordered by $\leqslant_T$. The *level* of t, denoted by $l(t)$, is the order type of $\{s\in T: s<_T t\}$, and the set of all elements of T of level α is denoted by T_α. The *height* of T is the smallest α such that $T_\alpha=0$. We shall be interested exclusively in trees of height ω_1. For convenience, we shall also work only with *normal* trees, e.i., trees such that each element has successors of arbitrarily high levels, and such that when α is a limit ordinal and $t\in T_\alpha$ then t is determined by $\{s\in T: s<_T t\}$ i.e., there is no other $t'\in T_\alpha$ with $\{s\in T: s<_T t\}=\{s\in T: s<_T t'\}$. All our results remain true for non-normal trees, as the reader may easily verify, but the proofs go more smoothly for normal trees.

For our purposes, if T is a (normal) tree of height ω_1 then a *subtree* of T is a subset S of T such that S itself, with the induced ordering, is a normal tree of

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height ω_1 and S is closed downward under $\leqslant_T$ (i.e., if $t\in S$ and $s<_T t$ then $s\in S$). The crucial point is that in a subtree every element has successors of arbitrarily high levels.

A *base* for T is a set B of subtrees such that for every subtree S of T there is $S'\in B$ with $S'\subseteq S$.

A *branch* through a tree T is a maximal linearly ordered subset.

An *Aronszajn tree* is a (normal) tree of height ω_1 such that for all $\alpha<\omega_1$, T_α is countable and T has no uncountable branches. An Aronszajn tree is *special* (sometimes called *Q-embeddable*) if there is a function $f: T\to\omega$ such that whenever $s<_T t$ we have $f(s)\neq f(t)$. A *Souslin tree* is an Aronszajn tree with the further property that every set of pairwise incomparable elements is countable. A *Kurepa tree* is a tree T of height ω_1 such that each T_α is countable for $\alpha<\omega_1$ and T has more than $\aleph_1$ uncountable branches.

In [3], Hanazawa defined a *non-Souslin base* for an Aronszajn tree T to be a collection C of uncountable antichains (pairwise incomparable sets) of T such that for any uncountable set $S\subseteq T$ there is $X\in C$ such that for all $s\in X$ there is $t\in S$ with $s\leq_T t$. It is apparent that if T is to have a non-Souslin base then in particular no subtree of T can be Souslin. Such trees were called by the author *non-Souslin* in [1]. This terminology is unfortunate since it distinguishes between trees which are non-Souslin and those which are not Souslin, so the author wishes to take this opportunity to suggest that trees with no Souslin subtrees be known henceforth as *anti-Souslin* trees.

There is a close connection between bases and non-Souslin bases. Suppose T is anti-Souslin. Then every non-Souslin base for T gives rise to a base of no greater cardinality, and conversely. If C is a non-Souslin base for T and if for each $X\in C$ we let $S(X)=\{t\in T:\{s\in X: t<_T s\}$ is uncountable$\}$ then it is easy to see that $S(X)$ is a subtree and that $\{S(X): X\in C\}$ is a base. If B is a base then form C by choosing an uncountable antichain from each element of B. It is straightforward to check that C is a non-Souslin base for T.

The advantage of the notion of a base, therefore, is that it applies to a larger class of trees. For anti-Souslin trees it is essentially equivalent to Hanazawa's definition.

We might remark that Souslin trees always have a base of cardinality $\aleph_1$. If T is Souslin, $t\in T$ and $T_t=\{s\in T: s\leqslant_T t$ or $t\leqslant_T s\}$, then $\{T_t: t\in T\}$ forms a base for T. Thus questions about the cardinality of a base are only interesting for trees which are not Souslin.

The rest of our set-theoretical terminology is fairly standard, and can generally be found in [4] or [5]. In independence proofs we consider forcing to be taking place over the universe V of set theory, and we write V^P for the generic extension of V obtained by forcing with P. If we have a particular P-generic set G then we replace V^P by $V[G]$.

Here are the main results of the paper.

THEOREM 1. *Suppose there is a Kurepa tree with at least κ branches. Then there is a special Aronszajn tree for which every base has cardinaity $\geqslant\kappa$.*

Theorem 1 is proved in Section 2, using a remark of Todorcevic which greatly simplifies the author's original proof. It improves Theorem 2 of [3], which asserts that if $\diamondsuit^+$ holds then there is a special Aronszajn tree such that every base has cardinality $\geqslant\aleph_2$, in view of Solovay's well known result (see [5, Corollary 7.11]) that $\diamondsuit^+$ implies the existence of a Kurepa tree.

THEOREM 2. *If it is consistent that there is an inaccessible cardinal then it is consistent that $\diamondsuit$ holds and every Aronszajn tree has a base of cardinality $\aleph_1$.*

The model we use is Levy's model in which a strongly inaccessible cardinal is collapsed to become ω_2. Of course, this is the same model in which Silver [6] proved there are no Kurepa trees. In view of Theorem 1, Silver's result is implied by Theorem 2. Theorem 2 also shows that Hanazawa's hypothesis $\diamondsuit^+$ cannot be reduced to $\diamondsuit$ alone.

THEOREM 3. *If it is consistent that there is an inaccessible cardinal then it is consistent that every Aronszajn tree has a base of cardinality $\aleph_1$ and the continuum is large.*

The model for Theorem 3 is obtained from the one for Theorem 2 by adjoining any number of Cohen reals. This gives a precise meaning to the phrase "the continuum is large".

Without the inaccessible one can still prove something:

THEOREM 4. *Suppose $2^{\aleph_1}=\kappa$. If one forces by adjoining λ Cohen reals, then in the extension every Aronszajn tree has a base of cardinality $\leqslant\kappa$.*

Thus, for example, if $\kappa=\aleph_2$ then it is possible to have $2^{\aleph_0}$ large while every Aronszajn tree has a relatively small base, namely one of cardinality $\leqslant\aleph_2$.

Theorems 2, 3 and 4 are proved in Section 3.

Theorems 2, 3 and 4 have consequences for certain linear orderings also. A linear ordering $(S, \leqslant_S)$ is called a *Specker ordering* (and its order type is a *Specker type*) if S is uncountable, has no uncountable well-ordered or conversely well-ordered subsets, and has no uncountable subsets order-embeddable in the real numbers.

See [2] for a discussion of Specker types. Every Specker ordering arises as a lexicographically ordered (not necessarily normal) Aronszajn tree. If each level T_α of T is linearly ordered by $\leqslant_\alpha$ then the lexicographic ordering of T is defined by setting $s \leqslant t$ iff either $s \leqslant_T t$ or else s and t are incomparable and if u, v are $\leqslant_T$-minimal such that $u \leqslant_T s$, $u \nleqslant_T t$ and $v \leqslant_T t$, $v \nleqslant_T s$ and $u, v \in T_\alpha$ then $u \leqslant_\alpha v$. A *Souslin ordering* is a Specker ordering with no uncountable pairwise disjoint set of nonempty open intervals, i.e., it arises from a Souslin tree. Let us call a Specker ordering *anti-Souslin* if it has no Souslin suborderings. Such orderings arise from anti-Souslin trees.

From Theorem 2 and the equivalence of bases with non-Souslin bases for anti-Souslin trees, we arrive at the following:

COROLLARY 5. *If it is consistent that there is an inaccessible cardinal, then it is consistent that for every anti-Souslin Specker ordering S there is a collection C of subsets of S such that C has cardinality* $\leqslant \aleph_1$ *and every uncountable subset of S contains an order-isomorphic copy of an element of C.*

Details are left to the reader. There are similar corollaries for Theorems 3 and 4.

One may wonder whether there are results similar to the ones above for trees of height ω_1 such that for each $\alpha < \omega_1$, $|T_\alpha| \leqslant \aleph_1$ rather than $|T_\alpha| = \aleph_0$. The answer, it turns out, is an emphatic no.

THEOREM 6. *Suppose T is a (normal) tree with height* ω_1 *such that every element of T has exactly* $\aleph_1$ *immediate successors. Then there is a family* $\langle S_\alpha : \alpha < 2^{\aleph_1} \rangle$ *of subtrees of T such that for all* α, β *if* $\alpha \neq \beta$ *then* $S_\alpha \cap S_\beta$ *does not contain a subtree. It follows that every base for T must have the maximum cardinality* $2^{\aleph_1}$.

Theorem 6 is proved in Section 4.

In view of the results here and in [3], it appears that the most interesting problem left open is the following.

PROBLEM. Is it consistent with $2^{\aleph_0} = \aleph_1$ that no Aronszajn tree has a base of cardinality $\aleph_1$?

2. Proof of Theorem 1.

The author wishes to thank Stevo Todorcevic for the following argument, presented here with his permission, which reduces Theorem 1 to a straightforward observation.

Let $(K, \leqslant_K)$ be a Kurepa tree with κ branches and let T be a special Aronszajn tree. Let KT denote $\{(s, t): s\in K, t\in T \text{ and } l(s)=l(t)\}$ with the coordinatewise ordering. Then KT is clearly an Aronszajn tree, for $(KT)_\alpha=K_\alpha\times T_\alpha$ for all $\alpha<\omega_1$ and if $B\subseteq KT$ were an uncountable branch then $\{t\in T: \exists s(s, t)\in B\}$ would be an uncountable branch through T, which is impossible.

If $f: T\to\omega$ witnesses that T is special then $g: KT\to\omega$ witnesses that KT is special, where $g(s, t)=f(t)$.

Finally, let $<B_\xi: \xi<\kappa>$ be a sequence of distinct uncountable branches through K. If $S_\xi=\{(s, t)\in KT: s\in B_\xi\}$ then it is easy to see that $S_\xi\cap S_\eta$ is countable whenever $\xi\neq\eta$, and of course each S_ξ is a subtree. It follows immediately that any base for KT must have cardinality at least κ.

3. Proof of Theorems 2, 3, and 4.

Whereas the proof of Theorem 3 really includes that of Theorem 2 as a special case, it will make the ideas clearer to prove Theorem 2 separately first. The principal tool in both arguments is Levy's partial ordering for collapsing an inaccessible to ω_2.

Let κ be strongly inaccessible, and let P consist of all countable functions p such that $\text{domain}(p)\subseteq\kappa\times\omega_1$ and $\forall(\alpha, \xi)\in\text{domain}(p)\ p(\alpha, \xi)<\alpha$, partially ordered by functional extension, i.e., $p\leqslant q$ iff $p\supseteq q$. Then, as is well known (see [4, e.g.]), P is countably closed and has the κ-chain condition, and in V^P all cardinals of V which lie strictly between ω_1 and κ are collapsed onto ω_1.

If $\alpha<\omega_1$, $P_\alpha=\{p\in P: \text{domain}(p)\subseteq\alpha\times\omega_1\}$ and $P^\alpha=\{p\in P: \text{domain}(p)\cap(\alpha\times\omega_1)=0\}$, then $P\cong P_\alpha\times P^\alpha$. Thus if G is P-generic, $G_\alpha=G\cap P_\alpha$ and $G^\alpha=G\cap P^\alpha$, then G_α is P_α-generic (over V) and G^α is P^α-generic over $V[G_\alpha]$. It follows that $V[G]$ is a Levy-generic extension of $V[G_\alpha]$.

If T is an Aronszajn tree in $V[G]$, then by the κ-chain condition there is $\alpha<\kappa$ such that $T\in V[G_\alpha]$. We will show that every subtree of T in $V[G]$ contains a subtree that lies in $V[G_\alpha]$, and this will suffice, since the subtrees of T in $V[G_\alpha]$ form a set of cardinality at most $[2^{\aleph_1}]^{V[G_\alpha]}$, and hence of cardinality $\aleph_1$ in $V[G]$. By the remark in the preceding paragraph, we may assume $T\in V$.

Thus it will suffice to prove:

LEMMA 3.1. *Suppose $T\in V$ is an Aronszajn tree. If S is a subtree of T lying in $V[G]$, then there is a subtree S' of S lying in V.*

PROOF. We work in V. Let $\dot{S}$ be a P-name for S, and assume $\Vdash_P \dot{S}$ is a subtree of T.

First observe that if $p\in P$ and $U=\{t\in T: p\Vdash t\in\dot{S}\}$ is uncountable, then $S'=\{t\in T: \{u\in U: t\leqslant_T u\}$ is uncountable$\}$ is a subtree and $p\Vdash S'\subseteq\dot{S}$. Thus we may assume that U is always countable, and hence that $\exists\alpha_p<\omega_1\ \forall\beta\geqslant\alpha_p\ \forall t\in T_\beta\ \exists q\leqslant p\ q\Vdash t\notin\dot{S}$. For convenience, take α_p minimal.

Now fix $p\in P$ and choose λ regular and so large that P, $\dot{S}\in H(\lambda)$, where $H(\lambda)$ denotes the collection of sets hereditarily of cardinality $<\lambda$. Let N be a countable elementary substructure of $H(\lambda)$ (with respect to $\in$) such that p, P, $\dot{S}\in N$, and let $\alpha=\omega_1\cap N$. Let $<t_n: n<\omega>$ enumerate T_α. Now define a descending sequence $<p_n: n<\omega>$ of elements of $P\cap N$ so that $p_0=p$ and $\forall n\exists s<_T t_n\ p_{n+1}\Vdash s\notin\dot{S}$. This is possible since, given p_n, we know that $\alpha_{p_n}\in N$ since $p_n\in N$, and hence $\alpha_{p_n}<\alpha$. Thus if $s<_T t_n$ is chosen with $s\in T_{\alpha_{p_n}}$ then $\exists p_{n+1}\leqslant p_n\ p_{n+1}\Vdash s\notin\dot{S}$.

But now if $q\leqslant\bigcup\{p_n: n<\omega\}$ and $q\Vdash t_n\in\dot{S}$ (which must be possible for some q and n), we arrive at a contradiction because $\exists s<_T t_n\ p_{n+1}\Vdash s\notin\dot{S}$ and hence $q\Vdash s\notin\dot{S}$.

Since P is countably closed and adjoins a subset of ω_1, it follows that $\diamondsuit$ is true in $V[G]$. Alternatively, one can argue easily that any $\diamondsuit$-sequence in V remains a $\diamondsuit$-sequence in $V[G]$.

Now we turn our attention to Theorem 3. The proof is similar but a trifle more complicated because of the need to adjoin many real numbers.

Let μ be a cardinal, and let Q be the partial ordering of finite functions mapping subsets of μ into 2. Then Q is the usual ordering for adjoining μ Cohen subsets of ω.

We will eventually force with $P\times Q$, but first let us make an observation about forcing with Q alone.

LEMMA 3.2. *Suppose T is an Aronszajn tree (in V) and H is Q-generic. Then any subtree of T which lies in $V[H]$ contains a subtree lying in V.*

PROOF. Let $\dot{S}$ be a Q-name such that $\Vdash\dot{S}$ is a subtree of T. As in the proof of Lemma 3.1, if $p\in P$ and $\{t\in T: p\Vdash t\in\dot{S}\}$ is uncountable then we are done, so assume otherwise. Then there is α_p so that $\forall\beta\geqslant\alpha_p\ \forall t\in T_\beta\ \exists q\leqslant p\ q\Vdash t\notin\dot{S}$.

It is now an easy matter to find $\alpha<\omega_1$ and a countable set $X\subseteq\mu$ such that, if $P|X=\{p\in P: \text{domain}(p)\subseteq X\}$, then $\forall p\in P|X\ \alpha_p<\alpha$ and $\forall t\in T_{\alpha_p}\ \exists q\in P|X\ q\leqslant p$ and $q\Vdash t\notin\dot{S}$.

But if $q\in P$ and $t\in T_\alpha$ with $q\Vdash t\in\dot{S}$, then $p=q|X\in P|X$ so $\exists p'\in P|X\ p'\leqslant p$ and $p'\Vdash s\notin\dot{S}$, where s is the unique predecessor of t of level α_p. But then clearly p' and q are compatible, and this contradiction completes the proof.

REMARK. Lemma 3.2 really completes the proof of Theorem 4. If $2^{\aleph_1}=\kappa$ and we adjoin λ Cohen reals then any Aronszajn tree T must lie in an intermediate model V_1 obtained by adjoining at most $\aleph_1$ of the Cohen reals, and the remaining Cohen reals are generic over V_1. Thus by Lemma 3.2 the subtrees of T lying in V_1 form a base for T, and there are at most $[2^{\aleph_1}]^{V_1}=[2^{\aleph_1}]^V=\kappa$ such subtrees in V_1.

Now suppose $G\times H$ is $(P\times Q)$-generic and T is an Aronszajn tree in $V[G][H]$. Then by the countable chain condition for Q, T is adjoined to $V[G]$ by at most ω_1 Cohen reals, and by Lemma 3.2 a base for T in this intermediate model is still a base for T in $V[G][H]$ so without loss of generality we may take $\mu=\omega_1$.

Also, it is not hard to see that by the κ-chain condition for P, we have $T\in V[G_\xi][H]$ for some $\xi<\kappa$. Since $P\times Q\cong P_\xi\times P^\xi\times Q$, we see that G^ξ is P^ξ-generic over $V[G_\xi][H]$. Let $V_1=V[G_\xi]$, $V_2=V_1[H]$. The following lemma will complete the proof.

LEMMA 3.3. *Suppose S is a subtree of T and $S\in V_2[G^\xi]$ $(=V[G][H])$. Then there is a subtree $S'\subseteq S$ such that $S'\in V_2$.*

PROOF. We work in V_2 and consider forcing with respect to P^ξ. Suppose $\Vdash_{P^\xi}\dot{S}$ is a subtree of T. As before, we may assume that for every $p\in P^\xi$ there is $\alpha_p<\omega_1$ such that $\forall\beta\geqslant\alpha_p\ \forall t\in T_\beta\ \exists q\leqslant p\ q\Vdash t\notin\dot{S}$. Also, since Q has the countable chain condition and $V_2=V_1[H]$, we may assume that the correspondence carrying p to α_p lies in V_1.

Now we work in V_1. Let $\dot{T}$ be a Q-name for T and suppose

$$\Vdash_Q \dot{T} \text{ is an Aronszajn tree.}$$

Without loss of generality we may suppose $T_\alpha\in V_1$; for example we may take $T_\alpha=\{\alpha\}\times\omega$. Let λ be regular and large enough that P^ξ, Q, $\dot{T}$, $\dot{S}\in H(\lambda)$ (here $\dot{S}$ is really a Q-name for the P^ξ-name $\dot{S}$), and let N be a countable elementary substructure of $H(\lambda)$ with P^ξ, Q, $\dot{T}$, $\dot{S}\in N$. Let $\alpha=\omega_1\cap N$, and let $\gamma\geqslant\alpha$ be large enough so that if $\beta<\alpha$, $s\in T_\beta$, $t\in T_\alpha$ and $p\Vdash s\leqslant_T t$, then $p|\gamma\Vdash s\leqslant_T t$.

Let $<(t_n, p_n):n<\omega>$ enumerate all pairs $(t, q)\in T_\alpha\times(Q|\gamma)$. Beginning with an arbitrary $p\in P^\xi\cap N$ (which we could have chosen before N, if necessary), we find a sequence $<p_n:n<\omega>$ of elements of $P^\xi\cap N$ much as in the proof of Lemma 3.1. Set $p_0=p$. Given p_n, we find p_{n+1} as follows. Let $\alpha_p=\alpha_{p_n}$, and find $r_n\leqslant q_n$, $r_n\in P|\gamma$ so that for some $s_n\in T_{\alpha_n}$, $r_n\Vdash s_n\leqslant_T t_n$.

But we also have $\Vdash_Q$ "$\exists p'\leqslant p_n\ p'\Vdash_{P^\xi}s_n\notin\dot{S}$", so there is $r'_n\in Q\cap N$, $r'_n\leqslant r_n|\alpha$, and $p_{n+1}\in P^\xi\cap N$ so that $r'_n\Vdash_Q$ "$p_{n+1}\Vdash_{P^\xi}s_n\notin\dot{S}$". But then r'_n is compatible with r_n, so $r'_n\cup r_n\leqslant q_n$.

Finally, suppose $p'\leqslant\bigcup\{p_n:n<\omega\}$ (note that the sequence $<p_n:n<\omega>$ lies in V_1, so the union is in P^ξ), $t\in T_\alpha$ and in V_2 $p'\Vdash_{P^\xi} t\in\dot{S}$. Then for some $q\in H$ we have

$$q\Vdash\text{“}p'\Vdash_{P^\xi} t\in\dot{S}\text{”}.$$

It is clear from the construction of the r'_n that $\{r'_n\cup r_n : t_n=t\}$ is dense in Q and lies in V_1, so $\exists n\ r'_n\cup r_n\in H$, $t_n=t$. For this n, $q\cup r'_n\cup r_n\in H$.

But now we are in trouble, for in V_2 we must have

$$p_{n+1}\Vdash_{P^\xi} s_n\notin\dot{S}$$

since this is forced by r'_n,

$$s_n\leqslant_T t$$

since this is forced by r_n, and

$$p'\Vdash_{P^\xi} t\in\dot{S}$$

since this is forced by q. All this, of course, adds up to a contradiction, and completes the proof of Theorems 2 and 3.

REMARK. One may complicate this argument still further and arrange for $2^{\aleph_1}$ to be arbitrarily large, independently of $2^{\aleph_0}$. Just use the usual (ground model) ordering to adjoin many subsets of ω_1 with countable conditions. Since this ordering is countably closed and has the $[2^{\aleph_0}]^+$-chain condition, hence the κ-chain condition, we may simply combine it with P in the argument above. Details are left to the reader.

4. Proof of Theorem 6.

Suppose now that T has height ω_1, every element of T has successors at every higher level, and every element of T has exactly $\aleph_1$ immediate successors. If $INC=\{s\in\bigcup\{{}^\alpha\omega_1 : \alpha<\omega_1\} : s$ is strictly increasing$\}$, then it is easy to see by induction on the levels of T that T is isomorphic to a subtree of INC, and that since each element of T has $\aleph_1$ immediate successors the subtree can be chosen so that whenever it contains s, then it also contains $s\alpha$ for every α such that $s\alpha\in INC$. Here by $s\alpha$ we mean the function t with domain equal to domain$(s)+1$ and such that $t|\text{domain}(s)=s$ and $t(\text{domain}(s))=\alpha$. Hence without loss of generality we may identify T with this subtree of INC. Thus $T\subseteq INC$.

Let $S=\{s\in T : \forall\alpha<\text{domain}(s)$ if α is a limit ordinal then $\sup\{s(\beta):\beta<\alpha\}>\alpha$, and $\forall\beta\in\text{domain}(s)\ s(\beta)>\beta\}$.

LEMMA 4.1. *S is a subtree of T.*

PROOF. It is clear that S is closed downward. Let $s\in S$ and let $\alpha>\text{level}(s)$ be fixed. There is some immediate successor of s in T of the form $s\beta$, where $\beta>\alpha$. Now let t be any element of T of level α extending $s\beta$. Clearly $t\in S$. Thus S is a subtree.

We will find all the S_α as subsets of S. Suppose $s\in S$, and consider the sequence 0, $s(0)$, $s^2(0)$, $s^3(0)$, $\cdots$. If all the $s^n(0)<\text{domain}(s)$ then if $\alpha=\sup\{s^n(0):n<\omega\}$ we have $\sup\{s(\beta):\beta<\alpha\}=\alpha$, a contradiction since $s\in S$. Thus there is an i such that $s^{i-1}(0)<\text{domain}(s)\leqslant s^i(0)$. We refer to i as the *depth* of s.

Next, let $<A_n:n<\omega>$ be a disjoint decomposition of ω into infinite sets such that $\forall n\ n\in\bigcup\{A_i:i<n\}$. Let X be an uncountable subset of ω_1 with uncountable complement. For each $\alpha<\omega_1$ and $n<\omega$, let $\phi_{\alpha n}:{}^{\alpha}2\to{}^{(A_n)}2$ be a bijection. Let $<f_\xi:\xi<2^{\aleph_1}>$ enumerate ${}^{\omega_1}2$.

Fix $\xi<2^{\aleph_1}$. We define S_ξ. Suppose $s\in S$ with depth i. If $1\leqslant j<i$, let us say that j is *s-good* for ξ provided that if j-$1\in A_n$ and $\alpha=s^{n+1}(0)$ then $s^{j+1}(0)\in X$ iff $\phi_{\alpha n}(f_\xi|\alpha)(j-1)=0$. (This assumes that $\alpha\geqslant\omega$; the case $\alpha<\omega$ is omitted.) Now let $s\in S_\xi$ iff for all j, if $1\leqslant j<i$ then j is s-good for ξ. Note in particular that if $i=1$ then $s\in S_\xi$.

LEMMA 4.2. *S_ξ is a subtree.*

PROOF. It is clear that S_ξ is closed downward. Fix $s\in S_\xi$ with depth i, and let $\beta>\text{level}(s)$ be given. We know that there is $t\geqslant s$ such that $t\in S$ and t has level $s^i(0)$. Then t also has depth i, so $t\in S_\xi$ as well. Now fix $\gamma\geqslant\beta$ such that $\gamma\in X$ iff $\phi_{\alpha n}(f_\xi|\alpha)(i-1)=0$ (where $i-1\in A_n$ and $\alpha=t^{n+1}(0)$) and γ is so large that $t\gamma\in S$. Then if $u=t\gamma$ we have $u\in S_\xi$ also since $u^j(0)=s^j(0)$ for all $j<i$ and $u^{i+1}(0)=\gamma$. But now if $v\geqslant u$ is an element of level β then v has depth $i+1$ so $v\in S_\xi$ also. Hence s is extended in S_ξ at level β.

The following lemma will now complete the proof.

LEMMA 4.3. *If $\xi\neq\eta$ then $S_\xi\cap S_\eta$ contains no subtree of T.*

PROOF. Suppose on the contrary that $U\subseteq S_\xi\cap S_\eta$ is a subtree. Let $\beta<\omega_1$ be arbitrary and choose $u\in U$ with $\text{level}(u)\geqslant\beta$. Say $i=\text{depth}(u)$. Then determine inductively a sequence in U, $u=u_i<u_{i+1}<u_{i+2}<\cdots$, such that for all $j\geqslant i$, level $(u_{j+1})>u_j^j(0)$. If we set $\bar{u}=\bigcup\{u_j:j\geqslant i\}$ then $\bar{u}^j(0)$ is defined for all $j\in\omega$. Let $\bar{\alpha}=\sup\{\bar{u}^j(0):j\in\omega\}$. Then $f_\xi|\bar{\alpha}$ may be recovered from $\bar{u}$ in the following way. If $\delta<\bar{\alpha}$ then for some $m\geqslant i$ we have $\bar{u}^m(0)>\delta$. Consider the function $g:A_{m-1}\to 2$ given by $g(j)=0$ iff $\bar{u}^{j+2}(0)\in X$. Let $\alpha_m=\bar{u}^m(0)$ and let $f=\phi^{-1}_{\alpha_m m-1}(g)$. Then since each of the $u_j\in S_\xi$ and $u_j^k(0)=\bar{u}^k(0)$ whenever $u_j^k(0)$ is defined, we must have $f=f_\xi|\alpha_m$. Thus $f_\xi|\bar{\alpha}$ is the union of the $f_\xi|\alpha_m$ and so is canonically determined from

$\bar{\alpha}$. But of course the same argument applies to determine $f_\eta|\bar{\alpha}$ in exactly the same way, so $f_\xi|\bar{\alpha}=f_\eta|\bar{\alpha}$. Finally, since β was arbitrary and $\bar{\alpha}>\beta$ we must have $f_\xi=f_\eta$, a contradiction since $\xi=\eta$.

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Dartmouth College
Hanover, NH 03755 USA