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A SUBSYSTEM OF CLASSICAL ANALYSIS PROPER TO TAKEUTI'S REDUCTION METHOD FOR Π_1^1-ANALYSIS

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After Gentzen's works for the pure number theory, G. Takeuti gave consistency proofs of some impredicative subsystems of classical analysis in [5], [7], [8] and [9] ([9] with N. Yasugi). In these proofs, the only 'Überschreitung' beyond the finitist standpoint in Hilbert's sense was the accessibility of some systems of o. d.'s (ordinal diagrams) which were also introduced by Takeuti in [4] and [6]. Thus these works may be regarded as nice extensions of Gentzen's. But, unfortunately, it was not shown that the system of o. d.'s $O(\omega+1, \omega^3)$ with respect to $<_0$ used in the consistency proof for **SINN**′*) (, which is equivalent to (Π_1^1-CA)+(BI)) is optimal.

In this paper, we will propose a subsystem of classical analysis **AII** which is equivalent to **SINN**′, and prove the consistency of **AII** by the accessibility of the system $O(\omega+1, 1)$ with respect to $<_0$, following Gentzen [2] and Takeuti [8]. Also in [1], we will show that the transfinite induction up to each o. d. from the system $O(\omega+1, 1)$ with respect to $<_0$ is derivable in **AII**. Thus we will complement Takeuti's consistency proof for (Π_1^1-CA)+(BI).

In § 1 the definition of **AII** and some preliminary definitions for a consistency proof will be given. In § 2 the main lemma will be proved and from which together with the accessibility of the system $O(\omega+1, 1)$ with respect to $<_0$, the consistency of **AII** follows immediately.

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§ 1. Preliminary Definitions

In this paper, we will use the terminology and notation in the same sense as those in [PT].

*) Usually this proof is said to be one for **SINN** which is equivrlent to (Π_2^1-CA), but as remarked in [8], foo1note 2, it is at the same time one for **SINN**′.

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DEFINITION 1.1. The system of second order arithmetic **INN′**, **AII** and **AIΠ_1^1** are obtained from **INN** (Definition 27.4, [PT], p. 320) by restricting the constants to 0, ′ and =, and modifying second order $\forall$: left

$$\frac{F(V)\Gamma\rightarrow\Delta}{\forall\phi F(\phi),\Gamma\rightarrow\Delta}$$

, as follows:

1.1.1. The system **INN′** (, which was called **SINN′** in [8, footnote 2]).

1.1.1.1. The principal formula $\forall\phi F(\phi)$ is isolated (Definition 27.2.(5), [PT, p. 322]),

or,

1.1.1.2. The abstract V in the auxiliary formula $F(V)$ is isolated.

1.1.2. The system **AII**.

1.1.2.1. The principal formula $\forall\phi F(\phi)$ is isolated,

or,

1.1.2.2. The abstract V in $F(V)$ is a second order free variable.

1.1.3. The system **AIΠ_1^1**.

1.1.3.1. The principal formula $\forall\phi F(\phi)$ is a Π_1^1-formula,

or,

1.1.3.2. The abstract V in $F(V)$ is a second order free variable.

AII (**AIΠ_1^1**) is an abbreviation of the Axiom of Instantiation $\forall\phi F(\phi)\supset F(V)$ with the Isolated formulae (Π_1^1-formulae) $\forall\phi F(\phi)$.

Observe that **AIΠ_1^1** contains (BI)+(Π_∞^0-CA), and (BI) contains **AIΠ_1^1**, hence **AIΠ_1^1**, (BI)+(Π_∞^0-CA) and (BI) are equivalent each other. Also note that a formula of the form,

$$\exists\phi\forall x_1\cdots x_n(\phi(x_1,\cdots,x_n)\equiv A(x_1,\cdots,x_n)),$$

is isolated provided that A is a Π_1^1-formula and the bound variable ϕ dose not occur in A. Hence the Π_1^1-comprehension axioms are derivable in **AII**, and so, **AII**, **INN′** and (Π_1^1-CA)+(BI) are equivalent each other.

In the rest of this section, we will give some preliminary definitions for a consistency proof of **AII**.

Following the idea of Takeuti, we add the rule of substitution to **AII**.

DEFINITION 1.2. Rule of substitution.

$$\frac{A_1,\cdots,A_n\rightarrow B_1,\cdots,B_m}{A_1\binom{\alpha}{V},\cdots,A_n\binom{\alpha}{V}\rightarrow B_1\binom{\alpha}{V},\cdots,B_m\binom{\alpha}{V}}$$

where α is a second order free variable, V is an arbitrary abstract with the same number of argument-places as α and $A_1, \cdots, A_n, B_1, \cdots, B_m$ are arbitrary formulae. Here α is called the eigenvariable of the substitution.

In what follows, a proof (-figure) will mean a proof tree which is locally correct with respect to the rules of **AII** and substitution.

DEFINITION 1.3. Let P be a proof of $\rightarrow$ and d a mapping (called an assignment of P) from the set of substitutions in P to the set of positive integers, where the value $d(J)$ is called the degree of J (with respect to d) for each substitution J in P.

We call the pair $\langle P, d\rangle$ a *proof with degree* if the following conditions are satisfied.

1.3.1. Every substitution is in the end-piece and there is no ind (induction rule) under a substitution.

1.3.2. Let A be a semi-formula in P. If we calculate the degree $d(A)$ of A by the following clauses 1.3.2.1.–1.3.2.4., then we have

$$d(B)<d(J)$$

for every substitution J in P and every formula B in the upper sequent of J.

1.3.2.1. $d(A)=\omega$ if A is not isolated.

Suppose A is isolated.

1.3.2.2. $d(A)=0$ if A contains no logical symbol.

1.3.2.3. $d(A)=d(B)$ if A is of the form $\neg B$;
$d(A)=\max\{d(A_1), d(A_2)\}$ if A is of the form $A_1\wedge A_2$;
$d(A)=d(B(x))$ if A is of the form $\forall x B(x)$,

1.3.2.4. $d(A)=\max\{d(F(\phi))+1, d(J)\}$ if A is of the form $\forall\phi F(\phi)$, where J ranges over substitutions which disturb $\forall\phi F(\phi)$.

DEFINITION 1.4.

1.4.1. Let A be a formula. We define the *grade of* A, denoted by $g(A)$, as follows:

1.4.1.1. $g(A)=0$ if A contains no logical symbol, or A is isolated and of the form $\forall\phi F(\phi)$.

1.4.1.2. $g(A)=g(B)+1$ if A is of the form $\neg B$.

1.4.1.3. $g(A)=\max\{g(B), g(C)\}+1$ if A is of the form $B\wedge C$.

1.4.1.4. $g(A)=g(B(0))+1$ if A is of the form $\forall x B(x)$.

1.4.1.5. $g(A)=g(F(\alpha))+1$ if A is not isolated and of the form $\forall\phi F(\phi)$.

1.4.2. A second order $\forall$: left is said to be isolated if the principal formula is isolated.

1.4.3. Let P be a proof and S a sequent in P. The *height of S in P*, denoted by $h(S;P)$ or simply $h(S)$, is defined inductively 'from below to above', as follows:

1.4.3.1. $h(S)=0$ if S is the end-sequent of P, or S is the upper sequent of a substitution in P.

1.4.3.2. $h(S)=h(S')$ if S is an upper sequent of an inference except substitution, cut, ind and isolated second order $\forall$: left, where S' is the lower sequent of the inference.

1.4.3.3. $h(S)=\max\{h(S'), g(D)\}$ if S is an upper sequent of cut, ind or isolated second order $\forall$: left, where D is the cut formula, induction formula or auxiliary formula of the inference, respectively, and S' is the lower sequent of the inference.

Next we will assign an o.d. from $O(\omega+1, 1)$ to a proof with degree. For simplicity, we write (i, μ) for a non-zero connected o.d. $(i, 0, \mu)$.

DEFINITION 1.5. For each o.d. μ from $O(\omega+1, 1)$ and natural number n, we define inductively an o.d. $\omega(n, \mu)$, as follows:

$$\omega(0, \mu)=\mu\,, \qquad \omega(n+1, \mu)=(\omega, \omega(n, \mu))\,.$$

DEFINITION 1.6. For each i such that $0\leq i<\omega$, we define two binary relations $\ll_i$ and $\underline{\ll}_i$ on the set of o.d.'s, as follows:

1.6.1. $\mu\ll_i\nu$ iff for each j such that $i\leq j\leq\omega$, $\mu<_j\nu$.

1.6.2. $\mu\underline{\ll}_i\nu$ iff $\mu\ll_i\nu$ or $\mu=\nu$.

By the definition, the following proposition is easily verified. (cf. Lemma 27.1, [PT], p. 320)

PROPOSITION 1.7.

1.7.1. $\mu\ll_i\nu$ implies $\omega(n, \mu)\ll_i\omega(n, \nu)$ and $\omega(n, \mu\#\theta)\ll_i\omega(n, \nu\#\theta)$ for every natural number n and every o.d. θ.

1.7.2. $\mu\ll_i\nu$ and $i\leq j<\omega$ imply $(j, \mu)\ll_i(j, \nu)$.

DEFINITION 1.8. Let $\langle P, d\rangle$ be a proof with degree. To each sequent S and each line (Schlussstrich) of an inference J in P, we will assign o.d.'s, denoted by $O(S;P,d)$ and $O(J;P,d)$, or simply $O(S)$ and $O(J)$, from $O(\omega+1, 1)$ inductively 'from above to below', as follows:

1.8.1. $O(S)=0$ if S is an initial sequent of P.

Suppose that the o.d.'s of the upper sequents of an inference J have been assigned. And let J be of the form:

$$\frac{S' \ (S'')}{S} J$$

The o. d.'s $O(J)$ and $O(S)$ are then determined, as follows:

1.8.2. If J is a weak structural inference, then $O(J)$ is $O(S')$.

1.8.3. If J is a logical inference except $\wedge$: right and isolated second order $\forall$: left, then $O(J)$ is $O(S')\#0$.

1.8.4. If J is a $\wedge$: right or cut, then $O(J)$ is $O(S')\#O(S'')$.

1.8.5. If J is an isolated second order $\forall$: left, then $O(J)$ is $(\omega, 0)\#O(S')$.

1.8.6. If J is an ind or substitution, then $O(J)$ is $(\omega, O(S'))$.

1.8.7. If J is not a substitution, then $O(S)$ is

$$\omega(h(S')-h(S), O(J)).$$

1.8.8. If J is a substitution, then $O(S)$ is $(d(J), O(J))$.

And the o. d. $O(P, d)$ of $\langle P, d\rangle$ is defined to be $(\omega, O(S; P, d))$ where S is the end-sequent of P.

The preliminary definitions have finished and now we can state the following main lemma.

Main lemma. If $\langle P, d\rangle$ is a proof with degree, then we can construct another proof with degree $\langle P', d'\rangle$ such that:

$$O(P', d') <_0 O(P, d),$$

and in fact,

$$O(P', d') \ll_0 O(P, d).$$

Assume that the main lemma has been proved finitistically. Since for any proof P of $\rightarrow$ in **AII** and the empty assignment ϕ, $\langle P, \phi\rangle$ is a proof with degree, the consistency of **AII** will follow from the accessibility of the system $O(\omega+1, 1)$ with respect to $<_0$.

A proof of the main lemma will be given in the next section.

§ 2. Proof of the Main Lemma.

The reduction step from $\langle P, d\rangle$ to $\langle P', d'\rangle$ is almost the same as in [PT]. Up to (3) in [PT], p. 328, the reduction steps are completely the same as in [PT], i. e., (1) substitution the individual constant 0 for redundant first order free variables, (2) '*VJ* Reduktion' in [2] and (3) eliminating equality axioms in the end-piece of P.

(4) By virtue of the above, we may assume that there are no applications of ind and no equality axioms as initial sequents in the end-piece of P. Suppose that the end-piece of P contains logical initial sequents.

Suppse P is of the following form and $D \to D$ is one of the initial sequents in the end-piece of P:

$$
P_0 \left\{ \begin{array}{cc} \vdots & D \xrightarrow{0} D \\ & \vdots \\ \Gamma \xrightarrow{\mu} \Delta, D' & D', \Pi \xrightarrow{\nu} \Lambda_1, D', \Lambda_2 \end{array} \right. \quad m
$$
$$
\frac{}{\Gamma, \Pi \xrightarrow{\omega(m-n,\, \mu \# \nu)} \Delta, \Lambda_1, D', \Lambda_2} \quad n
$$
$$
\begin{array}{c} \vdots \\ \longrightarrow \end{array}
$$

where m is $h(\Gamma \to \Delta, D'; P)$ and μ is $\mathrm{O}(\Gamma \to \Delta, D'; P, d)$, etc.

We reduce P to the following P' (and d' is defined to be the restriction of d to P'):

$$
P_0 \left\{ \begin{array}{c} \vdots \\ \Gamma \xrightarrow{\mu'} \Delta, D' \end{array} \right. \quad n
$$
$$
\frac{}{\Gamma, \Pi \longrightarrow \Delta, \Lambda_1, D', \Lambda_2} \quad n
$$
$$
\begin{array}{c} \vdots \\ \longrightarrow \end{array}
$$

We see easily that for every sequent S in P_0

$$\mathrm{O}(S; P', d') \underline{\ll}_0\, \omega(h(S; P) - h(S; P'), \mathrm{O}(S; P, d))$$

in particular $\mu' \underline{\ll}_0\, \omega(m-n, \mu)$, and so $\mu' \ll_0 \omega(m-n, \mu \# \nu)$.

Thus by proposition 1.7, we have $\mathrm{O}(P', d') \ll_0 \mathrm{O}(P, d)$.

REMARK. If we would use a 'potential' in [3] instead of the height of a sequent, then we could simplify the calculation of the o. d.'s in this and the next cases.

(5) We assume besides the conditions in (4) that the end-piece of P contains no logical initial sequents. Then let P^* be the proof obtained from P by eliminating weakenings in the end-piece of P and d^* be the restriction of d for P^*. Similarly in the case (4) we have $\mathrm{O}(P^*, d^*) \underline{\ll}_0\, \mathrm{O}(P, d)$.

(6) Suppose that the end-piece of P contains neither ind, weakening nor axiom other than mathematical ones. Then the end-piece of P contains a suitable cut J. (cf. Sublemma 12.9., [PT], p. 105)

(7) The case where the cut formula of J is of the form $\forall \phi F(\phi)$.

Case 1. $\forall \phi F(\phi)$ is isolated.

Let P be the following form:

$$
\begin{array}{c}
\dfrac{\begin{array}{c}\vdots\\ \Gamma_1 \xrightarrow{\lambda} \Delta_1, F(\alpha)\end{array}}{\begin{array}{c}\Gamma_1 \xrightarrow{\lambda\#0} \Delta_1, \forall\phi F(\phi)\\ \vdots\\ \Gamma_2 \xrightarrow{\tau} \Delta_2, \forall\phi F(\phi)\end{array}}
\quad
\dfrac{\begin{array}{c}\vdots\\ F(V), \Pi_1 \xrightarrow[(\omega,0)\#\mu]{\mu} \Lambda_1\end{array}}{\begin{array}{c}\forall\phi F(\phi), \Pi_1 \longrightarrow \Lambda_1\\ \vdots\\ \forall\phi F(\phi), \Pi_2 \xrightarrow{\rho} \Lambda_2\end{array}}\\
\hline
J \quad \Gamma_2, \Pi_2 \longrightarrow \Delta_2, \Lambda_2\\
\vdots\\
\Gamma_3 \xrightarrow[(\omega,\nu)]{\nu} \Delta_3\\
\vdots\\
\xrightarrow[(\omega,\sigma)]{\sigma}
\end{array}
$$

where $\Gamma_3 \rightarrow \Delta_3$ is the i-resolvent of $\Gamma_2, \Pi_2 \rightarrow \Delta_2, \Lambda_2$, i being $d(\forall\phi F(\phi))$. $(\omega, 0)\#\mu$ is the o. d. of the line of the inference which is the right boundary inference, etc. Let P' be the following:

$$
\begin{array}{c}
\dfrac{\begin{array}{c}\vdots\\ \Gamma_1 \xrightarrow{\lambda} \Delta_1, F(\alpha)\end{array}}{\begin{array}{c}\Gamma_1 \longrightarrow F(\alpha), \Delta_1, \forall\phi F(\phi)\\ \vdots\\ \Gamma_2 \xrightarrow{\tau'} F(\alpha), \Delta_2, \forall\phi F(\phi)\end{array}}\ (\text{double line})
\quad
\dfrac{\begin{array}{c}\vdots\\ F(V), \Pi_1 \xrightarrow[(\omega,0)\#\mu]{\mu} \Lambda_1\end{array}}{\begin{array}{c}\forall\phi F(\phi), \Pi_1 \longrightarrow \Lambda_1\\ \vdots\\ \forall\phi F(\phi), \Pi_2 \xrightarrow{\rho} \Lambda_2\end{array}}\\
\hline
\Gamma_2, \Pi_2 \longrightarrow F(\alpha), \Delta_2, \Lambda_2\\
\vdots
\end{array}
$$

$$
\begin{array}{c}
\dfrac{\begin{array}{c}\vdots\\ \Gamma_1 \longrightarrow \Delta_1, F(\alpha)\end{array}}{\begin{array}{c}\Gamma_1 \longrightarrow \Delta_1, \forall\phi F(\phi)\\ \vdots\\ \Gamma_2 \xrightarrow{\tau} \Delta_2, \forall\phi F(\phi)\end{array}}
\quad
\dfrac{\dfrac{J_1\ \dfrac{\Gamma_3 \xrightarrow[(\omega,\theta)]{\theta} \Delta_3, F(\alpha)}{\Gamma_3 \xrightarrow{(i,(\omega,\theta))} \Delta_3, F(V)} \quad \begin{array}{c}\vdots\\ F(V), \Pi_1 \xrightarrow{\mu} \Lambda_1\end{array}}{\Gamma_3, \Pi_1 \longrightarrow \Delta_3, \Lambda_1}\ (i,(\omega,0))\#\mu}{\begin{array}{c}\forall\phi F(\phi), \Pi_1, \Gamma_3 \longrightarrow \Delta_3, \Lambda_1\\ \vdots\\ \forall\phi F(\phi), \Pi_2, \Gamma_3 \xrightarrow{\rho'} \Delta_3, \Lambda_2\end{array}}\ (\text{double line})\\
\hline
\Gamma_2, \Pi_2, \Gamma_3 \longrightarrow \Delta_2, \Delta_3, \Lambda_2\\
\hline\hline
\Gamma_2, \Pi_2, \Gamma_3 \longrightarrow \Delta_3, \Delta_2, \Lambda_2\\
\vdots\\
\Gamma_3, \Gamma_3 \longrightarrow \Delta_3, \Delta_3\\
\hline\hline
\Gamma_3 \xrightarrow[(\omega,\nu')]{\nu'} \Delta_3\\
\vdots\\
\xrightarrow[(\omega,\sigma')]{\sigma'}
\end{array}
$$

where J_1 is a substitution with the eigenvariable α. $d'(J')$ for a substitution J' except J_1 is defined to be $d(J'')$ where J'' is the corresponding substitution to J' in P. $d'(J_1)$ is defined to be i.

Following propositions (7.1)-(7.6) are easily verified by proposition 1.7. (cf. [PT], pp. 332-333):

$$\lambda \ll_0 \lambda \# 0\,, \tag{7.1}$$

$$\tau' \ll_0 \tau\,, \tag{7.2}$$

$$(\omega, \theta) \ll_0 (\omega, \nu)\,, \tag{7.3}$$

$$(i, (\omega, \theta)) \# \mu \ll_{i+1} (\omega, 0) \# \mu\,, \tag{7.4}$$

$$\rho' \ll_{i+1} \rho\,, \tag{7.5}$$

$$(\omega, \nu') \ll_0 (\omega, \nu)\,. \tag{7.6}$$

It follows from (7.6) that $(\omega, \sigma') \ll_0 (\omega, \sigma)$.

Case 2. $\forall\phi F(\phi)$ is not isolated.

Let P be the following form:

$$
\begin{array}{c}
\begin{array}{cc}
\vdots & \vdots \\
\dfrac{\Gamma_1 \longrightarrow \Delta_1, F(\alpha)}{\Gamma_1 \longrightarrow \Delta_1, \forall\phi F(\phi)} & \dfrac{F(\beta), \Pi_1 \longrightarrow \Lambda_1}{\forall\phi F(\phi), \Pi_1 \longrightarrow \Lambda_1} \\
\vdots & \vdots
\end{array} \\
J\,\dfrac{\Gamma_2 \longrightarrow \Delta_2, \forall\phi F(\phi) \qquad \forall\phi F(\phi), \Pi_2 \longrightarrow \Lambda_2}{\Gamma_2, \Pi_2 \longrightarrow \Delta_2, \Lambda_2}\; m \\
\vdots \\
\Phi \longrightarrow \Psi \\
\vdots \\
\longrightarrow
\end{array}
$$

where $\Phi \rightarrow \Psi$ denotes the uppermost sequent below J whose height is less than m. Let P' be the following:

$$
\begin{array}{c}
\begin{array}{cc}
\begin{array}{c}
\vdots \\
\dfrac{\Gamma_1 \rightarrow \Delta_1, F(\beta)}{\Gamma_1 \rightarrow F(\beta), \Delta_1, \forall\phi F(\phi)} \\
\vdots \\
\dfrac{\Gamma_2 \rightarrow F(\beta), \Delta_2, \forall\phi F(\phi) \quad \forall\phi F(\phi), \Pi_2 \rightarrow \Lambda_2}{\Gamma_2, \Pi_2 \rightarrow F(\beta), \Delta_2, \Lambda_2} \\
\vdots \\
\dfrac{\Phi \rightarrow F(\beta), \Psi}{\Phi \rightarrow \Psi, F(\beta)}
\end{array}
&
\begin{array}{c}
\vdots \\
\dfrac{F(\beta), \Pi_1 \rightarrow \Lambda_1}{\forall\phi F(\phi), \Pi_1, F(\beta) \rightarrow \Lambda_1} \\
\vdots \\
\dfrac{\Gamma_2 \rightarrow \Delta_2, \forall\phi F(\phi) \quad \forall\phi F(\phi), \Pi_2, F(\beta) \rightarrow \Lambda_2}{\Gamma_2, \Pi_2, F(\beta) \rightarrow \Delta_2, \Lambda_2}\; m \\
\vdots \\
\dfrac{\Phi, F(\beta) \rightarrow \Psi}{F(\beta), \Phi \rightarrow \Psi}
\end{array}
\end{array} \\
\dfrac{}{\dfrac{\Phi, \Phi \rightarrow \Psi, \Psi}{\Phi \rightarrow \Psi}}\; n \\
\vdots \\
\rightarrow
\end{array}
$$

And for every substitution J' in P', $d'(J')$ is defined to be $d(J'')$ where J'' is the corresponding substitution to J' in P.

From $n<m$ we see easily that $O(P', d')<_0 O(P, d)$.

(8) The cases where the cut formula of J is of the form $F_1 \wedge F_2$, $\neg F$ or $\forall x F(x)$ are treated in the same way as the Case 2. in (7).

This completes the proof of the main lemma.

REMARK. A consistency proof of **AIΠ**$_1^1$ by the accessibility of the system O(2, 1) with respect to $<_0$ can be given similarly for the above consistency proof of **AII**.

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