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JUSTIFICATION OF PARTIALLY-MULTIPLICATIVE AVERAGING FOR A CLASS OF FUNCTIONAL-DIFFERENTIAL EQUATIONS WITH VARIABLE STRUCTURE AND IMPULSES

By

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1. Introduction

The averaging method of Bogoljubov-Mitropol'skij has been recognized as one of the most efficient mathematical methods in the nonlinear mechanics, cf. e. g. [1]-[2]. The generalization of the averaging method for asymptotic integration of systems of differential equations with impulses is substantiated by the following reasons:

—due to their complex structure, the qualitative investigation of these systems is subject to great difficulties while the averaged system is without impulse action;

—the solution of the averaged system approximates the solution of the original system with any prescribed accuracy on an asymptotically large time-interval.

This paper presents a justification of the method of partially multiplicative averaging for a class of functional-differential equations which, beside the impulse action, is of variable structure. For related papers, see [3]-[4] and their references, where the averaging method has been justified for certain other classes of functional-differential equations with impulses.

2. Statement of the Problem

We are given the following:

a) a set of hypersurfaces

$$\sigma_i:\ t=t_i(x)\,,\quad i=1, 2, \cdots,$$

which for $x\in D\subset R^n$ lie in the half space $t>0$ of the $(n+1)$-dimensional space (t, x) and satisfy the condition

$$t_i(x)<t_{i+1}(x)\,,\quad i=1, 2, \cdots;$$

b) a set of function $\Phi_i(t, x)$, $i=1, 2, \cdots$ defined for all points (t, x) belonging

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to the hypersurfaces σ_i, respectively;

c) a set of ordered pairs of matrix functions $A_i^{(1)}(t, x, y, z)$, $A_i^{(2)}(t, x, y, z)$, $i=0, 1, 2, \cdots$ defined in the domain $\{t\geq 0, x, y\in D, z\in D_1\subset R^n\}$;

d) a set of vector-functions $I_i(x)$, $i=1, 2, \cdots$ defined in the domain D;

Let a mapping point P_t with current coordinates $(t, x(t))$ move in the domain $\{t\geq 0, x\in D\}$. We assume that motion of the point P_t is governed by a law characterized by:

e) a system of differential equations of neutral type

$$\left.\begin{aligned}&\dot{x}(t)=\varepsilon A(t, x(t), x(\Delta(t, x(t))), \dot{x}(\Delta(t, x(t))))X(t, x(t)), \quad t>0, \quad t\neq t_i(x),\\ &x(t)=\varphi(t, \varepsilon), \quad t\in[-\delta, 0],\\ &\dot{x}(t)=\dot{\varphi}(t, \varepsilon), \quad t\in[-\delta, 0],\end{aligned}\right\} \tag{1}$$

where $\varepsilon>0$ is a small parameter, $A(t, x, y, z)=(a_{ij}(t, x, y, z))_{n,m}$, δ is a positive constant, $\Delta(t, x)$ is a transformed argument satisfying the condition

$$t-\delta\leq\Delta(t, x)\leq t \tag{2}$$

in the domain $\{t\geq-\delta, x\in D\}$, $\varphi(t, \varepsilon)$ is an initial value function defined and differentiable with respect to t in the domain $\{t\in[-\delta, 0], \varepsilon\in(0, \mathscr{E}], \mathscr{E}=\text{const}>0\}$.

The motion itself can be described as follows: Departing from the point $(t=\tau_0=0, x_0=\varphi(0, \varepsilon))$ the point P_t moves along the trajectory $(t, x(t))$ governed by the solution $x(t)$ of system (1), where $A(t, x, y, z)=A_0^{(1)}(t, x, y, z)=A_0^{(2)}(t, x, y, z)$ untill the moment $\tau_1>0$ at which the trajectory $(t, x(t))$ meets the hypersurface σ_1 at the point $(\tau_1, x_1^-=x(\tau_1))$. Then the point P_t instantly moves from the position (τ_1, x_1^-) to the position $(\tau_1, x_1^+=x_1^-+\varepsilon I_1(x_1^-))$ and further on follows the trajectory $(t, x(t))$, described by the solution $x(t)$ of system (1) where

$$A(t, x, y, z)=\begin{cases}A_1^{(1)}(t, x, y, z) & \text{if } \Phi_1(\tau_1, x_1^-)\geq 0,\\ A_1^{(2)}(t, x, y, z) & \text{if } \Phi_1(\tau_1, x_1^-)<0,\end{cases}$$

untill it meets the hypersurface σ_2, etc.

We suppose that the function $x(t)$ satisfies the agreement condition at the moment $\tau_0=0$

$$x(0+0)=x_0=\varphi(0, \varepsilon). \tag{3}$$

At the next points τ_i, $x(t)$ has first kind discontinuity and is continuous on the left satisfying the agreement condition

$$x(\tau_i+0)=x_i^+, \tag{4}$$

where

$$x_i^+=x_i^-+\varepsilon I_i(x_i^-), \quad i=1, 2, \cdots. \tag{5}$$

Note that the point (τ_i, x_i^+) does not necessarily belong to the hypersurface σ_i, $i=1, 2, \cdots$.

The relations a)–d) and e) in which the matrix-function A of equation (1) has the form

$$A(t, x, y, z)=\begin{cases} A_0^{(1)}(t, x, y, z)=A_0^{(2)}(t, x, y, z) \text{ for } \tau_0<t\leq\tau_1\,, \\ A_i^{(1)}(t, x, y, z) \text{ for } \tau_i<t\leq\tau_{i+1} \quad \text{if } \Phi_i(\tau_i, x_i^-)\geq 0\,, \\ A_i^{(2)}(t, x, y, z) \text{ for } \tau_i<t\leq\tau_{i+1} \quad \text{if } \Phi_i(\tau_i, x_i^-)<0\,, \end{cases} \tag{6}$$

will be called *a system of functional-differential equations with variable structure and with impulses* and will be denoted shortly by system (1). The curve described by the point P_t during its motion will be called an integral curve or a trajectory of this system in the space (t, x).

Suppose that the following limits exist

$$\left.\begin{aligned} &\lim_{T\to\infty}\frac{1}{T}\int_t^{t+T} A(\theta, x, x, 0)d\theta=A_0(x)\,, \\ &\lim_{T\to\infty}\frac{1}{T}\sum_{t<t_i<t+T} I_i(x)=I_0(x)\,. \end{aligned}\right\} \tag{7}$$

Then we juxtapose to (1) the following averaged system of ordinary differential equations

$$\dot{\bar{x}}(t)=\varepsilon[A_0(\bar{x}(t))X(t, \bar{x}(t))+I_0(\bar{x}(t))] \tag{8}$$

with initial condition

$$\bar{x}(0)=x_0\,. \tag{9}$$

We note that if $x=(x_1, \cdots, x_n)$ and $A=(a_{ij})_{nm}$, then by definition

$$\|x\|=\left[\sum_{i=1}^n x_i^2\right]^{1/2}, \qquad \|A\|=\left[\sum_{i=1}^n\sum_{j=1}^m a_{ij}^2\right]^{1/2}.$$

We denote by $\overline{1, n}$ the set of natural numbers $\{1, 2, \cdots, n\}$.

3. Main Result

The following theorem for closeness of the solutions of the systems (1) and (8), (9) holds true:

THEOREM 1. *Suppose the following conditions hold:*

1. *The functions $t_i(x)$ are twice continuous differentiable in the domain $D\subset R^n$, they are positive and satisfy the condition $t_i(x)<t_{i+1}(x)$, $i=1, 2, \cdots$. The function $\Phi_i(t, x)$ is defined on the hypersurface σ_i for $x\in D$, $i=1, 2, \cdots$. The function $A_i^{(k)}(t, x, y, z)$ is continuous in the domain $\{t_i(x)\leq t\leq t_{i+1}(x), x, y\in D, z\in D_1\subset R^n\}$; $t_0(x)$*

$\equiv 0$; $k=1,2$; $i=0,1,2,\cdots$. *The function* $\Delta(t, x)$ *is continuous and satisfies the condition* $t-\delta\leq\Delta(t, x)\leq t$, $\delta=\text{const}>0$ *in the domain* $\{t\geq 0, x\in D\}$. *The functions* $\varphi(t, \varepsilon)$ *and* $\dot{\varphi}(t, \varepsilon)$ *are continuous in the domain* $\{t\in[-\delta, 0], \varepsilon\in(0, \mathcal{E}], \mathcal{E}=\text{const}>0\}$ *and* $\varphi(t, \varepsilon)\in D$, $\dot{\varphi}(t, \varepsilon)\in D_1$. *The functions* $I_i(x)$, $i=1,2,\cdots$ *are continuous in* D.

2. *There exist positive constants* M, K, C *and a function* $\gamma(\varepsilon)$, *such that*

$$\left\|\frac{\partial t_i(x)}{\partial x}\right\|+\|A_i^{(k)}(t, x, y, z)\|+\|X(t, x)\|+\|I_i(x)\|\leq M,$$
$$\|A_i^{(k)}(t, x, y, z)-A_i^{(k)}(t, x', y', z')\|\leq K(\|x-x'\|+\|y-y'\|+\|z-z'\|),$$
$$\|X(t, x)-X(t, x')\|+\|I_i(x)-I_i(x')\|\leq K\|x-x'\|, \left\|\frac{\partial^2 t_i(x)}{\partial x^2}\right\|\leq C$$

for all $t\geq 0$, $x, x', y, y'\in D$, $z, z'\in D_1$, $k=1,2$, $i=1,2,\cdots$ *and* $\|\dot{\varphi}(t, \varepsilon)\|\leq\gamma(\varepsilon)$ *for* $\varepsilon\in(0, \mathcal{E}]$, *where* $\lim\limits_{\varepsilon\to 0}\frac{\gamma(\varepsilon)}{\varepsilon}=\text{const}>0$ *and* $\sup\limits_{\varepsilon\in(0,\mathcal{E}]}\frac{\gamma(\varepsilon)}{\varepsilon}=\text{const}>0$.

3. *The limits* (7) *exist uniformly in* $t\geq 0$ *and* $x\in D$ *and*

$$\lim_{T\to\infty}\frac{1}{T}\sum_{t<t_i<t+T}1=d, \qquad d=\text{const}>0.$$

4. *The functions* $a_{(i)jl}^{(k)}(t, x, y, z)-a_{jl}^{(0)}(x)$, $j=\overline{1, n}$, $l=\overline{1, m}$, $k=1,2$, $i=1,2,\cdots$, *where* $a_{(i)jl}^{(k)}(t, x, y, z)$ *and* $a_{jl}^{(0)}(x)$ *are elements of the matrices* $A_i^{(k)}(t, x, y, z)$ *and* $A_0(x)$, *respectively, do not change their sign in the entire domain* $\{t\geq 0, x, y\in D, z\in D_1\}$, *that is either* $a_{(i)jl}^{(k)}(t, x, y, z)-a_{jl}^{(0)}(x)\geq 0$ *or* $a_{(i)jl}^{(k)}(t, x, y, z)-a_{jl}^{(0)}(x)\leq 0$ *in this domain.*

5. *For every* $\varepsilon\in(0, \mathcal{E}]$ *the system of functional-differential equations* (1) *with variable structure and with impulses has continuous solution* $x(t)$ *for* $t>0$ *and* $t\neq\tau_i$ *which satisfies the agreement condition* (3) *and* (4).

6. *For every* $\varepsilon\in(0, \mathcal{E}]$ *the averaged system* (8) *with initial condition* (9) *has a solution* $\bar{x}(t)$ *which belongs to the domain* D *for* $t\geq 0$ *together with its neighbourhood of radius* $\rho=\text{const}>0$ *and satisfies the inequalities*

$$\frac{\partial t_i(\bar{x}(t))}{\partial x}I_i(\bar{x}(t))\leq\beta<0, \quad \beta=\text{const},$$
$$t\in(t_i', t_i''), \quad t_i'=\inf_{x\in D}t_i(x), \quad t_i''=\sup_{x\in D}t_i(x), \quad i=1,2,\cdots,$$

or $\frac{\partial t_i(x)}{\partial x}\equiv 0$, *i.e.* σ_i *is a hyperplane.*

Then for each $\eta>0$ *and* $L>0$ *there exists* $\varepsilon_0\in(0, \mathcal{E}]$ $(\varepsilon_0=\varepsilon_0(\eta, L))$ *such that for* $\varepsilon\leq\varepsilon_0$ *the inequality*

$$\|x(t)-\bar{x}(t)\|<\eta$$

holds for $0\leq t\leq L\varepsilon^{-1}$.

LEMMA 1. *Suppose that the conditions of Theorem* 1 *hold. Let* T *be a suffi-*

ciently large number. Then for every natural number $p\geq 1$ *the following inequality is fulfilled*

$$\|x(pT)-\bar{x}(pT)\|\leq\varepsilon\sum_{i=0}^{p-1}[1+\varepsilon(3M+d)KT]^i[\alpha(T)T+\varepsilon\bar{M}], \tag{10}$$

where $\bar{M}=(M+d)(3M+d)KMT^2+\max_{i=\overline{1,p}} M_i$ and $M_i=M_i(T, d_1, \cdots d_i)$ are constants depending on T and on the constants d_j, $j=\overline{1,i}$, $\alpha(T)$ and d_j being defined later.

PROOF OF LEMMA 1. The condition 3 of Theorem 1 implies that there exists a function $\alpha(t)$, which is monotonely decreasing to zero for t tending to infinity, such that for all $t\geq 0$ and $x\in D$ the following inequalities hold:

$$\left.\begin{aligned}&\left\|\int_t^{t+T}[A(\theta, x, x, 0)-A_0(x)]d\theta\right\|\leq\alpha(T)T/2,\\&\Big\|\sum_{t<t_i<t+T}I_i(x)-I_0(x)T\Big\|\leq\alpha(T)T/2.\end{aligned}\right\} \tag{11}$$

In the proof of Lemma 1 we use the method of full mathematical induction.

We prove the inequality (11) for $p=1$.

Consider the system (1) in the interval $[0, T]$. Let d_1 points

$$t_1(x_0)=t_1^{(0)},\quad \cdots,\quad t_{d_1}(x_0)=t_{d_1}^{(0)}$$

lie in the interval $(0, T)$ such that $t_i^{(0)}<t_{i+1}^{(0)}$ for $i=\overline{1,(d_1-1)}$.

We denote as $x_0^{(0)}(t, 0, x_0)$ the solution of the system

$$x_0^{(0)}(t, 0, x_0)=\begin{cases}x_0+\varepsilon\displaystyle\int_0^t A_0^{(1)}(\theta, x_0^{(0)}(\theta, 0, x_0), x_0^{(0)}(\varDelta_0^{(0)}(\theta), 0, x_0), \dot{x}_0^{(0)}(\varDelta_0^{(0)}(\theta), 0, x_0))d\theta, & t>0,\\ \varphi(t, \varepsilon), \quad -\delta\leq t\leq 0,\end{cases}$$

$$\dot{x}_0^{(0)}(t, 0, x_0)=\dot{\varphi}(t, \varepsilon), \quad -\delta\leq t\leq 0, \tag{12}$$

where $\varDelta_0^{(0)}(t)=\varDelta(t, x_0^{(0)}(t, 0, x_0))$.

The solution of (12) coincides with the solution $x(t)$ of the system (1) till the moment τ_1, at which the trajectory of this system meets the hypersurface δ_1, i.e. $x(t)=x_0^{(0)}(t, 0, x_0)$ for $t\in[-\delta, \tau_1]$.

Consider the function

$$\bar{x}_0^{(0)}(t, 0, x_0)=x_0+\varepsilon\int_0^t A_0^{(1)}(\theta, x_0, x_0, 0)X(\theta, x_0)d\theta.$$

Let us estimate the norm of the difference

$$R_0^{(0)}(t, 0, x_0, \varepsilon)=x_0^{(0)}(t, 0, x_0)-\bar{x}_0^{(0)}(t, 0, x_0).$$

For $0<t\leq T$ we have

$$\|R_0^{(0)}(t, 0, x_0, \varepsilon)\| \leq \varepsilon \int_0^t \|A_0^{(1)}(\theta, x_0^{(0)}(\theta, 0, x_0), x_0^{(0)}(\Delta_0^{(0)}(\theta), 0, x_0),$$
$$\dot{x}_0^{(0)}(\Delta_0^{(0)}(\theta), 0, x_0))X(\theta, x_0^{(0)}(\theta, 0, x_0)) - A_0(\theta, x_0, x_0, 0)X(\theta, x_0)\|d\theta$$
$$\leq \varepsilon \int_0^t \{\|A_0^{(1)}(\theta, x_0^{(0)}(\theta, 0, x_0), x_0^{(0)}(\Delta_0^{(0)}(\theta), 0, x_0), \dot{x}_0^{(0)}(\Delta_0^{(0)}(\theta), 0, x_0))$$
$$-A(\theta, x_0, x_0, 0)\| . \|X(\theta, x_0^{(0)}(\theta, 0, x_0))\| + \|A(\theta, x_0, x_0, 0)\| .$$
$$\|X(\theta, x_0^{(0)}(\theta, 0, x_0)) - X(\theta, x_0)\|\}d\theta \leq \varepsilon KM \int_0^t \{2\|x_0^{(0)}(\theta, 0, x_0) - x_0\|$$
$$+\|x_0^{(0)}(\Delta_0^{(0)}(\theta), 0, x_0) - x_0\| + \|\dot{x}_0^{(0)}(\Delta_0^{(0)}(\theta), 0, x_0)\|\}d\theta$$
$$\leq 2\varepsilon^2 KM \int_0^t d\theta \int_0^\theta \|A_0^{(1)}(l, x_0^{(0)}(l, 0, x_0), x_0^{(0)}(\Delta_0^{(0)}(l), 0, x_0),$$
$$\dot{x}_0^{(0)}(\Delta_0^{(0)}(l), 0, x_0))\| . \|X(l, x_0^{(0)}(l, 0, x_0))\|dl$$
$$+\varepsilon KM \Big\{ \int_{J_{0,t}^-} \|\varphi(\Delta_0^{(0)}(\theta), \varepsilon) - \varphi(0, \varepsilon)\|d\theta + \varepsilon \int_{J_{0,t}^+} d\theta \int_0^{\Delta_0^{(0)}(\theta)} \|A_0^{(1)}(l, x_0^{(0)}(l, 0, x_0),$$
$$x_0^{(0)}(\Delta_0^{(0)}(l), 0, x_0), \dot{x}_0^{(0)}(\Delta_0^{(0)}(l), 0, x_0))\| . \|X(l, x_0^{(0)}(l, 0, x_0))\|dl \Big\}$$
$$+\varepsilon KM \Big\{ \int_{J_{0,t}^-} \|\dot{\varphi}(\Delta_0^{(0)}(\theta), \varepsilon)\|d\theta + \varepsilon \int_{J_{0,t}^+} \|A_0^{(1)}(\Delta_0^{(0)}(\theta), x_0^{(0)}(\Delta_0^{(0)}(\theta), 0, x_0),$$
$$x_0^{(0)}(\Delta_0^{(0)}(\Delta_0^{(0)}(\theta), 0, x_0), \dot{x}_0^{(0)}(\Delta_0^{(0)}(\Delta_0^{(0)}(\theta)), 0, x_0))\| .$$
$$\|X(\Delta_0^{(0)}(\theta), x_0^{(0)}(\Delta_0^{(0)}(\theta), 0, x_0))\|d\theta \Big\} \leq 2\varepsilon^2 KM^3 \int_0^t d\theta \int_0^\theta dl$$
$$+\varepsilon\gamma(\varepsilon)\sqrt{n}\, KM \int_{J_{0,t}^-} |\Delta_0^{(0)}(\theta)|d\theta + \varepsilon^2 KM^3 \int_{J_{0,t}^+} \Delta_0^{(0)}(\theta)d\theta$$
$$+\varepsilon\gamma(\varepsilon)KM \int_{J_{0,t}^-} d\theta + \varepsilon^2 KM^3 \int_{J_{0,t}^+} d\theta \leq \varepsilon^2 KM^3 T^2$$
$$+\varepsilon\gamma(\varepsilon)(\delta\sqrt{n}+1)KM \int_0^t d\theta + \varepsilon^2 KM^3 \int_0^t \theta d\theta + \varepsilon^2 KM^3 \int_0^t d\theta$$
$$\leq 3\varepsilon^2 KM^3 T^2/2 + \varepsilon\gamma(\varepsilon)(\delta\sqrt{n}+1)KMT + \varepsilon^2 KM^3 T \equiv \omega_0^{(0)}(\varepsilon^2, T),$$

where

$$J_{0,t}^- \cup J_{0,t}^+ = (0, t],$$
$$J_{0,t}^- = \{\theta : \theta \in (0, t] \wedge \Delta_0^{(0)}(\theta) \in [-\delta, 0]\},$$
$$J_{0,t}^+ = (0, t] \setminus J_{0,t}^- .$$

The obtained estimate shows that the function $\tilde{x}_0^{(0)}(t, 0, x_0)$ approximates the solution $x_0^{(0)}(t, 0, x_0)$ of system (12) in the interval $(0, T]$ with accuracy of order ε^2.

The moment τ_1 at which the trajectory $(t, x(t))$ meets the hypersurface σ_1 is a solution of the equation

$$t = t_1(x_0^{(0)}(t, 0, x_0)) . \tag{13}$$

Since

$$
\begin{aligned}
t_1(x_0^{(0)}(t,0,x_0) &= t_1(\tilde{x}_0^{(0)}(t,0,x_0)+R_0^{(0)}(t,0,x_0,\varepsilon)) \\
&= t_1\left(x_0+\varepsilon\int_0^t A_0^{(1)}(\theta,x_0,x_0,0)X(\theta,x_0)d\theta+O(\varepsilon^2)\right) \\
&= t_1(x_0)+\varepsilon\frac{\partial t_1(x_0)}{\partial x}\int_0^t A_0^{(1)}(\theta,x_0,x_0,0)X(\theta,x_0)d\theta+O(\varepsilon^2) \\
&= t_1^{(0)}+\varepsilon\frac{\partial t_1(x_0)}{\partial x}\int_0^{t_1^{(0)}} A_0^{(1)}(\theta,x_0,x_0,0)X(\theta,x_0)d\theta \\
&\quad +\varepsilon\frac{\partial t_1(x_0)}{\partial x}\int_{t_1^{(0)}}^t A_0^{(1)}(\theta,x_0,x_0,0)X(\theta,x_0)d\theta+O(\varepsilon^2) \\
&= t_1^{(0)}+\varepsilon\frac{\partial t_1(x_0)}{\partial x}\int_0^{t_1^{(0)}} A_0^{(1)}(\theta,x_0,x_0,0)X(\theta,x_0)d\theta \\
&\quad +\varepsilon\frac{\partial t_1(x_0)}{\partial x}(t-t_1^{(0)})A_0^{(1)}(\tilde{t},x_0,x_0,0)X(\tilde{t},x_0)+O(\varepsilon^2)\,, \qquad (14) \\
&\quad \tilde{t}=t_1^{(0)}+\mu(t-t_1^{(0)})\,, \qquad 0\le\mu\le 1\,,
\end{aligned}
$$

then from (13) it follows that $\tau_1=t_1^{(0)}+\varepsilon\Theta_1^{(0)}+O(\varepsilon^2)$, where

$$\Theta_1^{(0)}=\frac{\partial t_1(x_0)}{\partial x}\int_0^{t_1^{(0)}} A_0^{(1)}(\theta,x_0,x_0,0)X(\theta,x_0)d\theta\,.$$

We note that in (14) the values of the constant μ are different in general for the different components of the vector $A_0^{(1)}(\tilde{t},x_0,x_0,0)X(\tilde{t},x_0)$.

From the inequality $t_1^{(0)}>0$ it follows that $\tau_1>\tau_0$ under the condition that ε is sufficiently small.

Hence

$$x(t)=x_0^{(0)}(t,0,x_0)=\tilde{x}_0^{(0)}(t,0,x_0)+R_0^{(0)}(t,0,x_0,\varepsilon)$$

for $\tau_0<t\le\tau_1=t_1^{(0)}+\varepsilon\Theta_1^{(0)}+O(\varepsilon^2)$.

Furthermore,

$$
\begin{aligned}
x_1^+ &= x_0^{(0)}(\tau_1,0,x_0)+\varepsilon I_1(x_0^{(0)}(\tau_1,0,x_0)) \\
&= \tilde{x}_0^{(0)}(\tau_1,0,x_0)+\varepsilon I_1^{(0)}+R_0^{(0)}(\tau_1,0,x_0,\varepsilon) \\
&= x_0+\varepsilon\int_0^{\tau_1} A_0^{(1)}(\theta,x_0,x_0,0)X(\theta,x_0)d\theta+\varepsilon I_1^{(0)}+R_0^{(0)}(\tau_1,0,x_0,\varepsilon)\,,
\end{aligned}
$$

where $I_1^{(0)}\equiv I_1(x_0^{(0)}(\tau_1,0,x_0))$.

In the general case $(s=\overline{1,d_1})$ we denote as $x_s^{(0)}(t,\tau_s,x_s^+)$ the solution of the system

$$x_s^{(0)}(t,\tau_s,x_s^+)=\begin{cases} x_s^+ +\varepsilon\int_{\tau_s}^{t} A_s^{(k_s)}(\theta, x_s^{(0)}(\theta,\tau_s,x_s^+), x_s^{(0)}(\Delta_s^{(0)}(\theta),\tau_s,x_s^+), \\ \dot{x}_s^{(0)}(\Delta_s^{(0)}(\theta),\tau_s,x_s^+))X(\theta, x_s^{(0)}(\theta,\tau_s,x_s^+))d\theta\,, \quad t>\tau_s\,, \\ x_{s-1}^{(0)}(t,\tau_{s-1},x_{s-1}^+)\,, \quad -\delta\leq t\leq\tau_s\,, \end{cases} \tag{15}$$

$$\dot{x}_s^{(0)}(t,\tau_s,x_s^+)=\dot{x}_{s-1}^{(0)}(t,\tau_{s-1},x_{s-1}^+)\,, \quad -\delta\leq t\leq\tau_s\,,$$

where k_s is equal to 1 or 2 depending on whether the number $\Phi_s(\tau_s, x_{s-1}^{(0)}(\tau_s,\tau_{s-1}, x_{s-1}^+))$ is nonnegative or less than zero, respectively, and $\Delta_s^{(0)}(t)=\Delta(t, x_s^{(0)}(t,\tau_s,x_s^+))$, and

$$\begin{aligned} x_s^+ &= x_{s-1}^{(0)}(\tau_s,\tau_{s-1},x_{s-1}^+)+\varepsilon I_{s-1}(x_{s-1}^{(0)}(\tau_s,\tau_{s-1},x_{s-1}^+)) \\ &= x_0+\varepsilon\sum_{i=1}^{s}\int_{\tau_{i-1}}^{\tau_i} A_{i-1}^{(k_{i-1})}(\theta, x_0, x_0, 0)X(\theta, x_0)d\theta+\varepsilon\sum_{i=1}^{s} I_i^{(0)} \\ &\quad +\sum_{i=1}^{s} R_{i-1}^{(0)}(\tau_i,\tau_{i-1},x_{i-1}^+,\varepsilon)\,, \quad k_0=1\,, \end{aligned}$$

$$I_s^{(0)}\equiv I_s(x_{s-1}^{(0)}(\tau_s,\tau_{s-1},x_{s-1}^+))\,, \quad x_0^+\equiv x_0\,.$$

The solution of (15) coincides with the solution of the system (1) in the interval $[-\delta,\tau_s]$.

Consider the function

$$\tilde{x}_s^{(0)}(t,\tau_s,x_s^+)=x_s^+ +\varepsilon\int_{\tau_s}^{t} A_s^{(k_s)}(\theta, x_0, x_0, 0)X(\theta, x_0)d\theta\,.$$

One can show, as we did for the case $s=0$, that the difference

$$R_s^{(0)}(t,\tau_s,x_s^+,\varepsilon)=x_s^{(0)}(t,\tau_s,x_s^+)-\tilde{x}_s^{(0)}(t,\tau_s,x_s^+)$$

satisfies the following inequality for $t\in(\tau_s, T]$

$$\begin{aligned} ||R_s^{(0)}(t,\tau_s,x_s^+,\varepsilon)|| &\leq 3\varepsilon^2 KM(MT+s)^2/2+\varepsilon\gamma(\varepsilon)(\sqrt{n}+1)KMT \\ &\quad +3\varepsilon KMT\sum_{i=1}^{s}\omega_{i-1}^{(0)}(\varepsilon^2, T)+\varepsilon^2 KM^3 T\equiv\omega_s^{(0)}(\varepsilon^2, T)\,. \end{aligned}$$

Hence the function $\tilde{x}_s^{(0)}(t,\tau_s,x_s^+)$ approximates the solution of (15) in the interval $(\tau_s, T]$ with accuracy of order ε^2.

We show that, after the moment τ_s the trajectory $(t, x(t))$ does not meet the hypersurface σ_s any more.

Indeed, since the root of the equation

$$t=t_s(x_s^{(0)}(t,\tau_s,x_s^+))$$

is $\bar{t}_s=\tau_s+\varepsilon\dfrac{\partial t_s(x_0)}{\partial x}I_s^{(0)}+O(\varepsilon^2)$, then from the condition 6 of Theorem 1 and from the continuity of the vector-function $I_s(x)$ it follows that for sufficiently small ε it holds that $\bar{t}_s<\tau_s$. Thus, the trajectory $(t, x(t))$ does not meet the hypersurface σ_s after the moment τ_s.

The moment τ_{s+1} at which the trajectory $(t, x(t))$ meets the hypersurface σ_{s+1} satisfies

$$\tau_{s+1}=t_{s+1}^{(0)}+\varepsilon\Theta_{s+1}^{(0)}+O(\varepsilon^2)\,, \qquad s=\overline{1,(d_1-1)}\,,$$

where

$$\Theta_{s+1}^{(0)}=\frac{\partial t_{s+1}(x_0)}{\partial x}\left[\int_0^{t_{s+1}^{(0)}} A(\theta, x_0, x_0, 0)X(\theta, x_0)+\sum_{i=1}^{s} I_i^{(0)}\right].$$

From $t_{s+1}^{(0)}>t_s^{(0)}$ it follows that for sufficiently small ε the inequality $\tau_{s+1}>\tau_s$ holds.

Combining the cases $s=0$ and $s=\overline{1,d_1}$ and using (6) we can write $x(t)$ in the form

$$\begin{aligned}
x(t)&=x_s^{(0)}(t, \tau_s, x_s^+)\\
&=x_0+\varepsilon\sum_{i=0}^{s}\int_{\tau_{i-1}}^{\tau_i} A_{i-1}^{(k_i-1)}(\theta, x_0, x_0, 0)X(\theta, x_0)d\theta\\
&\quad+\varepsilon\sum_{i=0}^{s} I_i^{(0)}+\sum_{i=0}^{s} R_{i-1}^{(0)}(\tau_i, \tau_{i-1}, x_{i-1}^+, \varepsilon)\\
&\quad+\varepsilon\int_{\tau_s}^{t} A_s^{(k_s)}(\theta, x_0, x_0, 0)X(\theta, x_0)d\theta+R_s^{(0)}(t, \tau_s, x_s^+, \varepsilon)\\
&=x_0+\varepsilon\int_0^t A(\theta, x_0, x_0, 0)X(\theta, x_0)d\theta+\varepsilon\sum_{i=0}^{s} I_i^{(0)}\\
&\quad+\sum_{i=0}^{s} R_{i-1}^{(0)}(\tau_i, \tau_{i-1}, x_{i-1}^+, \varepsilon)+R_s^{(0)}(t, \tau_s, x_s^+, \varepsilon)
\end{aligned}$$

for

$$t_s^{(0)}+\varepsilon\Theta_s^{(0)}+\gamma_s O(\varepsilon^2)=\tau_s<t\leq\tau_{s+1}=t_{s+1}^{(0)}+\varepsilon\Theta_{s+1}^{(0)}+O(\varepsilon^2) \qquad s=\overline{0,(d_1-1)}\,,$$

as well as for

$$t_{d_1}^{(0)}+\varepsilon\Theta_{d_1}^{(0)}+O(\varepsilon^2)=\tau_{d_1}<t\leq T\,, \qquad s=d_1\,,$$

where

$$A_{-1}^{(k-1)}(t, x, y, z)\equiv 0\,, \qquad I_0^{(0)}\equiv R_{-1}^{(0)}(\tau_0, \tau_{-1}, x_{-1}^+, \varepsilon)\equiv 0\,, \qquad k_0=1\,,$$

$$\Theta_s^{(0)}=\frac{\partial t_s(x_0)}{\partial x}\left[\int_0^{t_s^{(0)}} A(\theta, x_0, x_0, 0)X(\theta, x_0)d\theta+\sum_{i=0}^{s-1} I_i^{(0)}\right], \qquad s=\overline{1,(d_1-1)}\,,$$

$$t_0^{(0)}=\Theta_0^{(0)}=\gamma_0=0\,, \qquad \gamma_s=1\,, \qquad s=\overline{1,d_1}\,.$$

Hence

$$\begin{aligned}
x(T)&=x_{d_1}^{(0)}(T, \tau_{d_1}, x_{d_1}^+)=x_0+\varepsilon\int_0^T A(\theta, x_0, x_0, 0)X(\theta, x_0)d\theta\\
&\quad+\varepsilon\sum_{i=0}^{d_1} I_i^{(0)}+\sum_{t=0}^{d_1} R_{i-1}^{(0)}(\tau_i, \tau_{i-1}, x_{i-1}^+, \varepsilon)+R_{d_1}^{(0)}(T, \tau_{d_1}, x_{d_1}^+, \varepsilon)\,.
\end{aligned}$$

Let $\bar{x}(t)$ be the solution of the averaged system (8) with initial condition (9). Then for $t\geq 0$ the equality

$$\bar{x}(t)=x_0+\varepsilon\int_0^t [A_0(\bar{x}(\theta))X(\theta, \bar{x}(\theta))+I_0(\bar{x}(\theta))]d\theta$$

holds which gives us

$$\bar{x}(T)=x_0+\varepsilon\int_0^T [A_0(\bar{x}(\theta))X(\theta, \bar{x}(\theta))+I_0(\bar{x}(\theta))]d\theta$$

Further on we shall estimate the norm of the difference $x(T)-\bar{x}(T)$. To this end, taking into account (9) we write $x(T)$ in the form

$$\begin{aligned} x(T)&=x_0+\varepsilon I_0(x_0)T+\varepsilon A_0(x_0)\int_0^T X(\theta, x_0)d\theta \\ &+\varepsilon\int_0^T [A(\theta, x_0, x_0, 0)-A_0(x_0)]X(\theta, x_0)d\theta \\ &+\varepsilon\left[\sum_{i=0}^{d_1} I_i^{(0)}-I_0(x_0)T\right]+\sum_{i=0}^{d_1} R_{i-1}^{(0)}(\tau_i, \tau_{i-1}, x_{i-1}^+, \varepsilon) \\ &+R_{d_1}^{(0)}(T, \tau_{d_1}, x_{d_1}^+, \varepsilon)\,. \end{aligned} \tag{17}$$

Define the operator B_p ($p=1, 2, \cdots$)

$$B_p x=x+\varepsilon I_0(x)T+\varepsilon A_0(x)\int_{(p-1)T}^{pT} X(\theta, x)d\theta\,, \qquad x\in D\,.$$

From (17), according to (11), the conditions of Theorem 1, the generalized mean value theorem in the integral calculus and the Cauchy inequality in the discrete case we get

$$\begin{aligned} \|x(T)-B_1x_0\| &\leq\varepsilon\left\|\int_0^T [A(\theta, x_0, x_0, 0)-A_0(x_0)]X(\theta, x_0)d\theta\right\| \\ &+\varepsilon\left\|\sum_{i=0}^{d_1} I_i^{(0)}-I_0(x_0)T\right\|+\sum_{i=0}^{d_1+1}\omega_{i-1}^{(0)}(\varepsilon^2, T) \\ &\leq\varepsilon\alpha(T)T/2+\varepsilon\left\|\sum_{i=1}^{d_1} I_i(x_0)-I_0(x_0)T\right\| \\ &+\varepsilon\left\|\sum_{i=1}^{d_1}(I_i^{(0)}-I_i(x_0))\right\|+\sum_{i=0}^{d_1+1}\omega_{i-1}^{(0)}(\varepsilon^2, T) \\ &\leq\varepsilon\alpha(T)T+\varepsilon K\sum_{i=0}^{d_1}\|x_{i-1}^{(0)}(\tau_i, \tau_{i-1}, x_{i-1}^+)-x_0\| \\ &+\sum_{i=0}^{d_1+1}\omega_{i-1}^{(0)}(\varepsilon^2, T)\leq\varepsilon\alpha(T)T+\varepsilon K\sum_{i=1}^{d_1}\Bigg\| x_0 \\ &+\varepsilon\int_0^{\tau_i} A(\theta, x_0, x_0, 0)X(\theta, x_0)d\theta+\varepsilon\sum_{l=0}^{i-1} I_l^{(0)} \end{aligned}$$

$$
\begin{aligned}
&+\sum_{l=0}^{i} R_{l-1}^{(0)}(\tau_l, \tau_{l-1}, x_{l-1}^{+}, \varepsilon)-x_0\Big\| + \sum_{i=1}^{d_1+1} \omega_{i-1}^{(0)}(\varepsilon^2, T) \\
&\leq \varepsilon\alpha(T)T+\varepsilon^2 KM^2 Td_1+\varepsilon^2 K\sum_{i=1}^{d_1}\sum_{l=0}^{i-1}||I_l^{(0)}|| \\
&+\varepsilon K\sum_{i=1}^{d_1}\sum_{l=0}^{i}||R_{l-1}^{(0)}(\tau_l, \tau_{l-1}, x_{l-1}^{+}, \varepsilon)||+\sum_{i=0}^{d_1+1}\omega_{i-1}^{(0)}(\varepsilon^2, T) \\
&\leq \varepsilon\alpha(T)T+\varepsilon^2 KMd_1(2MT+d_1-1)/2 \\
&+\varepsilon K\sum_{i=1}^{d_1}\sum_{l=0}^{i}\omega_{l-1}^{(0)}(\varepsilon^2, T)+\sum_{i=0}^{d_1+1}\omega_{i-1}^{(0)}(\varepsilon^2, T)\leq \varepsilon\alpha(T)T+\varepsilon^2 M_1 ,
\end{aligned}
\tag{18}
$$

where $\omega_{-1}^{(0)}(\varepsilon^2, T)\equiv 0$ and $M_1=M_1(T, d_1)$ is a constant.

For $t\geq 0$, $\tau\in[0, T)$ and $x\in D$, we have

$$
\begin{aligned}
&||A_0(x)||\leq M, \qquad ||I_0(x)||\leq Md, \qquad ||\bar{x}(\tau)-x_0||\leq \varepsilon M(M+d)T, \\
&||A_0(\bar{x}(\tau))-A_0(x_0)||\leq 2\varepsilon(M+d)KMT, \\
&||I_0(\bar{x}(\tau))-I_0(x_0)||\leq \varepsilon d(M+d)KMT.
\end{aligned}
$$

Using these estimates we find

$$
\begin{aligned}
&||\bar{x}(T)-B_1x_0||\leq \varepsilon\int_0^T \{||A_0(\bar{x}(\theta))-A_0(x_0)||\,.\,||X(\theta, \bar{x}(\theta))|| \\
&+||A_0(x_0)||\,.\,||X(\theta, \bar{x}(\theta))-X(\theta, x_0)||+||I_0(\bar{x}(\theta))-I_0(x_0)||\}d\theta \\
&\leq \varepsilon^2(M+d)(3M+d)KMT^2.
\end{aligned}
\tag{19}
$$

The following inequality follows from (18) and (19)

$$
||x(T)-\bar{x}(T)||\leq ||x(T)-B_1x_0||+||\bar{x}(T)-B_1x_0||\leq \varepsilon\alpha(T)T+\varepsilon^2\bar{M}, \tag{20}
$$

where $\bar{M}=(M+d)(3M+d)KMT^2+M_1$.

Thus, we got an estimate for $||x(T)-\bar{x}(T)||$. Henceforth we showed the closeness of $x(T)$ and $\bar{x}(T)$. Moreover, since $\bar{x}(T)$ belongs to the domain D with its neighbourhood of radius ρ, then from (19) and (20) it follows that B_1x_0 and $x(T)$ belong to D too.

Therefore, the inequality (10) is established for $p=1$.

We introduce the notation

$$
\begin{aligned}
&\tau_i^{(j-1)}\equiv \tau_{d_0+d_1+\cdots+d_{j-1}+i},\ d_0+d_1+\cdots+d_{j-1}+i=d(j-1\,;i), \\
&x_i^{(j-1)+}\equiv x_{d(j-1;i)}^{+}, \qquad d_0=0, \qquad i=\overline{1, d_j}, \\
&\tau_0^{(j-1)}\equiv (j-1)T, \qquad \tau_{d_j+1}^{(j-1)}\equiv jT, \\
&x_0^{(j-1)+}\equiv x((j-1)T), \qquad x_{d_j+1}^{(j-1)+}\equiv x(jT), \qquad j=1, 2, \cdots.
\end{aligned}
$$

We note that according to the introduced notation

$$
\tau_0^{(j-1)}\equiv \tau_{d_{j-1}+1}^{(j-2)}, \qquad x_0^{(j-1)+}\equiv x_{d_{j-1}+1}^{(j-2)+}, \qquad j=2, 3\cdots.
$$

Assume that for $p=j$, $j\geq 2$ the inequality (10) and results of the type (16) and

(18)-(20) are fulfilled.

We shall prove the correctness of (10) for $p=j+1$.

Let d_{j+1} points

$$t_{d(j;1)}(\bar{x}(jT)), \cdots, t_{d(j;d_{j+1})}(\bar{x}(jT))$$

lie in the interval $(jT, (j+1)T)$ and the following inequilities

$$t_{d(j;i)}(\bar{x}(jT)) < t_{d(j;i+1)}(\bar{x}(jT))$$

hold for $i=\overline{1, (d_{j+1}-1)}$.

Then from the continuity of the functions $t_i(x)$, $i=1, 2, \cdots$ and from the supposition that (10) holds for $p=j$ it follows that if ε is sufficiently small, then the points

$$j_{d(j;1)}(x(jT))=t_1^{(j)}, \cdots, t_{d(j;d_{j+1})}(x(jT))=t_{d_{j+1}}^{(j)} \tag{21}$$

lie in the interval $(jT, (j+1)T)$, moreover $t_i^{(j)}<t_{i+1}^{(j)}$ for $i=\overline{1, (d_{j+1}-1)}$.

The solution $x(t)$ of system (1) which is assumed to be build in the intervals $((p-1)T, pT]$, $p=\overline{1, j}$ is estended in the next interval $(jT, (j+1)T]$. Denote $x(jT)$ by x_{jT}.

Let $x_0^{(j)}(t, jT, x_{jT})$ be a solution of thesystem

$$x_0^{(j)}(t, jT, x_{jT})=\begin{cases} x_{jT}+\varepsilon\displaystyle\int_{jT}^{t} A_{d(j;0)}^{(k_{d(j;0)})}(\theta, x_0^{(j)}(\theta, jT, x_{jT}), \\ x_0^{(j)}(\Delta_0^{(j)}(\theta), jT, x_{jT}), \dot{x}_0^{(j)}(\Delta_0^{(j)}(\theta), jT, x_{jT})) . \\ X(\theta, x_0^{(j)}(\theta, jT, x_{jT}))d\theta , \qquad t>jT , \\ x_{d_j}^{(j-1)}(t, \tau_{d_j}^{(j-1)}, x_{d_j}^{(j-1)+}) , \qquad -\delta\le t\le jT , \end{cases} \tag{22}$$
$$\dot{x}_0^{(j)}(t, jT, x_{jT})=\dot{x}_{d_j}^{(j-1)}(t, \tau_{d_j}^{(j-1)}, x_{d_j}^{(j-1)+}) , \qquad -\delta\le t\le jT ,$$

where $k_{d(j;0)}$ is equal to 1 or 2, $\Delta_0^{(j)}(t)=\Delta(t, x_0^{(j)}(t, jT, x_{jT}))$.

The solution of (22) coincides with the solution of the system (1) till the moment $\tau_1^{(j)}$ at which the trajectory $(t, x(t))$ meets the hypersurface $\sigma_{d(j;1)}$, i.e. $x(t)=x_0^{(j)}(t, jT, x_{jT})$ for $t\in[-\delta, \tau_1^{(j)}]$.

Consider the function

$$\bar{x}_0^{(j)}(t, jT, x_{jT})=x_{jT}+\varepsilon\int_{jT}^{t} A_{d(j;0)}^{(k_{d(j;0)})}(\theta, x_{jT}, x_{jT}, 0)X(\theta, x_{jT})d\theta .$$

For $jT<t\le(j+1)T$ we have

$$\begin{aligned} &\|R_0^{(j)}(t, jT, x_{jT}, \varepsilon)\|=\|x_0^{(j)}(t, jT, x_{jT})-\bar{x}_0^{(j)}(t, jT, x_{jT})\| \\ &\le\varepsilon\int_{jT}^{t} \|A_{d(j;0)}^{(k_{d(j;0)})}(\theta, x_0^{(j)}(\theta, jT, x_{jT}), x_0^{(j)}(\Delta_0^{(j)}(\theta), jT, x_{jT}), \\ &\dot{x}_0^{(j)}(\Delta_0^{(j)}(\theta), jT, x_{jT}))X(\theta, x_0^{(j)}(\theta, jT, x_{jT}))-A_{d(j;0)}^{(k_{d(j;0)})}(\theta, x_{jT}, x_{jT}, 0). \end{aligned}$$

$$X(\theta, x_{jT})\|d\theta \leq \varepsilon\int_{jT}^{t}\{\|A_{d(j;0)}^{(k_{d(j;0)})}(\theta, x_0^{(j)}(\theta, jT, x_{jT}), x_0^{(j)}(\Delta_0^{(j)}(\theta), jT, x_{jT}),$$
$$\dot{x}_0^{(j)}(\Delta_0^{(j)}(\theta), jT, x_{jT})) - A_{d(j;0)}^{(k_{d(j;0)})}(\theta, x_{jT}, x_{jT}, 0)\|.\|X(\theta, x_0^{(j)}(\theta, jT, x_{jT}))\|$$
$$+\|A_{d(j;0)}^{(k_{d(j;0)})}(\theta, x_{jT}, x_{jT}, 0)\|.\|X(\theta, x_0^{(j)}(\theta, jT, x_{jT}))-X(\theta, x_{jT})\|\}d\theta$$
$$\leq \varepsilon KM\int_{jT}^{t}\{2\|x_0^{(j)}(\theta, jT, x_{jT})-x_{jT}\|+\|x_0^{(j)}(\Delta_0^{(j)}(\theta), jT, x_{jT})-x_{jT}\|$$
$$+\|\dot{x}_0^{(j)}(\Delta_0^{(j)}(\theta), jT, x_{jT})\|\}d\theta \leq 2\varepsilon^2KM^3\int_{jT}^{t}d\theta\int_{jT}^{\theta}dl$$
$$+\varepsilon KM\Bigg\{\sum_{i=0}^{d_j}\int_{J_{jT,t}^{i}}[x_i^{(j-1)}(\Delta_0^{(j)}(\theta), \tau_i^{(j-1)}, x_i^{(j-1)+})-x_{(j-1)T}\|$$
$$+\|x_{jT}-x_{(j-1)T}\|d\theta+\varepsilon M^2\int_{\tilde{J}_{jT,t}^{+}}\int_{jT}^{\Delta_0^{(j)}(\theta)}dl\Bigg\}$$
$$+\varepsilon KM\Bigg\{\sum_{i=0}^{d_j}\int_{J_{jT,t}^{i}}\|\dot{x}_i^{(j-1)}(\Delta_0^{(j)}(\theta), \tau_i^{(j-1)}, x_i^{(j-1)+})\|d\theta$$
$$+\int_{\tilde{J}_{jT,t}^{+}}\|\dot{x}_0^{(j)}(\Delta_0^{(j)}(\theta), jT, x_{jT})\|d\theta\Bigg\} \leq \varepsilon^2KM^3T^2$$
$$+\varepsilon^2KM^2\sum_{i=0}^{d_j}\int_{J_{jT,t}^{i}}[M(\Delta_0^{(j)}(\theta)-(j-1)T)+i]d\theta$$
$$+\varepsilon KM\sum_{i=0}^{d_j}\sum_{l=0}^{i}\omega_{l-1}^{(j-1)}(\varepsilon^2, T)\int_{J_{jT,t}^{i}}d\theta+\varepsilon KMT[\|x_{jT}-B_jx_{(j-1)T}\|$$
$$+\|B_jx_{(j-1)T}-x_{(j-1)T}\|+\varepsilon^2KM^3\int_{\tilde{J}_{jT,t}^{+}}(\Delta_0^{(j)}(\theta)-jT)d\theta$$
$$+\varepsilon^2KM^3\left(\sum_{i=0}^{d_j}\int_{J_{jT,t}^{i}}d\theta+\int_{\tilde{J}_{jT,t}^{+}}d\theta\right)\leq \varepsilon^2KM^3T^2$$
$$+\varepsilon^2KM^2\int_{jT}^{t}[M(\theta-(j-1)T)+d_j]d\theta+\varepsilon KMT\sum_{l=0}^{d_j}\omega_{l-1}^{(j-1)}(\varepsilon^2, T)$$
$$+\varepsilon^2KMT[\alpha(T)T+\varepsilon M_j+M(M+d)T]+\varepsilon^2KM^3\int_{jT}^{t}(\theta-jT)d\theta$$
$$+\varepsilon^2KM^3T\leq 3\varepsilon^2KM^3T^2/2+\varepsilon^2KM^2T(2MT+d_j)$$
$$\varepsilon^2KM^2T^2(M+d)+\varepsilon^2KM^3T+\varepsilon^2KMT[\alpha(T)T+\varepsilon M_j]$$
$$+\varepsilon KMT\sum_{l=0}^{d_j}\omega_{l-1}^{(j-1)}(\varepsilon^2, T)\equiv\omega_0^{(j)}(\varepsilon^2, T),$$

where

$$\left(\bigcup_{i=0}^{d_j}J_{jT,t}^{i}\right)\cup\tilde{J}_{jT,t}^{+}=(jT, t],$$
$$J_{jT,t}^{i}=\{\theta:\theta\in(jT, t]\wedge\Delta_0^{(j)}(\theta)\in(\tau_i^{(j-1)}, \tau_{i+1}^{(j-1)}]\}, \qquad i=\overline{0, d_j},$$
$$\tilde{J}_{jT,t}^{+}=(jT, t]\diagdown\left(\bigcup_{i=0}^{d_j}J_{jT,t}^{i}\right)$$

$$(\tau_i^{(0)}\equiv\tau_i,\ i=\overline{0, d_1},\ \tau_{d_1+1}^{(0)}=T,\ \omega_{-1}^{(j-1)}(\varepsilon^2, T)\equiv 0).$$

Hence, the function $\tilde{x}_0^{(j)}(t, jT, x_{jT})$ approximates the solution of (22) in the interval $(jT, (j+1)T]$ with accuracy of order ε^2.

For the root $\tau_1^{(j)}$ of the equation

$$t=t_{d(j;1)}(x_0^{(j)}(t, jT, x_{jT}))$$

we find

$$\tau_1^{(j)}=t_1^{(j)}+\varepsilon\Theta_1^{(j)}+O(\varepsilon^2),$$

where

$$\Theta_1^{(j)}=\frac{\partial t_{d(j;1)}(x_{jT})}{\partial x}\int_{jT}^{t_1^{(j)}} A_{d(j;0)}^{(k_{d(j;0)})}(\theta, x_{jT}, x_{jT}, 0)X(\theta, x_{jT})d\theta .$$

From $t_1^{(j)}>jT$ and from (23) it follows that if ε is sufficiently small, then $\tau_1^{(j)}>jT$.

Thus, for $jT>t\leq\tau_1^{(j)}$

$$x(t)=x_0^{(j)}(t, jT, x_{jT})=\tilde{x}_0^{(j)}(t, jT, x_{jT})+R_0^{(j)}(t, jT, x_{jT}, \varepsilon).$$

Further on we find

$$\begin{aligned} x_1^{(j)+}&=x_0^{(j)}(\tau_1^{(j)}, jT, x_{jT})+\varepsilon I_{d(j;1)}(x_0^{(j)}(\tau_1^{(j)}, jT, x_{jT}))\\ &=\tilde{x}_0^{(j)}(\tau_1^{(j)}, jT, x_{jT})+\varepsilon I_1^{(j)}+R_0^{(j)}(\tau_1^{(j)}, jT, x_{jT}, \varepsilon)\\ &=x_{jT}+\varepsilon\int_{jT}^{\tau_1^{(j)}} A_{d(j;0)}^{(k_{d(j;0)})}(\theta, x_{jT}, x_{jT}, 0)X(\theta, x_{jT})d\theta\\ &\quad+\varepsilon I_1^{(j)}+R_0^{(j)}(\tau_1^{(j)}, jT, x_{jT}, \varepsilon), \end{aligned}$$

where $I_1^{(j)}\equiv I_{d(j;1)}(x_0^{(j)}(\tau_1^{(j)}, jT, x_{jT}))$.

In the general case $(s=\overline{1, d_{j+1}})$ we denote as $x_s^{(j)}(t, \tau_s^{(j)}, x_s^{(j)+})$ the solution of the system

$$x_s^{(j)}(t, \tau_s^{(j)}, x_s^{(j)+})=\begin{cases} x_s^{(j)+}+\varepsilon\int_{\tau_s^{(j)}}^{t} A_{d(j;s)}^{(k_{d(j;s)})}(\theta, x_s^{(j)}(\theta, \tau_s^{(j)}, x_s^{(j)+}), \\ x_s^{(j)}(\Delta_s^{(j)}(\theta), \tau_s^{(j)}, x_s^{(j)+}), \dot{x}_s^{(j)}(\Delta_s^{(j)}(\theta), \tau_s^{(j)}, x_s^{(j)+})) \cdot \\ X(\theta, x_s^{(j)}(\theta, \tau_s^{(j)}, x_s^{(j)+}))d\theta, \qquad t>\tau_s^{(j)}, \\ x_{s-1}^{(j)}(t, \tau_{s-1}^{(j)}, x_{s-1}^{(j)+}), \qquad -\delta\leq t\leq\tau_s^{(j)}, \end{cases} \tag{24}$$

$$\dot{x}_s^{(j)}(t, \tau_s^{(j)}, x_s^{(j)+})=\dot{x}_{s-1}^{(j)}(t, \tau_{s-1}^{(j)}, x_{s-1}^{(j)+}), \qquad -\delta\leq t\leq\tau_s^{(j)},$$

where $\Delta_s^{(j)}(t)=\Delta(t, x_s^{(j)}(t, \tau_s^{(j)}, x_s^{(j)+}))$ and

$$x_s^{(j)+}=x_{s-1}^{(j)}(\tau_s^{(j)}, \tau_{s-1}^{(j)}, x_{s-1}^{(j)+})+\varepsilon I_{d(j;s)}(x_{s-1}^{(j)}(\tau_s^{(j)}, \tau_{s-1}^{(j)}, x_{s-1}^{(j)+}))$$

$$= x_{jT} + \varepsilon \sum_{i=1}^{s} \int_{\tau_{i-1}^{(j)}}^{\tau_i^{(j)}} A_{d(j;i-1)}^{(k_{d(j;i-1)})}(\theta, x_{jT}, x_{jT}, 0) X(\theta, x_{jT}) d\theta$$

$$+ \varepsilon \sum_{i=1}^{s} I_i^{(j)} + \sum_{i=1}^{s} R_{i-1}^{(j)}(\tau_i^{(j)}, \tau_{i-1}^{(j)}, x_{i-1}^{(j)}, \varepsilon) ,$$

$$I_s^{(j)} \equiv I_{d(j;s)}(x_{s-1}^{(j)}(\tau_s^{(j)}, \tau_{s-1}^{(j)}, x_{s-1}^{(j)+})) .$$

The solution of (24) coincides with the solution of the system (1) in the in-interval $[-\delta, \tau_{s+1}^{(j)}]$.

Consider the function

$$\tilde{x}_s^{(j)}(t, \tau_s^{(j)}, x_s^{(j)+}) = x_s^{(j)+} + \varepsilon \int_{\tau_s^{(j)}}^{t} A_{d(j;s)}^{(k_{d(j;s)})}(\theta, x_{jT}, x_{jT}, 0) X(\theta, x_{jT}) d\theta .$$

One can show that the following inequality holds in the interval $jT < \tau_s^{(j)} < t \leq (j+1)T$

$$\|R_s^{(j)}(t, \tau_s^{(j)}, x_s^{(j)+}, \varepsilon)\| = \|x_s^{(j)}(t, \tau_s^{(j)}, x_s^{(j)+}) - \tilde{x}_s^{(j)}(t, \tau_s^{(j)}, x_s^{(j)+})\|$$

$$\leq 3\varepsilon^2 KM(MT+s)^2/2 + \varepsilon^2 KM^2 T(2MT+d_j)$$

$$+ \varepsilon^2 KM^2 T^2 (M+d) + \varepsilon^2 KMT[\alpha(T)T + \varepsilon M_j]$$

$$+ \varepsilon^2 KM^3 T + \varepsilon KMT \sum_{i=0}^{d_i} \omega_{i-1}^{(j-1)}(\varepsilon^2, T)$$

$$+ 3\varepsilon KMT \sum_{i=1}^{s} \omega_{i-1}^{(j)}(\varepsilon^2, T) \equiv \omega_s^{(j)}(\varepsilon^2, T)$$

etc.

Combined the cases $s=0$ and $s=\overline{1, d_{j+1}}$ and using (6) we write $x(t)$ in the form

$$x(t) = x_s^{(j)}(t, \tau_s^{(j)}, x_s^{(j)+}) = x_{jT} + \varepsilon \sum_{i=0}^{s} \int_{\tau_{i-1}^{(j)}}^{\tau_i^{(j)}} A_{d(j;i-1)}^{(k_{d(j;i-1)})}(\theta, x_{jT}, x_{jT}, 0)$$

$$X(\theta, x_{jT}) d\theta + \varepsilon \sum_{i=1}^{s} I_i^{(j)} + \sum_{i=1}^{s} R_{i-1}^{(j)}(\tau_i^{(j)}, \tau_{i-1}^{(j)}, x_{i-1}^{(j)+}, \varepsilon)$$

$$+ \varepsilon \int_{\tau_s^{(j)}}^{t} A_{d(j;s)}^{(k_{d(j;s)})}(\theta, x_{jT}, x_{jT}, 0) X(\theta, x_{jT}) d\theta + R_s^{(j)}(t, \tau_s^{(j)}, x_s^{(j)+}, \varepsilon)$$

$$= x_{jT} + \varepsilon \int_{jT}^{t} A(\theta, x_{jT}, x_{jT}, 0) X(\theta, x_{jT}) d\theta + \varepsilon \sum_{i=0}^{s} I_i^{(j)}$$

$$+ \sum_{i=0}^{s} R_{i-1}^{(j)}(\tau_i^{(j)}, \tau_{i-1}^{(j)}, x_{i-1}^{(j)+}, \varepsilon) + R_s^{(j)}(t, \tau_s^{(j)}, x_s^{(j)+}, \varepsilon) .$$

The presentation (25) holds for

$$t_s^{(j)} + \varepsilon \Theta_s^{(j)} + \gamma_s O(\varepsilon^2) = \tau_s^{(j)} < t \leq \tau_{s+1}^{(j)} + \varepsilon \Theta_{s+1}^{(j)} + O(\varepsilon^2) ,$$

$$s = \overline{0, (d_{j+1}-1)} ,$$

as well as for

$$t_{d_{i+1}}^{(j)}+\varepsilon\Theta_{d_{i+1}}^{(j)}+O(\varepsilon^2)=\tau_{d_{j+1}}^{(j)}<t\leq(j+1)T\,,\qquad s=d_{j+1}\,,$$

where

$$A_{d(j;i-1)}^{(k_{d(j;i-1)})}(t,x,y,z)\equiv 0\,,\qquad I_0^{(j)}\equiv R_{-1}^{(j)}(\tau_0^{(j)},\tau_{-1}^{(j)},x_{-1}^{(j)+},\varepsilon)\equiv 0\,,$$

$$\Theta_s^{(j)}=\frac{\partial t_{d(j;s)}(x_{jT})}{\partial x}\left[\int_{jT}^{t_s^{(j)}}A(\theta,x_{jT},x_{jT},0),X(\theta,x_{jT})d\theta+\sum_{i=0}^{s-1}I_i^{(j)}\right],$$

$$t_0^{(j)}=jT\,,\qquad \Theta_0^{(0)}=\gamma_0=0\,,\qquad \gamma_s=1\,,\qquad s=\overline{1,d_{j+1}}\,.$$

We derive $x((j+1)T)$ and $\bar{x}((j+1)T)$

$$\begin{aligned}x((j+1)T)&=x_{d_{j+1}}^{(j)}((j+1)T,\tau_{d_{j+1}}^{(j)},x_{d_{j+1}}^{(j)+})\\&=x_{jT}+\varepsilon\int_{jT}^{(j+1)T}A(\theta,x_{jT},x_{jT},0)X(\theta,x_{jT})d\theta+\varepsilon\sum_{i=0}^{d_{j+1}}I_i^{(j)}\\&\quad+\sum_{i=0}^{d_{j+1}+1}R_{i-1}^{(j)}(\tau_i^{(j)},\tau_{i-1}^{(j)},x_{i-1}^{(j)+},\varepsilon)\\&=x_{jT}+\varepsilon I_0(x_{jT})T+\varepsilon A_0(x_{jT})\int_{jT}^{(j+1)T}X(\theta,x_{jT})d\theta\\&\quad+\varepsilon\int_{jT}^{(j+1)T}[A(\theta,x_{jT},x_{jT},0)-A_0(x_{jT})]X(\theta,x_{jT})d\theta\\&\quad+\varepsilon\left[\sum_{i=0}^{d_{j+1}}I_i^{(j)}-I_0(x_{jT})\right]+\sum_{i=1}^{d_{j+1}+1}R_{i-1}^{(j)}(\tau_i^{(j)},\tau_{i-1}^{(j)},x_{i-1}^{(j)+},\varepsilon)\,,\\ \bar{x}((j+1)T)&=x_0+\varepsilon\int_{jT}^{(j+1)T}[A_0(\bar{x}(\theta))X(\theta,\bar{x}(\theta))+I_0(\bar{x}(\theta))]d\theta\\&=\bar{x}(jT)+\varepsilon\int_{jT}^{(j+1)T}[A_0(\bar{x}(\theta))X(\theta,\bar{x}(\theta))+I_0(\bar{x}(\theta))]d\theta\,.\end{aligned}$$

In order to estimate the norm of the difference $x((j+1)T)-\bar{x}((j+1)T)$ we use the inequality

$$\begin{aligned}&\|x((j+1)T)-\bar{x}((j+1)T)\|\leq\|x((j+1)T)-B_{j+1}x_{jT}\|\\&\quad+\|B_{j+1}x_{jT}-B_{j+1}\bar{x}(jT)\|+\|B_{j+1}\bar{x}(jT)-\bar{x}((j+1)T)\|\,.\end{aligned}\tag{26}$$

By repeating the argument in (18) we get the following for the first summand in the right-hand side of (26)

$$\begin{aligned}&\|x((j+1)T)-B_{j+1}x_{jT}\|\leq\varepsilon\alpha(T)T+\varepsilon^2KMd_{j+1}(2MT+d_{j+1}-1)/2\\&\quad+\varepsilon K\sum_{i=1}^{d_{j+1}}\sum_{l=0}^{i}\omega_{l-1}^{(j)}(\varepsilon^2,T)+\sum_{i=0}^{d_{j+1}+1}\omega_{i-1}^{(j)}(\varepsilon^2,T)\leq\varepsilon\alpha(T)T+\varepsilon^2M_{j+1}\,,\end{aligned}\tag{27}$$

where $\omega_{-1}^{(j)}(\varepsilon^2,T)\equiv 0$ and $M_{j+1}=M_{j+1}(T,d_1,\cdots,d_{j+1})$ is a constant.

For the second summand in the right-hand side of (26) we have

$$\begin{aligned}&\|B_{j+1}x_{jT}-B_{j+1}\bar{x}(jT)\|=\|x_{jT}+\varepsilon I_0(x_{jT})T\\&\quad+\varepsilon A_0(x_{jT})\int_{jT}^{(j+1)T}X(\theta,x_{jT})d\theta-\bar{x}(jT)-\varepsilon I_0(\bar{x}(jT))T\end{aligned}$$

$$+\varepsilon A_0(\bar{x}(jT))\int_{jT}^{(j+1)T} X(\theta, \bar{x}(jT))d\theta\| \leq \|x_{jT}-\bar{x}(jT)\|$$
$$+\varepsilon T\|I_0(x_{jT})-I_0(\bar{x}(jT)\|+\varepsilon\|A_0(xj_T)-A_0(\bar{x}(jT))\|.$$
$$.\int_{jT}^{(j+1)T}\|X(\theta, x_{jT})\|d\theta+\varepsilon\|A_0(\bar{x}(jT)\|\int_{jT}^{(j+1)T}\|X(\theta, x_{jT})-X(\theta, \bar{x}(iT))\|d\theta$$
$$\leq[1+\varepsilon(3M+d)KT].\|x_{jT}-\bar{x}(jT)\|$$
$$\leq[1+\varepsilon(3M+d)KT]\sum_{i=0}^{j-1}[1+\varepsilon(3M+d)KT]^i.[\varepsilon\alpha(T)T+\varepsilon^2\bar{M}]$$

where $\bar{M}=(M+d)(3M+d)KMT^2+\max_{i=\overline{1,j}} M_i$.

Since the following inequality holds for $t\in(jT,(j+1)T]$

$$\|\bar{x}(t)-\bar{x}(jT)\|\leq\varepsilon\int_{jT}^{t}[\|A_0(\bar{x}(\theta))\|.\|X(\theta, \bar{x}(\theta))\|$$
$$+\|I_0(\bar{x}(\theta))]d\theta\leq\varepsilon(M+d)MT,$$

then for the third summand in the right-hand side of (26) one gets

$$\|B_{j+1}\bar{x}(jT)-\bar{x}((j+1)T)\|=\|\bar{x}(jT)+\varepsilon I_0(\bar{x}(jT))T$$
$$+\varepsilon A_0(\bar{x}(jT))\int_{jT}^{(j+1)T} X(\theta, \bar{x}(jT))d\theta-\bar{x}(jT)$$
$$-\varepsilon\int_{jT}^{(j+1)T}[A_0(\bar{x}(\theta))X(\theta, \bar{x}(\theta))+I_0(\bar{x}(\theta))]d\theta\|$$
$$\leq\varepsilon\|A_0(\bar{x}(jT))\|\int_{jT}^{(j+1)T}\|X(\theta, \bar{x}(jT))-X(\theta, \bar{x}(\theta))\|d\theta$$
$$+\varepsilon\int_{jT}^{(j+1)T}\|A_0(\bar{x}(jT))-A_0(\bar{x}(\theta))\|.\|X(\theta, \bar{x}(\theta))\|d\theta$$
$$+\varepsilon\int_{jT}^{(j+1)T}\|I_0(\bar{x}(jT))-I_0(\bar{x}(\theta))\|d\theta\leq\varepsilon(3M+d)K\int_{jT}^{(j+1)T}\|\bar{x}(jT)-\bar{x}(\theta)\|d\theta$$
$$\leq\varepsilon^2(M+d)(3M+d)KMT^2. \tag{29}$$

The relations (26)-(29) imply

$$\|x((j+1)T-\bar{x}((j+1)T)\|\leq\sum_{i=0}^{j}[1+\varepsilon(3M+d)KT]^i.[\varepsilon\alpha(T)T+\varepsilon^2\bar{M}], \tag{30}$$

where $\bar{M}=(M+d)(3M+d)KMT^2+\max_{i=\overline{1,(j+1)}} M_i$.

The last inequality shows that (10) holds for $p=j+1$ and $x((j+1)T)$ belongs to the domain D. This proves Lemma 1.

Proof of Theorem 1. According to condition 3 of Theorem 1 there exists a constant $C(T)<\infty$ such that for all $i=1, 2, \cdots$ the inequality $d_i<C(T)$ holds. Hence there exists a constant $M_0(T)<\infty$ such that

$$\bar{M}=(M+d)(3M+d)KMT^2+\max_{i=1,2,\cdots} M_i\leq M_0(T). \tag{31}$$

Let q be the integer part of the number $L/\varepsilon T$. Then for every $p=\overline{1,q}$, according to (31) and Lemma 1, we have

$$\begin{aligned}
&\|x(pT)-\bar{x}(pT)\\
&\leq\varepsilon\sum_{i=0}^{p-1}[1+\varepsilon(3M+d)KT]^{i}.[\alpha(T)T+\varepsilon M_0(T)]\\
&\leq[\alpha(T)T+\varepsilon M_0(T)][1+\varepsilon(3M+d)KT]^{p}/(3M+d)KT\\
&\leq[e^{(3M+d)KL}+O(\varepsilon)][\alpha(T)T+\varepsilon M_0(T)]/(3m+d)KT.
\end{aligned}$$

Choose T large enough such that

$$e^{(3M+d)KL}\alpha(T)/(3M+d)K<\eta/4$$

and then choose ε so small that

$$O(\varepsilon)\alpha(T)/(3M+d)K+\varepsilon[e^{(3M+d)KL}+O(\varepsilon)]M_0(T)/(3M+d)KT<\eta/4.$$

Then for every $p\in\overline{1,q}$ the following inequality holds

$$\|x(pT)-\bar{x}(pT)\|<\eta/2. \tag{32}$$

Further on we estimate $\|\bar{x}(t)-\bar{x}((p-1)T)\|$ and $\|x(t)-x((p-1)T)\|$ in the interval $(p-1)T\leq t\leq pT$.

We have

$$\begin{aligned}
&\|\bar{x}(t)-\bar{x}((p-1)T)\|\\
&\leq\varepsilon\int_{(p-1)T}^{t}[\|A_0(\bar{x}(\theta))\|.\|X(\theta,\bar{x}(\theta))\|+\|I_0(\bar{x}(\theta))\|]d\theta\\
&\leq\varepsilon(M+d)MT.
\end{aligned}$$

According to (25) we get

$$\begin{aligned}
&\|x(t)-x((p-1)T)\|\\
&=\|x_s^{(p-1)}(t,\tau_s^{(p-1)},x_s^{(p-1)+})-x((p-1)T)\|\\
&\leq\varepsilon\int_{(p-1)T}^{t}\|A(\theta,x_{(p-1)T},x_{(p-1)T},0)\|.\|X(\theta,x_{(p-1)T})\|d\theta\\
&+\varepsilon\sum_{i=0}^{s}\|I_i^{(p-1)}\|+\sum_{i=0}^{s}\|R_{i-1}^{(p-1)}(\tau_i^{(p-1)},\tau_{i-1}^{(p-1)},x_{i-1}^{(p-1)+},\varepsilon)\|\\
&+\|R_s^{(p-1)}(t,\tau_s^{(p-1)},x_s^{(p-1)+},\varepsilon)\|\leq\varepsilon M(MT+s)\\
&+\sum_{i=0}^{s+1}\omega_{i-1}^{(p-1)}(\varepsilon^2T)\leq\varepsilon M[MT+C(T)]+\varepsilon^2M_0(T)\equiv\Psi(\varepsilon,T).
\end{aligned}\tag{34}$$

One can see that the choice of T provides

$$\Psi(\varepsilon,T)<\eta/2 \tag{35}$$

for sufficiently small ε. From (32)-(35) it follows that the choice of T yields

$$\|x(t)-\bar{x}(t)\|<\eta$$

for ε sufficiently small and for t from the interval $[(p-1)T, pT]$, where $p=1, 2, \cdots, q$.

Hence, fixing T as above, if ε is sufficiently small $(0<\varepsilon\leq\varepsilon_0\leq\mathcal{E})$ the inequality

$$||x(t)-\bar{x}(t)||<\eta$$

will hold in the entile interval $0\leq t\leq L\varepsilon^{-1}$.

Thus, Theorem 1 is proved.

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