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ON CL-ISOCOMPACTNESS AND WEAK BOREL COMPLETENESS

By

Masami SAKAI

Introduction.

A space X is said to be isocompact [1] if every countably compact closed subset of X is compact. In this paper we introduce a new class of spaces called CL-isocompact spaces. We call a space X CL-isocompact if the closure of each countably compact subset of X is compact. CL-isocompact spaces are isocompact. The class of CL-isocompact spaces behaves well with respect to topological operations. For example the class is productive and closed hereditary. After showing various properties of CL-isocompact spaces, we investigate the relationship between CL-isocompact spaces, weakly θ-refinable spaces [6] and weakly Borel complete spaces [3]. We show that every weakly θ-refinable space of non-measurable cardinal is weakly Borel complete and every weakly Borel complete space is CL-isocompact.

All spaces are assumed to be completely regular. But this is not always needed.

§ 1. Fundamental properties.

DEFINITION 1.1. A space X is said to be CL-isocompact if the closure of each countably compact subset of X is compact.

Obviously CL-isocompact spaces are isocompact.

PROPOSITION 1.2. *The following facts hold.*

(a) *Let f be a perfect map from X onto Y. Then, X is CL-isocompact iff Y is CL-isocompact.*

(b) *Let X be CL-isocompact, and Y be an F_σ-subset of X. Then, Y is CL-isocompact.*

(c) *If $X=\prod_\alpha X_\alpha$, with X_α CL-isocompact for $\alpha\in A$, then X is CL-isocompact.*

(d) *If $X=\bigoplus_\alpha X_\alpha$, with X_α CL-isocompact for $\alpha\in A$, then X is CL-isocompact.*

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(e) *If each* X_α *is a CL-isocompact subset of* X, *then* $\bigcap_\alpha X_\alpha$ *is CL-isocompact.*

(f) *The following* (1), (2) *and* (3) *are equivalent.*

(1) *X is hereditarily CL-isocompact.*

(2) *X is hereditarily isocompact.*

(3) *For each* $x \in X$, $X-\{x\}$ *is CL-isocompact.*

PROOF. (a) Compactness and countably compactness are preserved by perfect maps. From this fact, it is easy to show (a). (b) We set $Y=\bigcup_{i=1}^{\infty} Y_i$, each Y_i is closed in X. Let E be any countably compact subset of Y. Since each Y_i is CL-isocompact, $\mathrm{Cl}(E\cap Y_i)$ is compact. $\bigcup_i \mathrm{Cl}(E\cap Y_i)$ contains E as a dense subset. Since $\bigcup_i \mathrm{Cl}(E\cap Y_i)$ is pseudocompact σ-compact, it is compact. We get $\mathrm{Cl}_Y E=\bigcup_i \mathrm{Cl}(E\cap Y_i)$. (c) Let E be any countably compact subset of X. Since each $Pr_\alpha E$ is countably compact, $\mathrm{Cl}(Pr_\alpha E)$ is compact. Here Pr_α is the projection of X onto X_α. The closure of E in X is contained in the compact space $\prod_\alpha \mathrm{Cl}(Pr_\alpha E)$. $\mathrm{Cl}\, E$ must be compact. (d) is trivial. (e) $\bigcap_\alpha X_\alpha$ can be naturally embedded as a closed subspace into $\prod_\alpha X_\alpha$. By (b) and (c), $\bigcap_\alpha X_\alpha$ is CL-isocompact. (f) The equivalence of (1) and (2) is obvious. We assume (3). Let Y be any subspace of X. Since $Y=\cap\{X-\{x\}\mid x\in X-Y\}$, Y is CL-isocompact by (e). ∎

Bacon proved in [1] that the product of an isocompact space and a hereditarily isocompact space is isocompact. The following result generalizes it.

PROPOSITION 1.3. *Let X be CL-isocompact, and Y be isocompact. Then* $X\times Y$ *is isocompact.*

PROOF. Let E be any countably compact closed subset of $X\times Y$. Since $Pr_X E$ is countably compact, $\mathrm{Cl}(Pr_X E)$ is compact. Therefore $Pr_Y E$ is closed countably compact in Y. So, $Pr_Y E$ must be compact. E is contained in the compact space $\mathrm{Cl}(Pr_X E)\times Pr_Y E$. The proof is complete. ∎

PROPOSITION 1.4. *The following* (a) *and* (b) *hold.*

(a) *For each space X, there exists a CL-isocompact space pX with the following properties.*

(1) $X\subset pX\subset \beta X$. *Here* βX *is the Stone-Čech compactification of X.*

(2) *If f is a map from X onto a CL-isocompact space Y, then f has a continuous extention* f^p *that maps pX onto Y.*

(b) *If X has a dense countably compact subspace, then* $pX=\beta X$. *Conversely,* *if $pX=\beta X$, then X is pseudocompact.*

PROOF. (a) is obtained from Proposition 1.2. (b), (c) and Theorem 2.1. in [7]. (b) is trivial. Note that $pX\subset\upsilon X$, υX is the Hewitt's realcompactification. ■

§2. Weak Borel completeness.

A space X is said to be weakly Borel complete [3] if each Borel ultrafilter $\mathscr{B}$ on X with c.i.p. (countable intersection property) has the property that $\cap\{Z\mid Z\in\mathscr{B}\cap\mathscr{Z}(X)\}=\cap\{F\mid F\in\mathscr{B},\ F$ is closed in $X.\}$ is non-void. Here $\mathscr{Z}(X)$ is the set of zero sets of X.

THEOREM 2.1. *Weakly Borel complete spaces are CL-isocompact.*

PROOF. Weak Borel completeness is closed hereditary [3]. So, we show that a weakly Borel complete space which has a dense countably compact subset is compact. Let X be weakly Borel complete, and Y be a dense countably compact subset of X.

Suppose that X is not compact. Since X is pseudocompact, X is not realcompact. We take a free zero ultrafilter $\mathscr{Z}$ on X with c.i.p.. Each element of $\mathscr{Z}$ must intersect with Y. Put $\mathscr{A}=\{\mathscr{H}\mid\mathscr{H}$ is a closed family such that (1) $\mathscr{Z}\subset\mathscr{H}$. (2) If $H\in\mathscr{H}$, then $H\cap Y\neq\emptyset$. (3) $\mathscr{H}$ is closed under the finite intersections.$\}$. Let $\mathscr{H}$ be a maximal element of $\mathscr{A}$. It is easily showed that $\mathscr{H}$ is closed under the countable intersections, and $X\in\mathscr{H}$ by the maximality.

Put $\mathscr{D}=\{B\in Bo(X)\mid B\supset H\cap Y$ for some $H\in\mathscr{H}\}$. Here $Bo(X)$ is the set of Borel sets of X. We take a Borel ultrafilter $\mathscr{B}$ on X containing $\mathscr{D}$. Put $\mathscr{E}=\{B\in Bo(X)\mid$If $B\not\supset H\cap Y$ for any $H\in\mathscr{H}$, then $B\cap H\cap Y=\emptyset$ for some $H\in\mathscr{H}.\}$.

Now, $\mathscr{E}$ satisfies the following conditions.

(a) If F is closed in X, then $F\in\mathscr{E}$.

(b) If $B\in\mathscr{E}$, then $X-B\in\mathscr{E}$.

(c) If $\mathscr{E}\supset\{B_i\}_{i=1}^{\infty}$, then $\bigcap_i B_i\in\mathscr{E}$.

Firstly we show (a). Let F be a closed subset of X, and suppose that $F\not\supset H\cap Y$ for any $H\in\mathscr{H}$. Obviously $F\notin\mathscr{H}$. Put $\mathscr{L}=\mathscr{H}\cup\{F\cap H\mid H\in\mathscr{H}\}$. $\mathscr{L}$ satisfies (1), (3) of $\mathscr{A}$, and $\mathscr{H}\neq\mathscr{L}$, because $F\in\mathscr{L}$. By the maximality of $\mathscr{H}$, there exists $H\in\mathscr{H}$ such that $F\cap H\cap Y=\emptyset$. This shows that $F\in\mathscr{E}$. The proof of (b) and (c) is a routine matter. We omit the proof.

Since $Bo(X)$ is the smallest σ-field containing the set of closed subsets of X, we get $\mathscr{E}=Bo(X)$.

Suppose that $B\in\mathscr{B}$, and $B\cap H\cap Y=\emptyset$ for some $H\in\mathscr{H}$. Then $X-B\in\mathscr{D}\subset\mathscr{B}$.

It is a contradiction that $\mathcal{B}$ is a filter. Therefore, for each $B\in\mathcal{B}$, $B\cap H\cap Y\neq\emptyset$ for any $H\in\mathcal{H}$. It follows from $\mathcal{E}=Bo(X)$ that for each $B\in\mathcal{B}$ there exists some $H(B)\in\mathcal{H}$ such that $B\supset H(B)\cap Y$. This fact gives that $\mathcal{B}$ has c.i.p.. Since $\mathcal{Z}\subset\mathcal{B}$, we obtain that $\cap\{Z\mid Z\in\mathcal{B}\cap\mathcal{Z}(X)\}=\emptyset$. This is a contradiction that X is weakly Borel complete. ■

COROLLARY 2.2. *If X has a countably compact dense subset, then $wX=\beta X$. Here wX is the weak Borel completion of X.*

PROOF. Apply Proposition 1.4. (b) and Theorem 2.1. ■

COROLLARY 2.3. *If X is a perfect image of a weakly Borel complete space, then X is CL-isocompact.*

PROOF. Apply Proposition 1.2. (a) and Theorem 2.1. ■

It is not known whether perfect images of weakly Borel complete spaces are weakly Borel complete.

THEOREM 2.4. *If X is a weakly θ-refinable space of non-measurable cardinal, then X is weakly Borel complete.*

PROOF. Hardy proved in [2] that a weakly θ-refinable space of non-measurable cardinal is a-realcompact. The procedure of the proof is valid for this theorem.

Let $\mathcal{B}$ be a Borel ultrafilter on X with c.i.p.. Let $\mathcal{H}=\{H\mid H\in\mathcal{B},\ H$ is closed in $X.\}$. Suppose that $\cap\mathcal{H}=\emptyset$. Since $\mathcal{U}=\{X-H\mid H\in\mathcal{H}\}$ is an open cover of X, there exists a weak θ-refinement $\mathcal{V}=\bigcup_{n=1}^{\infty}\mathcal{V}_n$ of $\mathcal{U}$. For n, j, let $H_{nj}=\{x\in X\mid 1\leqq\mathrm{ord}(x, \mathcal{V}_n)\leqq j\}$. Then obviously $X=\bigcup_{n,j}H_{nj}$. By c.i.p. of $\mathcal{B}$, there exist natural numbers n, j such that $H_{nj}\cap B\neq\emptyset$ for any $B\in\mathcal{B}$. We fix these n, j.

By virtue of Zorn's lemma, we can find a discrete subspace $D\subset H_{nj}$ such that

(a) $\{\mathrm{St}(x, \mathcal{V}_n)\mid x\in D\}$ covers H_{nj},

(b) If $V\in\mathcal{V}_n$, then $|V\cap D|\leqq 1$.

Since $|X|<m_1$, D is realcompact. Here m_1 is the first measurable cardinal.

For each $F\in\mathcal{H}$, let $F^*=\{x\in D\mid \mathrm{St}(x, \mathcal{V}_n)\cap F\cap H_{nj}\neq\emptyset\}$. Then $\mathcal{M}=\{F^*\mid F\in\mathcal{H}\}$ is a free filter base on D. Take a ultrafilter $\mathcal{K}$ on D such that $\mathcal{M}\subset\mathcal{K}$. Since D is realcompact, there exists a countable subcollection $\{K_i\}_{i=1}^{\infty}$

$\subset \mathcal{K}$ such that $\bigcap_i K_i = \emptyset$. Let $U_i = \cup\{\mathrm{St}(x, \mathcal{V}_n) \mid x \in K_i\}$. If $x \in \bigcap_i U_i$, then for each i there exist $x_i \in K_i$ and $V_i \in \mathcal{V}_n$ with $x, x_i \in V_i$. Since this shows that $\mathrm{ord}(x, \mathcal{V}_n) = \omega$, we have $x \notin H_{nj}$. Consequently $H_{nj} \cap (\bigcap_i U_i) = \emptyset$.

If $X - U_i \in \mathcal{H}$ for some i, we can consider $(X - U_i)^*$. But it is easily showed that $K_i \cap (X - U_i) = \emptyset$. Since K_i, $(X - U_i)^* \in \mathcal{K}$, this is a contradiction. It must be $X - U_i \notin \mathcal{H}$ for every i. Therefore $X - U_i \notin \mathcal{B}$ for every i. Since it must be $U_i \in \mathcal{B}$ for every i, we have $\bigcap_i U_i \in \mathcal{B}$. It follows that $H_{nj} \cap (\bigcap_i U_i) \neq \emptyset$. This is a contradiction. ■

By the similar procedure of the proof of Theorem 2.4, we can show that each θ-refinable space [6] is weakly Borel complete if the cardinality of each closed discrete subspace is non-measurable.

REMARK 2.5. Hardy conjectured in [2, Remark 2.8.] that there exists an a-realcompact space of non-measurable cardinal which is not weakly θ-refinable. Rudin's Dowker space in [4] is, in fact, such a space. Because Simon proved in [5] that the Rudin's Dowker space is a-realcompact, and not weakly Borel complete. This fact answers the third question posed in [9].

COROLLARY 2.6. *A quasi-developable space of non-measurable cardinal is Borel complete.*

PROOF. It is known that a quasi-developable space is hereditarily weakly θ-refinable, and that Borel completeness is equivalent to be hereditarily weakly Borel complete [3]. ■

Addendum

Theorem 2.4 is extendable to the class of θ-penetrable spaces. Namely each θ-penetrable space of non-measurable cardinal is weakly Borel complete. For θ-penetrable spaces, refer to [8]. For the proof, we use the fact that, for a free closed filter $\mathcal{F}$ on X with c.i.p. which is extendable to a Borel ultrafilter on X with c.i.p., $\{X - F \mid F \in \mathcal{F}\}$ has a weak θ-refinement if it has a θ-penetration. This fact is proved by the quite similar way of [8, Lemma 2.2].

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Institute of Mathematics
University of Tsukuba
Ibaraki, 305 Japan