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INJECTIVE DIMENSION OF GENERALIZED MATRIX RINGS

By

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A Morita context $\langle M, N\rangle$ consists of two rings R and S with identity, two bimodules ${}_RN_S$ and ${}_SM_R$, and two bimodule homomorphisms called the pairings $(-, -): N\otimes_S M\rightarrow R$ and $[-, -]: M\otimes_R N\rightarrow S$ satisfying the associativity conditions $(n, m)n'=n[m, n']$ and $[m, n]m'=m(n, m')$. The images of the pairings are called the trace ideals of the context and are denoted by ${}_RI_R$ and ${}_SJ_S$.

Let Λ be the generalized matrix ring defined by the Morita context $\langle M, N\rangle$, i.e.,

$$\Lambda=\begin{bmatrix} R & N \\ M & S \end{bmatrix},$$

where the addition is given by element-wise and the multiplication by

$$\begin{bmatrix} r & n \\ m & s \end{bmatrix}\begin{bmatrix} r' & n' \\ m' & s' \end{bmatrix}=\begin{bmatrix} rr'+(n, m') & rn'+ns' \\ mr'+sm' & [m, n']+ss' \end{bmatrix}.$$

For a right R-module U, id-U_R(fd-U_R) denotes the injective (flat) dimension of U_R, respectively.

Let

$$\Gamma=\begin{bmatrix} R & 0 \\ M & S \end{bmatrix}$$

be the generalized matrix ring defined by the trivial context $\langle M, 0\rangle$. In a previous paper [9], we have established a theorem concerning the estimation of the injective dimension of Γ_Γ in terms of those of R_R, M_R and S_S as follows:

THEOREM. *Assume that ${}_SM$ is flat. Then we have*

$$\max(\text{id-}R_R, \text{id-}M_R, \text{id-}S_S)\leqq\text{id-}\Gamma_\Gamma\leqq\max(\text{id-}R_R, \text{id-}M_R, \text{id-}S_S-1)+1.$$

The main purpose of this paper is to extend a part of results in the previous paper [9] to Λ under some additional conditions on the Morita context $\langle M, N\rangle$. In Section 1, we decide a lower bound of id-Λ_Λ using id-R_R, id-M_R, id-S_S

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and id-N_S. In Section 2, we investigate an upper bound of id-Λ_Λ as well as a lower bound of id-Λ_Λ in terms of id-R_R, id-M_R, id-S_S and id-N_S under the condition that $N=NJ$, both ${}_SM$ and ${}_RN$ are flat, and the natural maps $I\otimes_R I\to I^2$ and $J\otimes_S J\to J^2$ are isomorphisms. The estimation of id-Λ_Λ is as follows:

THEOREM 2.6. *If $N=NJ$, both ${}_SM$ and ${}_RN$ are flat, and the natural maps $I\otimes_R I\to I^2$ and $J\otimes_S J\to J^2$ are isomorphisms, then we have*

$$\max(\text{id-}R_R, \text{id-}M_R, \text{id-}S_S, \text{id-}N_S)$$
$$\leqq \text{id-}\Lambda_\Lambda \leqq \max(\text{id-}R_R, \text{id-}M_R, \text{id-}S_S, \text{id-}N_S)+1.$$

In Section 3, we examine the condition for Λ to be a right self-injective ring. Section 4 is devoted to study id-Λ_Λ in case of the derived context. Furthermore, we show that id-R_R=id-Λ_Λ, if M_R is finitely generated projective, which is the extension of the well-known fact that id-$\begin{bmatrix} R & R \\ R & R \end{bmatrix}$=id-$R$, In the final Section 5, we exhibit some example when the left-hand side or the right-hand side equality holds in Theorem 2.6.

Throughout this paper, uniess otherwise specified, Λ denotes the generalized matrix ring defined by the Morita context $\langle M, N\rangle$ with pairings $(-, -)$ and $[-, -]$, and the trace ideals ${}_RI_R$ and ${}_SJ_S$. For a right R-module U, id-U_R(fd-U_R) denotes the injective (flat) dimension of U_R, respectively. Moreover, we set $e=\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\in\Lambda$ and $e'=\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\in\Lambda$.

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1. General cases.

The following lemma is essentially in [3, p. 346].

LEMMA 1.1. *Let A_R, ${}_RB_\Lambda$ and C_Λ be modules such that* $\mathrm{Ext}^i_\Lambda(B, C)=0$ $(i>0)$ *and* $\mathrm{Tor}^R_i(A, B)=0$ $(i>0)$. *Then there holds*

$$\mathrm{Ext}^n_R(A, \mathrm{Hom}_\Lambda(B, C))\cong\mathrm{Ext}^n_\Lambda(A\otimes_R B, C).$$

THEOREM 1.2. *Assume that* fd-${}_SM$ *and* fd-${}_RN$ *are finite. Then we have*

$$\max(\max(\text{id-}R_R, \text{id-}M_R)-\text{fd-}{}_RN, \max(\text{id-}S_S, \text{id-}N_S)-\text{fd-}{}_SM)$$
$$\leqq \text{id-}\Lambda_\Lambda.$$

PROOF. Let L be a right ideal of R. Since

$$\mathrm{Hom}_\Lambda(R/L\otimes_R e\Lambda,\ \Lambda)\cong\mathrm{Hom}_R(R/L,\ \mathrm{Hom}_\Lambda(e\Lambda,\ \Lambda))$$
$$\cong\mathrm{Hom}_R(R/L,\ \Lambda e)$$
$$\cong\mathrm{Hom}_R(R/L,\ R\oplus M)$$

and $\mathrm{Ext}^i_\Lambda(e\Lambda,\ \Lambda)=0$ $(i>0)$, the resulting spectral sequence is

$$\mathrm{E}_2^{p,q}=\mathrm{Ext}^q_\Lambda(\mathrm{Tor}^R_p(R/L,\ e\Lambda),\ \Lambda)\underset{q}{\Longrightarrow}\mathrm{Ext}^n_R(R/L,\ R\oplus M)\,.$$

Since $\mathrm{E}_2^{p,q}=0$ for either $q>\text{id-}\Lambda_\Lambda$ or $p>\text{fd-}_RN$, we have $\mathrm{Ext}^n_R(R/L,\ R\oplus M)=0$ for $n>\text{id-}\Lambda_\Lambda+\text{fd-}_RN$. Thus we have $\max(\text{id-}R_R,\ \text{id-}M_R)\text{-fd-}_RN\leqq\text{id-}\Lambda_\Lambda$. In the similar manner, we also obtain $\max(\text{id-}S_S,\ \text{id-}N_S)-\text{fd-}_SM\leqq\text{id-}\Lambda_\Lambda$, completing the proof.

2. Trace accessible cases.

We prepare some lemmas needed after.

LEMMA 2.1. *Every right ideal of Λ has the form of $[X\ Y]$ with X_R a submodule of $\begin{bmatrix}R\\M\end{bmatrix}_R$ and Y_S a submodule of $\begin{bmatrix}N\\S\end{bmatrix}_S$ satisfying $\left\{\begin{bmatrix}(n,m)\\sm\end{bmatrix}\middle|\begin{bmatrix}n\\s\end{bmatrix}\in Y,\ m\in M\right\}\subseteqq X$ and $\left\{\begin{bmatrix}rn\\ [m,n]\end{bmatrix}\middle|\begin{bmatrix}r\\m\end{bmatrix}\in X,\ n\in N\right\}\subseteqq Y$.*

PROOF. Let P be a right ideal of Λ. Put $X=\left\{\begin{bmatrix}r\\m\end{bmatrix}\middle|\begin{bmatrix}r&0\\m&0\end{bmatrix}\in P\right\}$ and $Y=\left\{\begin{bmatrix}n\\s\end{bmatrix}\middle|\begin{bmatrix}0&n\\0&s\end{bmatrix}\in P\right\}$. Then X and Y satisfy the above conditions. The converse part is obvious.

The following lemmas are well-known.

LEMMA 2.2.

(1) $I\,\mathrm{Ker}(-,\ -)=\mathrm{Ker}(-,\ -)I=0$.

(2) $J\,\mathrm{Ker}[-,\ -]=\mathrm{Ker}[-,\ -]J=0$.

LEMMA 2.3. *Assume that $N=NJ$. Then*

(1) $NJ=IN=N$.

(2) $I=I^2$ *and* $J=J^2$.

Following [10], a right R-module W is called L-accessible for an ideal L of R if $W=WL$.

LEMMA 2.4. *Assume that $N=NJ$ and that $_RN$ are flat. Then the following are equivalent:*

(1) *The natural maps* $I\otimes_R I\to I^2$ *and* $J\otimes_S J\to J^2$ *are isomorphisms.*

(2) *The pairings* $(-,\ -)$ *and* $[-,\ -]$ *are monic.*

PROOF. (1)⇒(2). The exact sequences

$$0\longrightarrow \operatorname{Ker}(-,\ -)_R \xrightarrow{\ \nu_1\ } N\otimes_S M_R \xrightarrow{(-,\ -)} I_R\longrightarrow 0$$

and

$$0\longrightarrow \operatorname{Ker}[-,\ -]_S \xrightarrow{\ \nu_2\ } M\otimes_R N_S \xrightarrow{[-,\ -]} J_S\longrightarrow 0$$

induce the following commutative diagrams with exact rows and columns

$$\begin{array}{ccccccc}
 & I\otimes_R\operatorname{Ker}(-,\ -) & \xrightarrow{I\otimes\nu_1} & I\otimes_R N\otimes_S M & \xrightarrow{I\otimes(-,\ -)} & I\otimes_R I & \longrightarrow 0\\
 & \downarrow{\scriptstyle\alpha_1} & & \downarrow{\scriptstyle\beta_1} & & \downarrow{\scriptstyle\gamma_1} & \\
0\longrightarrow & \operatorname{Ker}(-,\ -)\cap I(N\otimes_S M) & \xrightarrow{\subseteq} & I(N\otimes_S M) & \xrightarrow{\ \delta_1\ } & I^2=I & \\
 & & & \downarrow & & \downarrow & \\
 & & & 0 & & 0 &
\end{array} \qquad (*)$$

and

$$\begin{array}{ccccccc}
 & \operatorname{Ker}[-,\ -]\otimes_S J & \xrightarrow{\nu_2\otimes J} & M\otimes_R N\otimes_S J & \xrightarrow{[-,\ -]\otimes J} & J\otimes_S J & \longrightarrow 0\\
 & \downarrow{\scriptstyle\alpha_2} & & \downarrow{\scriptstyle\beta_2} & & \downarrow{\scriptstyle\gamma_2} & \\
0\longrightarrow & \operatorname{Ker}[-,\ -]\cap(M\otimes_R N)J & \xrightarrow{\subseteq} & (M\otimes_R N)J & \xrightarrow{\ \delta_2\ } & J^2=J, & \\
 & & & \downarrow & & \downarrow & \\
 & & & 0 & & 0 &
\end{array} \qquad (**)$$

where α_i, β_i and γ_i $(i=1,\ 2)$ are the natural maps, $\delta_1=(-,\ -)\,|\,I(N\otimes_S M)$, and $\delta_2=[-,\ -]\,|\,(M\otimes_R N)J$. Since γ_i is an isomorphism by assumption, α_i is epic by the 5-lemma. Since $\operatorname{Im}\alpha_1=I\operatorname{Ker}(-,\ -)=0$ and $\operatorname{Im}\alpha_2=\operatorname{Ker}[-,\ -]J=0$ by Lemma 2.2, δ_1 and δ_2 are monic. Since $N=IN=NJ$ by Lemma 2.3, it is easy to see that $\delta_1=(-,\ -)$ and $\delta_2=[-,\ -]$. Hence the pairings $(-,\ -)$ and $[-,\ -]$ are monic.

(2)⇒(1). Since $_RN$ is flat, $N=IN$ and $(-,\ -)$ is monic, it is easily verified that γ_1 is an isomorphism in view of the commutative diagram $(*)$. Moreover, since $(-,\ -)$ and $[-,\ -]$ are monic and $N=NJ$, it is easily checked that β_2 is the following comdosition of maps

$$M\otimes_R N\otimes_S J \xrightarrow[\sim]{M\otimes N\otimes[-,\ -]^{-1}} M\otimes_R N\otimes_S M\otimes_R N \xrightarrow[\sim]{M\otimes(-,\ -)\otimes N}$$

$$M\otimes_R I\otimes_R N \xrightarrow{\ \sim\ } M\otimes_R IN=M\otimes_R N.$$

It follows from the commutative diagram $(**)$ that γ_2 is an isomorphism.

In the remainder of this section, we assume that both ${}_SM$ and ${}_RN$ are flat and that the natural maps $I\otimes_R I\to I^2$ and $J\otimes_S J\to J^2$ are isomorphisms.

LEMMA 2.5. *Assume further that* $N=NJ$. *Let* $[X_0\ Y_0]$ *be a right ideal of* Λ *and put* $X_i=\left\{\sum_j\begin{bmatrix}(n_j, m_j)\\ s_jm_j\end{bmatrix}\middle|\begin{bmatrix}n_j\\ s_j\end{bmatrix}\in Y_{i-1},\ m_j\in M\right\}$ *and* $Y_i=\left\{\sum_k\begin{bmatrix}r_kn_k\\ [m_k, n_k]\end{bmatrix}\middle|\begin{bmatrix}r_k\\ m_k\end{bmatrix}\in X_{i-1},\ n_k\in N\right\}$ $(i=1, 2, 3)$. *Then*

(1) $Y_{i-1}\otimes_S M\cong X_i$ *as a right R-module and* $X_{i-1}\otimes_R N\cong Y_i$ *as a right S-module*

(2) $[X_{i-1}\ 0]\otimes_R e\Lambda\cong[X_{i-1}\ Y_i]$ *and* $[0\ Y_{i-1}]\otimes_S e'\Lambda\cong[X_i\ Y_{i-1}]$ *as right* Λ-*modules.*

PROOF. (1) Since ${}_SM$ is flat, and $(-, -)$ is monic by Lemma 2.4, the homomorphism $Y_{i-1}\otimes_S M\to X_i$ defined by $\begin{bmatrix}n\\ s\end{bmatrix}\otimes m\mapsto\begin{bmatrix}(n, m)\\ sm\end{bmatrix}$ for $\begin{bmatrix}n\\ s\end{bmatrix}\in Y_{i-1}$, $m\in M$, is an isomorphism. Similarly, we can show that $X_{i-1}\otimes_R N\cong Y_i$.

(2) It is easily seen that $[X_{i-1}\ Y_i]$ and $[X_i\ Y_{i-1}]$ are right ideals of Λ. Since $X_{i-1}\otimes_R N\cong Y_i$ by (1), the homomorphism $[X_{i-1}\ 0]\otimes_R e\Lambda\to[X_{i-1}\ Y_i]$ defined via

$$\begin{bmatrix}r & 0\\ m & 0\end{bmatrix}\otimes\begin{bmatrix}r' & n\\ 0 & 0\end{bmatrix}\longmapsto\begin{bmatrix}rr' & rn\\ mr' & [m, n]\end{bmatrix}\quad\text{for}\quad\begin{bmatrix}r\\ m\end{bmatrix}\in X_i,\ \begin{bmatrix}r & n\\ 0 & 0\end{bmatrix}\in e\Lambda,$$

is an isomorphism. By the similar manner as above, we obtain $[0\ Y_{i-1}]\otimes_S e'\Lambda\cong[X_i\ Y_{i-1}]$.

THEOREM 2.6. *Assume further that* $N=NJ$. *Then we have*

$$\max(\text{id-}R_R,\ \text{id-}M_R,\ \text{id-}S_S,\ \text{id-}N_S)$$
$$\leqq\text{id-}\Lambda_\Lambda\leqq\max(\text{id-}R_R,\ \text{id-}M_R,\ \text{id-}S_S,\ \text{id-}N_S)+1\,.$$

PROOF. Let $[X_0\ Y_0]$ be a right ideal of Λ and put $X_i=\left\{\sum_j\begin{bmatrix}(n_j, m_j)\\ s_jm_j\end{bmatrix}\middle|\begin{bmatrix}n_j\\ s_j\end{bmatrix}\in Y_{i-1},\ m_j\in M\right\}$ and $Y_i=\left\{\sum_k\begin{bmatrix}r_kn_k\\ [m_k, n_k]\end{bmatrix}\middle|\begin{bmatrix}r_k\\ n_k\end{bmatrix}\in X_{i-1},\ n_k\in N\right\}$ $(i=1, 2, 3)$. Then we consider the following exact sequence of right Λ-modules:

$$0\longrightarrow[X_1\ Y_0]\longrightarrow[X_0\ Y_0]\longrightarrow[X_0\ Y_0]/[X_1\ Y_0]\longrightarrow 0\,. \qquad (*)$$

Since $N=NJ$, it is easy to see that $Y_1=Y_1J$, from which it follows that $Y_1=Y_2=Y_3$. Therefore, we have $[X_0\ Y_0]/[X_1\ Y_0]\cong[X_0\ Y_1]/[X_1\ Y_1]=[X_0\ Y_1]/[X_1\ Y_2]$. Moreover, since both ${}_RN$ and ${}_SM$ are flat, and both $(-, -)$ and $[-, -]$ are monic by Lemma 2.4, we have $[X_1\ Y_0]\cong[0\ Y_0]\otimes_S e'\Lambda$ and $[X_0\ Y_0]/[X_1\ Y_0]\cong[X_0\ Y_1]/[X_1\ Y_2]\cong([X_0\ 0]/[X_1\ 0])\otimes_R e\Lambda$ by Lemma 2.5. Now, we put $\max(\text{id-}R_R,\ \text{id-}M_R,\ \text{id-}S_S,\ \text{id-}N_S)=t$. The exact sequence $(*)$ yields the following exact sequence

$$\mathrm{Ext}_\Lambda^{t+1}([X_0\ Y_0]/[X_1\ Y_0],\ \Lambda)\longrightarrow\mathrm{Ext}_\Lambda^{t+1}([X_0\ Y_0],\ \Lambda)\longrightarrow\mathrm{Ext}_\Lambda^{t+1}([X_1\ Y_0],\ \Lambda)\,,$$

from which it follows that $\mathrm{Ext}_\Lambda^{t+1}([X_0\ Y_0], \Lambda)=0$ together with the fact that

$$\begin{aligned}\mathrm{Ext}_\Lambda^{t+1}([X_0\ Y_0]/[X_1\ Y_0], \Lambda) &\cong \mathrm{Ext}_\Lambda^{t+1}([X_0\ Y_1]/[X_1\ Y_1], \Lambda)\\ &\cong \mathrm{Ext}_\Lambda^{t+1}([X_0\ Y_1]/[X_1\ Y_2], \Lambda)\\ &\cong \mathrm{Ext}_\Lambda^{t+1}((X_0/X_1)\otimes_R e\Lambda, \Lambda)\\ &\cong \mathrm{Ext}_R^{t+1}(X_0/X_1, \mathrm{Hom}_\Lambda(e\Lambda, \Lambda))\\ &\cong \mathrm{Ext}_R^{t+1}(X_0/X_1, \Lambda e)=0\end{aligned}$$

and that

$$\begin{aligned}\mathrm{Ext}_\Lambda^{t+1}([X_1\ Y_0], \Lambda) &\cong \mathrm{Ext}_\Lambda^{t+1}([0\ Y_0]\otimes_S e'\Lambda, \Lambda)\\ &\cong \mathrm{Ext}_S^{t+1}(Y_0, \mathrm{Hom}_\Lambda(e'\Lambda, \Lambda))\\ &\cong \mathrm{Ext}_S^{t+1}(Y_0, \Lambda e')=0\end{aligned}$$

in view of Lemma 1.1. Hence we have $t\leqq \text{id-}\Lambda_\Lambda \leqq t+1$ together with Theorem 1.2.

REMARK. If we assume that $M=MI$ instead of $N=NJ$ in Lemma 2.5 and Theorem 2.6, we obtain the same results by the symmetry of the Morita context $\langle M, N\rangle$.

THEOREM 2.7. *Assume further that* $NJ=N$.

(1) *If* $\max(\text{id-}R_R, \text{id-}M_R)<\max(\text{id-}S_S, \text{id-}N_S)=i\neq 0$, *then* $\text{id-}\Lambda_\Lambda=i$ *if and only if* $\mathrm{Ext}_S^i(N, S\oplus N)=0$.

(2) *If* $\max(\text{id-}S_S, \text{id-}N_S)<\max(\text{id-}R_R, \text{id-}M_R)=i\neq 0$ *and if* $\mathrm{Ext}_R^i(M/JM, R\oplus M)\neq 0$, *then* $\text{id-}\Lambda_\Lambda=i+1$.

(3) *Suppose that* $\max(\text{id-}R_R, \text{id-}M_R)=\max(\text{id-}S_S, \text{id-}N_S)=i\neq 0$.

(i) *If* $\mathrm{Ext}_R^i(X, R\oplus M)\neq 0$ *for some* $X_R\subsetneqq (R\oplus M)_R$, *then* $\text{id-}\Lambda_\Lambda=i+1$.

(ii) *If* $\text{id-}S_S>\text{id-}N_S$ *and if* $\mathrm{Ext}_R^i(M/JM, R)\neq 0$, *then* $\text{id-}\Lambda_\Lambda=i+1$.

(iii) *If* $\text{id-}N_S>\text{id-}S_S$ *and if* $\mathrm{Ext}_R^i(M/JM, M)\neq 0$, *then* $\text{id-}\Lambda_\Lambda=i+1$.

PROOF. (1) Let $[X_0\ Y_0]$ be a right ideal of Λ and put $X_i=\left\{\sum_k \begin{bmatrix}(n_k, m_k)\\ s_k m_k\end{bmatrix} \middle| \begin{bmatrix}n_k\\ s_k\end{bmatrix}\in Y_{i-1}, m_k\in M\right\}$ and $Y_i=\left\{\sum_j \begin{bmatrix}r_j n_j\\ [m_j, n_j]\end{bmatrix} \middle| \begin{bmatrix}r_j\\ m_j\end{bmatrix}\in X_{i-1}, n_j\in N\right\}$ $(i=1, 2, 3)$. Since $NJ=N$, it is easy to see that $Y_1=Y_2$. Moreover, since

$$\begin{aligned}\mathrm{Ext}_\Lambda^i([X_0\ Y_0]/[X_1\ Y_0], \Lambda) &\cong \mathrm{Ext}_\Lambda^i([X_0\ Y_1]/[X_1\ Y_1], \Lambda)\\ &= \mathrm{Ext}_\Lambda^i([X_0\ Y_1]/[X_1\ Y_2], \Lambda)\\ &\cong \mathrm{Ext}_\Lambda^i(([X_0\ 0]\otimes_R e\Lambda)/([X_1\ 0]\otimes_R e\Lambda), \Lambda)\\ &\cong \mathrm{Ext}_\Lambda^i(X_0/X_1\otimes_R e\Lambda, \Lambda)\end{aligned}$$

$$\cong \mathrm{Ext}_R^i(X_0/X_1,\ R\oplus M)=0$$

and

$$\mathrm{Ext}_\Lambda^i([X_1\ Y_0],\ \Lambda)\cong \mathrm{Ext}_\Lambda^i([0\ Y_0]\otimes_S e'\Lambda,\ \Lambda)$$
$$\cong \mathrm{Ext}_S^i(Y_0,\ S\oplus N)$$

by Lemmas 1.1 and 2.5, we have $\mathrm{Ext}_\Lambda^i([X_0\ Y_0],\ \Lambda)\cong \mathrm{Ext}_S^i(Y_0,\ S\oplus N)$ from the following exact sequence

$$0=\mathrm{Ext}_\Lambda^i([X_0\ Y_0]/[X_1\ Y_0],\ \Lambda)\longrightarrow \mathrm{Ext}_\Lambda^i([X_0\ Y_0],\ \Lambda)\longrightarrow \mathrm{Ext}_\Lambda^i([X_1\ Y_0],\ \Lambda)$$
$$\longrightarrow \mathrm{Ext}_\Lambda^{i+1}([X_0\ Y_0]/[X_1\ Y_0],\ \Lambda)=0\,.$$

It follows that id-$\Lambda_\Lambda=i$ if and only if $\mathrm{Ext}_\Lambda^i([X_0\ Y_0],\ \Lambda)\cong \mathrm{Ext}_S^i(Y_0,\ N\oplus S)=0$ for every right ideal $[X_0\ Y_0]$ of Λ if and only if $\mathrm{Ext}_S^i(N,\ S\oplus N)=0$ from the following exact sequence

$$\mathrm{Ext}_S^i(N,\ S\oplus N)=\mathrm{Ext}_S^i(S\oplus N,\ S\oplus N)\longrightarrow \mathrm{Ext}_S^i(Y_0,\ S\oplus N)$$
$$\longrightarrow \mathrm{Ext}_S^{i+1}((S\oplus N)/Y_0,\ S\oplus N)=0\,.$$

(2) The exact sequence of right Λ-modules

$$0\longrightarrow \begin{bmatrix}0 & 0\\ JM & J\end{bmatrix}\longrightarrow \begin{bmatrix}0 & 0\\ M & J\end{bmatrix}\longrightarrow \begin{bmatrix}0 & 0\\ M & J\end{bmatrix}\Big/\begin{bmatrix}0 & 0\\ JM & J\end{bmatrix}\longrightarrow 0$$

yields the following exact sequence

$$\mathrm{Ext}_\Lambda^{i-1}\left(\begin{bmatrix}0 & 0\\ JM & J\end{bmatrix},\ \Lambda\right)\longrightarrow \mathrm{Ext}_\Lambda^i\left(\begin{bmatrix}0 & 0\\ M & J\end{bmatrix}\Big/\begin{bmatrix}0 & 0\\ JM & J\end{bmatrix},\ \Lambda\right)$$
$$\longrightarrow \mathrm{Ext}_\Lambda^i\left(\begin{bmatrix}0 & 0\\ M & J\end{bmatrix},\ \Lambda\right)\longrightarrow \mathrm{Ext}_\Lambda^i\left(\begin{bmatrix}0 & 0\\ JM & J\end{bmatrix},\ \Lambda\right).$$

Since $J=J^2$ by Lemma 2.3, we have $\begin{bmatrix}0 & 0\\ JM & J\end{bmatrix}=\begin{bmatrix}0 & 0\\ JM & J^2\end{bmatrix}$. Since

$$\mathrm{Ext}_\Lambda^i\left(\begin{bmatrix}0 & 0\\ M & J\end{bmatrix}\Big/\begin{bmatrix}0 & 0\\ JM & J\end{bmatrix},\ \Lambda\right)=\mathrm{Ext}_\Lambda^i\left(\begin{bmatrix}0 & 0\\ M & J\end{bmatrix}\Big/\begin{bmatrix}0 & 0\\ JM & J^2\end{bmatrix},\ \Lambda\right)$$
$$\cong \mathrm{Ext}_\Lambda^i\left(\left(\begin{bmatrix}0 & 0\\ M & 0\end{bmatrix}\otimes_R e\Lambda\right)\Big/\left(\begin{bmatrix}0 & 0\\ JM & 0\end{bmatrix}\otimes_R e\Lambda\right),\ \Lambda\right)$$
$$\cong \mathrm{Ext}_\Lambda^i(M/JM\otimes_R e\Lambda,\ \Lambda)$$
$$\cong \mathrm{Ext}_R^i(M/JM,\ R\oplus M)\neq 0$$

and

$$\mathrm{Ext}_\Lambda^k\left(\begin{bmatrix}0 & 0\\ JM & J\end{bmatrix},\ \Lambda\right)\cong \mathrm{Ext}_\Lambda^k(J\otimes_S e'\Lambda,\ \Lambda)$$

$$\cong \mathrm{Ext}_S^k(J, S \oplus N) = 0 \qquad (k = i-1, i),$$

by Lemmas 1.1 and 2.5, we have $\mathrm{Ext}_\Lambda^i\left(\begin{bmatrix} 0 & 0 \\ M & J \end{bmatrix}, \Lambda\right) \cong \mathrm{Ext}_R^i(M/JM, R \oplus M) \neq 0$. Hence id-$\Lambda_\Lambda = i+1$ together with Theorem 2.6.

(3) (i) Let X_R be a submodule of $(R \oplus M)_R$ such that $\mathrm{Ext}_R^i(X, R \oplus M) \neq 0$ and $Y_1 = \left\{\sum_j \begin{bmatrix} r_j n_j \\ [m_j, n_j] \end{bmatrix} \middle| \begin{bmatrix} r_j \\ m_j \end{bmatrix} \in X,\ n_j \in N\right\}$. Since $[X\, Y_1]$ is a right ideal of Λ and

$$\mathrm{Ext}_\Lambda^i([X\, Y_1], \Lambda) \cong \mathrm{Ext}_\Lambda^i([X\, 0 \otimes_R]\, e\Lambda, \Lambda)$$
$$\cong \mathrm{Ext}_R^i(X, R \oplus M) \neq 0$$

by Lemmas 1.1 and 2.5, we have id-$\Lambda_\Lambda = i+1$ by Theorem 2.6.

(ii) Let

$$h_i^\#: \mathrm{Ext}_\Lambda^i\left(\Lambda\Big/\begin{bmatrix} R & N \\ JM & J \end{bmatrix}, e\Lambda\right) \oplus \mathrm{Ext}_\Lambda^i\left(\Lambda\Big/\begin{bmatrix} R & N \\ JM & J \end{bmatrix}, e'\Lambda\right)$$
$$\longrightarrow \mathrm{Ext}_\Lambda^i\left(\begin{bmatrix} R & N \\ M & J \end{bmatrix}\Big/\begin{bmatrix} R & N \\ JM & J \end{bmatrix}, e\Lambda\right) \oplus \mathrm{Ext}_\Lambda^i\left(\begin{bmatrix} R & N \\ M & J \end{bmatrix}\Big/\begin{bmatrix} R & N \\ JM & J \end{bmatrix}, e'\Lambda\right)$$

be the induced map by the inclusion map

$$h: \begin{bmatrix} R & N \\ M & J \end{bmatrix}\Big/\begin{bmatrix} R & N \\ JM & J \end{bmatrix} \hookrightarrow \Lambda\Big/\begin{bmatrix} R & N \\ JM & J \end{bmatrix}.$$

Since

$$\mathrm{Ext}_\Lambda^i\left(\Lambda\Big/\begin{bmatrix} R & N \\ JM & J \end{bmatrix}, e\Lambda\right) \cong \mathrm{Ext}_\Lambda^i(S/J \otimes_S e'\Lambda, e\Lambda)$$
$$\cong \mathrm{Ext}_S^i(S/J, N) = 0$$

by Lemma 1.1, we have $\mathrm{Im}\, h_i^\# \subseteq \mathrm{Ext}_\Lambda^i\left(\begin{bmatrix} R & N \\ M & J \end{bmatrix}\Big/\begin{bmatrix} R & N \\ JM & J \end{bmatrix}, e'\Lambda\right)$. Since $NJ = N$, we have $J = J^2$ by Lemma 2.3. Therefore, if

$$\mathrm{Ext}_\Lambda^i\left(\begin{bmatrix} R & N \\ M & J \end{bmatrix}\Big/\begin{bmatrix} R & N \\ JM & J \end{bmatrix}, e\Lambda\right) = \mathrm{Ext}_\Lambda^i\left(\begin{bmatrix} R & N \\ M & J \end{bmatrix}\Big/\begin{bmatrix} R & N \\ JM & J^2 \end{bmatrix}, e\Lambda\right)$$
$$\cong \mathrm{Ext}_\Lambda^i\left(\left(\begin{bmatrix} R & 0 \\ M & 0 \end{bmatrix} \otimes_R e\Lambda\right)\Big/\left(\begin{bmatrix} R & 0 \\ JM & 0 \end{bmatrix} \otimes_R e\Lambda\right), e\Lambda\right)$$
$$\cong \mathrm{Ext}_\Lambda^i(M/JM \otimes_R e\Lambda, e\Lambda)$$
$$\cong \mathrm{Ext}_R^i(M/JM, R) \neq 0,$$

then $h_i^\#$ is not epic. It follows that $\mathrm{Ext}_\Lambda^{i+1}\left(\Lambda\Big/\begin{bmatrix} R & N \\ M & J \end{bmatrix}, \Lambda\right) \neq 0$ from the exactness of the following sequence

$$\mathrm{Ext}_\Lambda^i\left(\Lambda\Big/\begin{bmatrix} R & N \\ JM & J\end{bmatrix}, \Lambda\right) \xrightarrow{h_i^\#} \mathrm{Ext}_\Lambda^i\left(\begin{bmatrix} R & N \\ M & J\end{bmatrix}\Big/\begin{bmatrix} R & N \\ JM & J\end{bmatrix}, \Lambda\right)$$

$$\longrightarrow \mathrm{Ext}_\Lambda^{i+1}\left(\Lambda\Big/\begin{bmatrix} R & N \\ M & J\end{bmatrix}, \Lambda\right) \longrightarrow \mathrm{Ext}_\Lambda^{i+1}\left(\Lambda\Big/\begin{bmatrix} R & N \\ JM & J\end{bmatrix}, \Lambda\right) = 0,$$

hence id-$\Lambda_\Lambda = i+1$ together with Theorem 2.6.

(iii) This can be proved by the similar manner as in (ii).

If we assume that $MI=M$ instead of $NJ=N$, Theorem 2.7 can be rewrited as follows:

THEOREM 2.8. *Assume further that* $MI=M$.

(1) *If* $\max(\text{id-}S_S, \text{id-}N_S) < \max(\text{id-}R_R, \text{id-}M_R) = i \neq 0$, *then* id-$\Lambda_\Lambda = i$ *if and only if* $\mathrm{Ext}_R^i(M, R\oplus M) = 0$.

(2) *If* $\max(\text{id-}R_R, \text{id-}M_R) < \max(\text{id-}S_S, \text{id-}N_S) = i \neq 0$ *and if* $\mathrm{Ext}_S^i(N/IN, S\oplus N) \neq 0$, *then* id-$\Lambda_\Lambda = i+1$.

(3) *Suppose that* $\max(\text{id-}S_S, \text{id-}N_S) = \max(\text{id-}R_R, \text{id-}M_R) = i \neq 0$.

(i) *If* $\mathrm{Ext}_S^i(Y, S\oplus N) \neq 0$ *for some* $Y_S \subseteq (S\oplus N)_S$, *then* id-$\Lambda_\Lambda = i+1$.

(ii) *If* id-$R_R >$ id-M_R *and if* $\mathrm{Ext}_S^i(N/IN, S) \neq 0$, *then* id-$\Lambda_\Lambda = i+1$.

(iii) *If* id-$M_R >$ id-R_R *and if* $\mathrm{Ext}_S^i(N/IN, N) \neq 0$, *then* id-$\Lambda_\Lambda = i+1$.

3. Self-injective rings.

In this section, we consider the condition for Λ to be right self-injective.

Let $\alpha: N \to \mathrm{Hom}_R(M, R)$ be a map defined by $n \mapsto (m \mapsto (n, m))$ for $n \in N$, $m \in M$ and $\sigma: S \to \mathrm{End}(M_R)$ the canonical map. Then we have the following theorem:

THEOREM 3.1. *If*

(1) R_R, M_R, N'_S *and* $\boldsymbol{l}_S(M)_S$ *are injective, where* $N' = \mathrm{Ker}\,\alpha$ *and* $\boldsymbol{l}_S(M) = \{s \in S \mid sm = 0$ *for every* $m \in M\}$,

(2) α *and* σ *are epic,*

(3) $\mathrm{Hom}_S(N, N' \oplus \boldsymbol{l}_S(M)) = 0$

are satisfied, then Λ_Λ *is injective.*

PROOF. Let $[X\ Y]$ be a right ideal of Λ. The exact seqence of right Λ-modules

$$0 \longrightarrow \begin{bmatrix} 0 & N' \\ 0 & \boldsymbol{l}_S(M)\end{bmatrix} \longrightarrow \Lambda \longrightarrow \begin{bmatrix} R & M^* \\ M & \mathrm{End}(M_R)\end{bmatrix} \longrightarrow 0,$$

where $M^* = \mathrm{Hom}_R(M, R)$, induces the following exact sequence

$$\mathrm{Ext}^1_\Lambda\left(\Lambda/[X\,Y],\ \begin{bmatrix} 0 & N' \\ 0 & l_S(M) \end{bmatrix}\right) \longrightarrow \mathrm{Ext}^1_\Lambda(\Lambda/[X\,Y],\ \Lambda)$$
$$\longrightarrow \mathrm{Ext}^1_\Lambda\left(\Lambda/[X\,Y],\ \begin{bmatrix} R & M^* \\ M & \mathrm{End}(M_R) \end{bmatrix}\right).$$

Since

$$\mathrm{Ext}^1_\Lambda\left(\Lambda/[XY],\ \begin{bmatrix} 0 & N' \\ 0 & l_S(M) \end{bmatrix}\right) \cong \mathrm{Ext}^1_\Lambda(\Lambda/[X\,Y],\ \mathrm{Hom}_S(\Lambda e',\ N' \oplus l_S(M)))$$
$$\cong \mathrm{Ext}^1_S(\Lambda/[X\,Y] \otimes_\Lambda \Lambda e',\ N' \oplus l_S(M)) = 0$$

and

$$\mathrm{Ext}^1_\Lambda\left(\Lambda/[X\,Y],\ \begin{bmatrix} R & M^* \\ M & \mathrm{End}(M_R) \end{bmatrix}\right) \cong \mathrm{Ext}^1_R(\Lambda/[X\,Y] \otimes_\Lambda \Lambda e,\ \Lambda e) = 0,$$

we have $\mathrm{Ext}^1_\Lambda(\Lambda/[X\,Y],\ \Lambda) = 0$, that is, Λ_Λ is injective.

THEOREM 3.2. *If*

(1) ${}_SM$ *and* ${}_RN$ *are flat,*

(2) *The natural maps* $I \otimes_R I \to I^2$ *and* $J \otimes_S J \to J^2$ *are isomorphisms,*

(3) $N = JN$,

(4) ${}_S(S/J)$ *is flat,*

then the converse of Theorem 3.1 *holds.*

PROOF. The exact sequence of right Λ-modules

$$0 \longrightarrow \begin{bmatrix} I & N \\ 0 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} R & N \\ 0 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} R & N \\ 0 & 0 \end{bmatrix} \Big/ \begin{bmatrix} I & N \\ 0 & 0 \end{bmatrix} \longrightarrow 0$$

yields the following exact sequence

$$\mathrm{Hom}_\Lambda\left(\begin{bmatrix} R & N \\ 0 & 0 \end{bmatrix},\ \begin{bmatrix} 0 & N' \\ 0 & l_S(M) \end{bmatrix}\right) \longrightarrow \mathrm{Hom}_\Lambda\left(\begin{bmatrix} I & N \\ 0 & 0 \end{bmatrix},\ \begin{bmatrix} 0 & N' \\ 0 & l_S(M) \end{bmatrix}\right)$$
$$\longrightarrow \mathrm{Ext}^1_\Lambda\left(\begin{bmatrix} R & N \\ 0 & 0 \end{bmatrix} \Big/ \begin{bmatrix} I & N \\ 0 & 0 \end{bmatrix},\ \begin{bmatrix} 0 & N' \\ 0 & l_S(M) \end{bmatrix}\right).$$

Since $\mathrm{Hom}_\Lambda\left(\begin{bmatrix} R & N \\ 0 & 0 \end{bmatrix},\ \begin{bmatrix} 0 & N' \\ 0 & l_S(M) \end{bmatrix}\right) \cong \begin{bmatrix} 0 & N' \\ 0 & l_S(M) \end{bmatrix} e = 0$ and

$$\mathrm{Ext}^1_\Lambda\left(\begin{bmatrix} R & N \\ 0 & 0 \end{bmatrix} \Big/ \begin{bmatrix} I & N \\ 0 & 0 \end{bmatrix},\ \begin{bmatrix} 0 & N' \\ 0 & l_S(M) \end{bmatrix}\right) = \mathrm{Ext}^1_\Lambda\left(\begin{bmatrix} R & N \\ 0 & 0 \end{bmatrix} \Big/ \begin{bmatrix} I & IN \\ 0 & 0 \end{bmatrix},\ \begin{bmatrix} 0 & N' \\ 0 & l_S(M) \end{bmatrix}\right)$$

$$\cong \mathrm{Ext}^1_\Lambda\left(R/I \otimes_R \begin{bmatrix} R & N \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & N' \\ 0 & \boldsymbol{l}_S(M) \end{bmatrix}\right)$$

$$\cong \mathrm{Ext}^1_R\left(R/I, \mathrm{Hom}_\Lambda\left(\begin{bmatrix} R & N \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & N' \\ 0 & \boldsymbol{l}_S(M) \end{bmatrix}\right)\right) = 0\,,$$

we have $\mathrm{Hom}_\Lambda\left(\begin{bmatrix} I & N \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & N' \\ 0 & \boldsymbol{l}_S(M) \end{bmatrix}\right) = 0$. Since $(-, -)$ is monic by Lemma 2.4, we obtain (3) of Theorem 3.1 by

$$\mathrm{Hom}_S(N, N' \oplus \boldsymbol{l}_S(M)) \cong \mathrm{Hom}_S\left(N, \mathrm{Hom}_\Lambda\left(\begin{bmatrix} 0 & 0 \\ M & S \end{bmatrix}, \begin{bmatrix} 0 & N' \\ 0 & \boldsymbol{l}_S(M) \end{bmatrix}\right)\right)$$

$$\cong \mathrm{Hom}_\Lambda\left(\begin{bmatrix} 0 & N \\ 0 & 0 \end{bmatrix} \otimes_S \begin{bmatrix} 0 & 0 \\ M & S \end{bmatrix}, \begin{bmatrix} 0 & N' \\ 0 & \boldsymbol{l}_S(M) \end{bmatrix}\right)$$

$$\cong \mathrm{Hom}_\Lambda\left(\begin{bmatrix} I & N \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & N' \\ 0 & \boldsymbol{l}_S(M) \end{bmatrix}\right) = 0\,.$$

Let $\nu : \begin{bmatrix} 0 & 0 \\ M & J \end{bmatrix} \subsetneq \begin{bmatrix} 0 & 0 \\ M & S \end{bmatrix}$ and put $g = \mathrm{Hom}_\Lambda(\nu, \Lambda)$. Then the diagram

$$\begin{array}{ccccc}
\mathrm{Hom}_\Lambda\left(\begin{bmatrix} 0 & 0 \\ M & S \end{bmatrix}, \Lambda\right) & \xrightarrow{g} & \mathrm{Hom}_\Lambda\left(\begin{bmatrix} 0 & 0 \\ M & J \end{bmatrix}, \Lambda\right) & \longrightarrow & \mathrm{Ext}^1_\Lambda(\mathrm{Coker}\,\nu, \Lambda) = 0 \\
\Big\downarrow \wr & & \Big\downarrow \wr & & \\
 & & \mathrm{Hom}_\Lambda\left(\begin{bmatrix} 0 & 0 \\ M & 0 \end{bmatrix} \otimes_R e\Lambda, \Lambda\right) & & \\
 & & \Big\downarrow \wr & & \\
S \oplus N & \xrightarrow{\sigma \oplus \alpha} & \mathrm{Hom}_R(M, M) \oplus \mathrm{Hom}_R(M, R) & &
\end{array}$$

commutes. Hence σ and α are epic. Let K be a right ideal of S. Since ${}_S(S/J)$ is flat, ${}_S\begin{bmatrix} 0 & 0 \\ M & J \end{bmatrix}$ is a pure submodule of ${}_S\begin{bmatrix} 0 & 0 \\ M & S \end{bmatrix}$ (see, e.g., [11, Proposition 11.1, p. 37]). Therefore ν induces $\tilde{\nu} = S/K \otimes_S \nu : S/K \otimes_S \begin{bmatrix} 0 & 0 \\ M & J \end{bmatrix} \subsetneq S/K \otimes_S \begin{bmatrix} 0 & 0 \\ M & S \end{bmatrix}$. Since Λ_Λ is injective and ${}_SM$ is flat, S_S and N_S are injective by Theorem 1.2. Consider the following commutative diagram

$$\begin{array}{ccccc}
\mathrm{Hom}_\Lambda\left(S/K \otimes_S \begin{bmatrix} 0 & 0 \\ M & S \end{bmatrix}, \Lambda\right) & \xrightarrow{g_1} & \mathrm{Hom}_\Lambda\left(S/K \otimes_S \begin{bmatrix} 0 & 0 \\ M & J \end{bmatrix}, \Lambda\right) & \longrightarrow & \mathrm{Ext}^1_\Lambda(\mathrm{Coker}\,\tilde{\nu}, \Lambda) = 0 \\
\Big\downarrow \wr & & \Big\downarrow \wr & & \\
 & & \mathrm{Hom}_S\left(S/K, \mathrm{Hom}_\Lambda\left(\begin{bmatrix} 0 & 0 \\ M & 0 \end{bmatrix} \otimes_R e\Lambda, \Lambda\right)\right) & & \\
 & & \Big\downarrow \wr & & \\
\mathrm{Hom}_S(S/K, S \oplus N) & \xrightarrow{g_2} & \mathrm{Hom}_S(S/K, \mathrm{Hom}_R(M, M) \oplus \mathrm{Hom}_R(M, R)) & \longrightarrow &
\end{array}$$

$$\longrightarrow \mathrm{Ext}^1_S(S/K, \boldsymbol{l}_S(M) \oplus N') \longrightarrow \mathrm{Ext}^1_S(S/K, S \oplus N) = 0\,,$$

where $g_1 = \mathrm{Hom}_\Lambda(\mathfrak{v}, \Lambda)$ and $g_2 = \mathrm{Hom}_S(S/K, \sigma \oplus \alpha)$, from which it follows that $\mathrm{Ext}_S^1(S/K, \boldsymbol{l}_S(M) \oplus N') = 0$. Hence N'_S and $\boldsymbol{l}_S(M)_S$ are injective. Moreover, R_R and M_R are injective by Theorem 1.2.

4. Derived contexts.

In this section, we suppose that $\langle M, N\rangle$ is the derived context of M_R. Then we have the following theorem.

THEOREM 4.1. *If* $\mathrm{Ext}_R^l(M, R \oplus M) = 0$ $(l>0)$, *then* $\text{id-}\Lambda_\Lambda = \max(\text{id-}R_R, \text{id-}M_R)$. *Furthermore, assuming that* ${}_SM$ *is flat, then* $\max(\text{id-}S_S, \text{id-}N_S) = \max(\text{id-}R_R, \text{id-}M_R)$.

PROOF. If both M_R and R_R are injective, then $\Lambda \cong \mathrm{Hom}_R(\Lambda e, \Lambda e)$ is right self-injective, for ${}_\Lambda\Lambda e$ is flat. Suppose that $\max(\text{id-}R_R, \text{id-}M_R) = i \neq 0$. Then there exists a right ideal L of R such that $\mathrm{Ext}_R^i(R/L, R \oplus M) \neq 0$. Now, let $[X\ Y]$ be a right ideal of Λ. Since ${}_\Lambda\Lambda e$ is flat and $\mathrm{Ext}_R^l(\Lambda e, \Lambda e) = 0$ $(l>0)$, we have

$$\mathrm{Ext}_\Lambda^{i+1}(\Lambda/[X\ Y], \Lambda) \cong \mathrm{Ext}_\Lambda^{i+1}(\Lambda/[X\ Y], \mathrm{Hom}_R(\Lambda e, \Lambda e))$$
$$\cong \mathrm{Ext}_R^{i+1}(\Lambda/[X\ Y] \otimes_\Lambda \Lambda e, \Lambda e) = 0$$

and

$$\mathrm{Ext}_\Lambda^i\left(\Lambda\Big/\begin{bmatrix} L & LN \\ M & S \end{bmatrix}, \Lambda\right) \cong \mathrm{Ext}_R^i\left(\Lambda\Big/\begin{bmatrix} L & LN \\ M & S \end{bmatrix} \otimes_\Lambda \Lambda e, \Lambda e\right)$$
$$\cong \mathrm{Ext}_R^i(R/L, R \oplus M) \neq 0$$

by Lemma 1.1. Hence $\text{id-}\Lambda_\Lambda = i$. Let V be a right S-module. Since ${}_SM$ is flat and $\mathrm{Ext}_R^l(M, R \oplus M) = 0$ $(l>0)$, we have

$$\mathrm{Ext}_S^{i+1}(V, S) = \mathrm{Ext}_S^{i+1}(V, \mathrm{Hom}_R(M, M)) \cong \mathrm{Ext}_R^{i+1}(V \otimes_S M, M) = 0$$

and

$$\mathrm{Ext}_S^{i+1}(V, N) = \mathrm{Ext}_S^{i+1}(V, \mathrm{Hom}_R(M, R)) \cong \mathrm{Ext}_R^{i+1}(V \otimes_S M, R) = 0$$

by Lemma 1.1. Hence $\max(\text{id-}S_S, \text{id-}N_S) \leqq i$. Let

$$0 \longrightarrow R \oplus M \longrightarrow E_0 \longrightarrow E_1 \longrightarrow \cdots \longrightarrow E_i \longrightarrow 0$$

be an injective resolution of $(R \oplus M)_R$. Then

$$0 \longrightarrow \mathrm{Hom}_R(M, R \oplus M) \longrightarrow \mathrm{Hom}_R(M, E_0) \longrightarrow \cdots \longrightarrow \mathrm{Hom}_R(M, E_i) \longrightarrow 0$$

is an injective resolution of $\mathrm{Hom}_R(M, R \oplus M)_S = (N \oplus S)_S$, for ${}_SM$ is flat and $\mathrm{Ext}_R^l(M, R \oplus M) = 0$ $(l>0)$. Thus $\max(\text{id-}S_S, \text{id-}N_S) = i$.

COROLLARY 4.2. *If* M_R *is finitely generated projective, then* $\text{id-}\Lambda_\Lambda = \text{id-}R_R$.

PROOF. This directly follows from Theorem 4.1.

5. Examples.

The following Examples are given to show the possibility that the equalities in both sides of Theorem 2.6 hold. In this section, $\boldsymbol{Z}$ denotes the ring of rational integers and $\boldsymbol{Q}$ the field of rational numbers.

EXAMPLE 5.1. Let

$$\Lambda=\begin{pmatrix} \boldsymbol{Q} & 0 & 0 & 0 \\ \boldsymbol{Q} & \boldsymbol{Q} & \boldsymbol{Q} & \boldsymbol{Q} \\ 0 & 0 & \boldsymbol{Z} & 0 \\ \boldsymbol{Q} & \boldsymbol{Q} & \boldsymbol{Q} & \boldsymbol{Q} \end{pmatrix},\ R=\begin{bmatrix} \boldsymbol{Q} & 0 \\ \boldsymbol{Q} & \boldsymbol{Q} \end{bmatrix},\ S=\begin{bmatrix} \boldsymbol{Z} & 0 \\ \boldsymbol{Q} & \boldsymbol{Q} \end{bmatrix},\ {}_SM_R=\begin{bmatrix} 0 & 0 \\ \boldsymbol{Q} & \boldsymbol{Q} \end{bmatrix},\ {}_RN_S=\begin{bmatrix} 0 & 0 \\ \boldsymbol{Q} & \boldsymbol{Q} \end{bmatrix}.$$

We define the pairings $(-,-)\colon N\otimes_S M\to R$ and $[-,-]\colon M\otimes_R N\to S$ via the multiplication in the ring R. Then the trace ideals are ${}_RI_R=\begin{bmatrix} 0 & 0 \\ \boldsymbol{Q} & \boldsymbol{Q} \end{bmatrix}$ and ${}_SJ_S=\begin{bmatrix} 0 & 0 \\ \boldsymbol{Q} & \boldsymbol{Q} \end{bmatrix}$, and the natural maps $I\otimes_R I\to I^2$ and $J\otimes_S J\to J^2$ are isomorphisms. Moreover, ${}_SM$ and ${}_RN$ are flat and $NJ=N$. Since id-$S_S=2$ (cf. [9, Proposition 7]), we have $\max(\text{id-}R_R, \text{id-}M_R)=1<\max(\text{id-}S_S, \text{id-}N_S)=1$. Furthermore, since $\mathrm{Ext}_S^2(N, S\oplus N)=0$, we have id-$\Lambda_\Lambda=2$ by Theorem 2.7(1).

EXAMPLE 5.2. Let

$$\Lambda=\begin{pmatrix} \boldsymbol{Z} & 0 & \boldsymbol{Z} & 0 \\ \boldsymbol{Q} & \boldsymbol{Z} & \boldsymbol{Q} & 0 \\ \boldsymbol{Z} & 0 & \boldsymbol{Z} & 0 \\ \boldsymbol{Q} & \boldsymbol{Q} & \boldsymbol{Q} & \boldsymbol{Q} \end{pmatrix},\ R=\begin{bmatrix} \boldsymbol{Z} & 0 \\ \boldsymbol{Q} & \boldsymbol{Z} \end{bmatrix},\ S=\begin{bmatrix} \boldsymbol{Z} & 0 \\ \boldsymbol{Q} & \boldsymbol{Q} \end{bmatrix},\ {}_SM_R=\begin{bmatrix} \boldsymbol{Z} & 0 \\ \boldsymbol{Q} & \boldsymbol{Q} \end{bmatrix},\ {}_RN_S=\begin{bmatrix} \boldsymbol{Z} & 0 \\ \boldsymbol{Q} & 0 \end{bmatrix}.$$

We define the pairings $(-,-)\colon N\otimes_S M\to R$ and $[-,-]\colon M\otimes_R N\to S$ via the multiplication in the ring S. Then the trace ideals are ${}_RI_R=\begin{bmatrix} \boldsymbol{Z} & 0 \\ \boldsymbol{Q} & 0 \end{bmatrix}$ and ${}_SJ_S=\begin{bmatrix} \boldsymbol{Z} & 0 \\ \boldsymbol{Q} & 0 \end{bmatrix}$, and the natural maps $I\otimes_R I\to I^2$ and $J\otimes_S J\to J^2$ are isomorphisms. Moreover, ${}_SM$ and ${}_RN$ are flat and $NJ=N$. Since id-$R_R=$ id-$S_S=2$ (cf. [9, Proposition 7]), we have $\max(\text{id-}R_R, \text{id-}M_R)=\max(\text{id-}S_S, \text{id-}N_S)=2$ and id-$S_S>$ id-$N_S=1$. Since

$$\mathrm{Ext}_R^2(M/JM, R)=\mathrm{Ext}_R^2\left(\begin{bmatrix} \boldsymbol{Z} & 0 \\ \boldsymbol{Q} & \boldsymbol{Q} \end{bmatrix}\Big/\begin{bmatrix} \boldsymbol{Z} & 0 \\ \boldsymbol{Q} & 0 \end{bmatrix},\ R\right)\cong\mathrm{Ext}_R^2([0\ \boldsymbol{Q}], R)\neq 0,$$

we get id-$\Lambda_\Lambda=3$ by Theorem 2.7(3) (ii).

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