TSUKUBA J. MATH.
Vol. 8 No. 2 (1984). 319–331

ON MAXIMAL GRADINGS OF SIMPLY CONNECTED ALGEBRAS

(Dedicated to Prof. N. Sone on his 65-th birthday)

By

Masahisa SATO

Recently K. Bongartz and P. Gabriel [2] introduced simply connected algebras and also proved that these algebras are completely determined by their trees and their gradings. Also they showed that each tree admits only a finite number of representation finite gradings.

In this paper, we are concerned with the maximal value $G(n)$ of gradings through all simply connected algebras with n simple modules for each n. This is accomplished by determining a maximal length $F(n)$ of the Auslander-Reiten quivers of these algebras, because we have $G(n+1)=F(n)+1$ in Lemma 1. (For the definition, see §1 or [2].) Further we shall show that in order to estimate the value $F(n)$, the following facts are essential.

(i) The Auslander-Reiten quiver of an algebra with the maximal length $F(n)$ is fully embedded in the Auslander-Reiten quiver of a suitable algebra whose graded tree admits $G(n+1)$.

(ii) In the latter quiver, there is a path from a vertex $P(p)$ to $P(t)$, where $P(p)$ and $P(t)$ are projective vertices which correspond respectively to vertices p and t of its graded tree such that a grading at t is $G(n+1)$ and a grading of p is maximal among gradings of vertices except t.

Finally we have the following result.

$G(2)=1,\ G(3)=3,\ G(4)=5,\ G(5)=7,\ G(6)=11,\ G(7)=15,\ G(8)\leqq 41$ and

$$G(n)\leqq\begin{cases}60n-469 & (9\leqq n\leqq 32)\\ n^2-4n+615 & (n\geqq 33)\end{cases}$$

$F(2)=2,\ F(3)=4,\ F(4)=6,\ F(5)=10,\ F(6)=14,\ F(7)\leqq 40$ and

$$F(n)\leqq\begin{cases}60n-410 & (8\leqq n\leqq 31)\\ n^2-2n+611 & (n\geqq 32)\end{cases}$$

It follows from our theorem that an upper bound of the number of

Received December 5, 1983.

indecomposable modules over simply connected algebra with n simple module is $\frac{(n-1)F(n)}{2}+4\left[\frac{n+2}{3}\right]$ where $[m]$ means a maximal natural number not exceeding m. We are sure that our result will help us to know the precise number $G(n)$.

§ 1. Preliminaries and Notations.

Let K be an algebraically closed field. Here we recall the definitions introduced in [2]. An algebra over K is called simply connected iff it is representation-finite, connected, basic, finite dimensional with a simply connected Auslander-Reiten quiver.

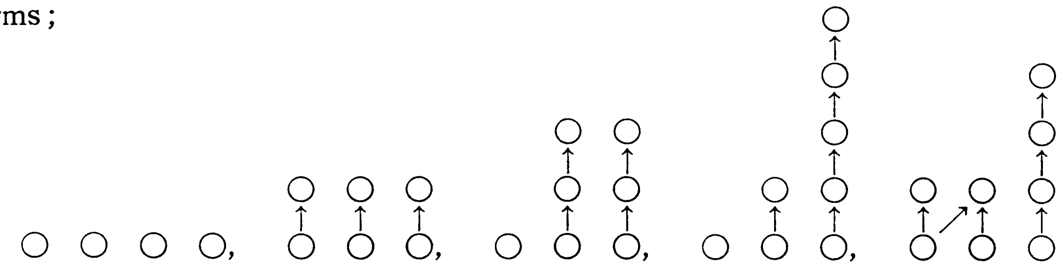
In the following, we use freely the results in [2] stated below.

Let (T, g) be a representation-finite graded tree. We consider the algebra $A^T=\bigoplus_{p,q} K(R_T)(p, q)$, where $K(R_T)$ is the mesh category of the Auslander-Reiten quiver R_T of the graded tree (T, g) and p, q run through all projective vertices in R_T.

THEOREM A. (Bongartz, Gabriel [2]) *The map* $(T, g)\mapsto A^T$ *yields a bijection between the isomorphism classes of representation-finite graded trees and isomorphism classes of simply connected algebras. Further in this case,* $K(R_T)\cong Ind(A^T)$ *and* $R_T\cong\Gamma_{A^T}$, *here* Γ_{A^T} *is the Auslander-Reiten qviver of the algebra* A^T.

According to this theorem, we shall identify A^T and (T, g) as follows. For a connected graded tree (T, g) with a maximal grading at a vertex x having r neighbours $\{x_i\}_{1\leq i\leq r}$, we can reconstruct graded trees (T^i, g^i)'s for $1\leqq i\leqq r$ by removing a vertex x as in [2]. Further we define a starting function at x by $s_x^T(y)=\dim_K K(R_T)(x, y)$ for each vertex y in R_T and also denote by $S_i^{T^i}$ the support of the starting functions $s_{x_i}^{T^i}$ in R_{T^i}, which is endowed with a partial order as a full subquiver of R_{T^i}.

The partially ordered set is representation-finite in the sense of Nazarova-Roiter [3] iff it does not contain as a full subposet one of the following five forms;



[1, 1, 1, 1]   [2, 2, 2]   [1, 3, 3]   [1, 2, 5]   [N, 4].

Next for a graded tree (T, g), we define a length function $L^T:(R_T)_0\rightarrow$

$N\cup\{0\}$ by $L^T(t)=0$ if $g(t)=0$ and $L^T(t)=L^T(s)+1$ if there is an arrow $s\to t$ in R_T.

Related to this, we define the length $L^T(R)$ of a full subtranslation quiver R of R_T as the maximal value of $L^T(z)$ where z runs over all vertices in R. We sometimes use the notation L instead of L^T if the meaning is clear. Also we put $F(n)=\max L^T(R_T)$ where (T, g) runs over all representation-finite graded trees with n vertices.

Then the next theorem is very useful for our classification in § 3.

THEOREM B. (Bongartz, Gabriel [2]) *Let (T, g) be an admissible tree. Then the following statements are equivalent.*

(1) *(T, g) is representation-finite.*

(2) *The following three conditions* (a), (b) *and* (c) *are satisfied.*

  (a) *Each (T^i, g^i) is representation-finite.*

  (b) *The value of each $s_{x_i}^{T^i}$ is $\leqq 1$.*

  (c) *The partially ordered set $S_{x_1}^{T^1}\amalg\cdots\amalg S_{x_r}^{T^r}$ is representation-finite in the sense of Nazarova-Roiter* [3].

§ 2. Simply Connected Algebras with Maximal Grading.

In this section, we shall study the Auslander-Reiten quivers of simply connected algebras in order to give an upper bound of the values of gradings. Let (T_n, g_n) be one of the representation-finite graded trees with n vertices such that there is a vertex t in T_n whose grading $g_n(t)$ is maximal among all possible values of gradings of representation-finite graded trees with n vertices. We put $G(n)=g_n(t)$.

Then we have the following lemmas. Here a vertex x of a tree T is called a tip if x has only one neighbour, clearly which is equivalent that $T\setminus\{x\}$ is connected.

LEMMA 1. *$G(n+1)=F(n)+1$ and $F(n+1)\geqq F(n)+2$.*

PROOF. Let (T, g) be a representation-finite graded tree with n vertices such that there is a vertex x in R_T with $L^T(t)=F(n)$. We construct a new translation quiver R as follows.

$$R_0=(R_T)_0\cup\{p, \tau^{-1}t\} \qquad \text{where } \tau \text{ is a translation,}$$

$$R_1=(R_T)_1\cup\{t\to p,\ p\to\tau^{-1}t\}.$$

The tree T' of R is the tree linked one vertex corresponding to p with T at a vertex corresponding to the τ-orbit of t by a path. Further

$$g'(z)=\begin{cases} g(z) & \text{if } z\in T, \\ F(n)+1 & \text{if } z=p, \end{cases}$$

is a grading of T'. Then (T', g') is a representation-finite graded tree with $R_{T'}=R$. Hence $F(n+1)\geqq L(R_{T'})=L(R_T)+2=F(n)+2$. Next we must show that $g'(p)=G(n+1)$. Let (T^*, g^*) be any representation-finite graded tree with $n+1$ vertices and let z be a vertex in T^* whose grading is maximal. Consider a connected component T_1^* of $T^*\setminus\{z\}$ which contains a vertex whose grading is 0. By Theorem B, $(T_1^*, g^*|T_1^*)$ is representation-finite, hence $L(R_{T_1^*})\leqq F(n)$. Also $g^*(z)\leqq L(R_{T_1^*})+1\leqq F(n)+1$, then $g'(p)=G(n+1)$.

LEMMA 2. *For the graded tree (T_{n+1}, g_{n+1}), t is a tip of T_{n+1}.*

PROOF. Assume the contrary t has at least two neighbours. Let T^* be a connected component of $T_{n+1}\setminus\{t\}$ which contains a vertex whose grading is 0. Since $(T^*, g|T^*)$ is representation-finite and $|T^*|\leqq n-1$, we can construct two representation-finite graded trees (T_1^*, g_1^*) and (T_2^*, g_2^*) in the following way;

$$T_1^*=T^*\cup\{t\} \qquad g_1^*=g^*|T_1^*$$

$$T_2^*=T_1^*\cup\{p\} \qquad g_2^*|T_1^*=g_1^* \quad \text{and} \quad g_2^*(p)=L(R_{T_1^*})+1.$$

Hence $G(n+1)\geqq g^*(p)>L(R_{T_1^*})\geqq g_1^*(t)=g^*(t)=G(n+1)$, which is a contradiction.

We put $T_n^*=T_{n+1}\setminus\{t\}$ and $g_n^*=g_{n+1}|T_n^*$, then T_n^* is connected tree from Lemma 2 and (T_n^*, g_n^*) is a representation-finite graded tree.

In the following, $P(t)$ denotes an indecomposable projective module corresponding to a vertex t in a tree and B_n denotes an algebra $A^{T_n^*}$.

LEMMA 3. *$\operatorname{rad} P(t)$ is simple injective as B_n-module.*

PROOF. Let L be a length function with respect to (T_n^*, g_n^*). By Lemma 2, $\operatorname{rad} P(t)$ is indecomposable, hence the canonical inclusion map $\operatorname{rad} P(t)\to P(t)$ is a irreducible map and $L(\operatorname{rad} P(t))+1=L(P(t))$. On the other hand, $g_{n+1}(t)=G(t)=F(n)+1$, thus $L(\operatorname{rad} P(t))=F(n)$. This means there is no irreducible map starting from $\operatorname{rad} P(t)$ in $R_{T_n^*}$, so $\operatorname{rad} P(t)$ is a simple injective B_n-module.

LEMMA 4. *Assume $p\in T_n^*$ is a vertex with a maximal grading in (T_n^*, g_n^*). Then there exists a path from $P(p)$ to $P(t)$ in $P_{T_{n+1}}$.*

PROOF. Assume there are no paths stated above. We consider a full subtranslation quiver R (it may be non-connected) of $R_{T_n^*\setminus\{p\}}$ consisting of vertices which are not successors of $P(p)$. So we put R^1 a connected component of R which contains $\operatorname{rad} P(t)$, further q a neighbour of p in T_n^* such that $P(q)$ belongs to R^1. For length functions L_1 and L with respect to R^1 and R respectively, $L-L_1$ has the constant value a for every vertex in R^1, where a is equal to the value of a minimal grading of projective vertices in R^1. We remark that

$$F(n)=G(n+1)-1=L(\operatorname{rad} P(t))=L_1(\operatorname{rad} P(t))+a\,.$$

If $a=0$ or R has at least three connected component, then as constructed in Lemma 1, there is a simply connected algebra whose maximal grading is larger than $F(n+1)$.

So we may assume $a>0$ and R has two connected component. Let R^2 be another connected component of R which contains a vertex with zero grading and M a neighbour of $P(p)$ such that M is contained in R^2. We remark $L(R^2) \geqq a$, since $L(R^2)\geqq L(M)=g_n^*(p)-1\geqq L(P(q))=L_1(P(q))+a\geqq a$.

Now we consider the following trees and their gradings.

$$T_n^*\setminus\{p\}=T_1\cup T_2 \quad \text{(disjoint union of connected trees)},$$

We may assume that q is a vertex of T_1. Under this assumption, we define

$$g_1=g_n^*-a\,|\,T_1 \quad \text{(a grading of } T_1),$$

$$g_2=g_n^*\,|\,T_2 \quad \text{(a grading of } T_2).$$

We can check the facts that (T_1, g_1) and (T_2, g_2) are representation-finite graded trees and R^1 and R^2 are full subtranslation quivers of R_{T_1} and R_{T_2} respectively. Choose a simple injective module S_2 in R_{T_2} and $S_1=P(z_1)$ a simple projective module in R_{T_1}, here z_1 is a vertex of T_1 such that $g_n^*(z_1)=a$. Then we can define a representation-finite translation quiver Q with $n-1$ vertices as follows.

$$Q_0=(R_{T_1})_0\cup\{P\}\cup(R_{T_2})_0 \quad \text{(set of vertices)},$$

$$Q_1=(R_{T_1})_1\cup(R_{T_2})_1\cup\{S_2\to P,\ P\to S_1\} \quad \text{(set of arrows)},$$

$$\tau^{-1}S_2=S_1 \quad \text{(new translation)}.$$

We put L^Q a length function with respect to Q, then we have $L^Q(\operatorname{rad} P(t)) =L_1(\operatorname{rad} P(t))+2+L^Q(S_2)=L_1(\operatorname{rad} P(t))+2+L_2(S_2)\geqq L_1(\operatorname{rad} P(t))+2+a=F(n)+2$, this is a contradiction.

The following corollary is useful to calculate an upper bound of $G(n+1)-G(n)$.

COROLLARY 5. *Assume T_1 is a connected component of $T_n^*\setminus\{p\}$ such that R_{T_1} has maximal length among the translation quivers corresponding to other connected* *components of* $T_n^*\backslash\{p\}$. *We put* $m=n-|T_1|-1$, *then it holds that*

(1) $F(n)=\max\{L(M)\,|\,M$ *is a successar of* $P(p)$ *in* $R_{T_n^*}.\}$.

(2) $F(n-1)\geqq L_1(R_{T_1})+2m$.

PROOF. The first statement follows immediately from Lemma 4. Since $|T_1|=n-m-1$, $F(n-m-1)\geqq L_1(R_{T_1})$ and $F(n-1)\geqq F(n-m-1)+2m$ by Lemma 1. Hence the second inequality holds.

Here we remark, by the above fact, it holds that

$$G(n+1)-g_n^*(p)=F(n)-L(\mathrm{rad}\,P(p))$$

and

$$G(n)-g_n^*(p)=F(n-1)-L(\mathrm{rad}\,P(p))\geqq L_1(R_{T_1})-L_1(\mathrm{rad}\,P(p))+2m\,,$$

hence

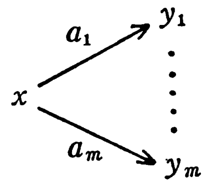
$$G(n+1)-G(n)\leqq\{F(n)-L(\mathrm{rad}\,P(p))\}-\{L_1(R_{T_1})-L_1(\mathrm{rad}\,P(p))\}-2m\,.$$

Now we must define some quiver which we need to estimate the value $F(n)-L(\mathrm{rad}\,P(p))=\max\{L(M)\}$ stated in Corollary 5.

We denote $\boldsymbol{r}(\mathrm{rad}\,P(p))$ and $\boldsymbol{r}^*(\mathrm{rad}\,P(p))$ full subtranslation quivers of $R_{T_n^*\backslash\{p\}}$ and $R_{T_n^*}$ consisting of successors of some indecomposable direct summand of $\mathrm{rad}\,P(p)$. Further we put $\boldsymbol{s}(\mathrm{rad}\,P(p))$ a full subtranslation quiver of $\boldsymbol{r}(\mathrm{rad}\,P(p))$ consisting of vertices m in $\boldsymbol{r}(\mathrm{rad}\,P(p))$ such that $\tau m\notin\boldsymbol{r}(\mathrm{rad}\,P(p))$. This is a union of some connected sections. We define $\boldsymbol{s}^*(\mathrm{rad}\,P(p))$ by a section in $\boldsymbol{r}^*(\mathrm{rad}\,P(p))$ linked the sections in $\boldsymbol{s}(\mathrm{rad}\,P(p))$ at p. Next, for a section $\boldsymbol{s}$ in a quiver R_T, we define a quiver $S(\boldsymbol{s})$ associated with a vector to each vertex.

Let $x_1,\cdots,x_n$ be vertices in $\boldsymbol{s}$. Inductively we define $S(\boldsymbol{s})$ and it's vector $d(x)\in Q^n$ for each vertex x in $S(\boldsymbol{s})$, here Q is the rational field.

First $d(x_i)=(\delta_{i,j})$, $(1\leqq i,\,j\leqq n$, and δ is the Kronecker $\delta)$.

Let  be a diagram already defined, here $a_1,\cdots,a_m$ are all arrows which start from x. $\tau^{-1}x$ is defined in the case that a vector $-(\sum_{i=1}^m d(y_i)-d(x))$ doesn't appear in vectors already defined and also we put $d(\tau^{-1}x)=\sum_{i=1}^m d(y_i)-d(x)$.

The following lemmas follow easily from definitions.

LEMMA 6. $\boldsymbol{r}(\mathrm{rad}\,(P(p))$ *and* $\boldsymbol{r}^*(\mathrm{rad}\,P(p))$ *are embeded into* $S(\boldsymbol{s}(\mathrm{rad}\,P(p)))$ *and* $S(\boldsymbol{s}^*(\mathrm{rad}\,P(p)))$ *respectively as full subtranslation quivers.*

LEMMA 7. *In the same notations of the above remark, it holds that* $F(n)-L(\text{rad}\,P(p))\leqq$ *the length of* $S(\boldsymbol{s}^*(\text{rad}\,P(p)))$.

[REMARK] From lemma 7, in order to calculate the value $F(n)-L(\text{rad}\,P(p))$, we need only to get a quiver $S(\boldsymbol{s})$ whose length is maximal for a possible section $\boldsymbol{s}=\boldsymbol{s}(\text{rad}\,P(p))$. So, we classify the possible $\boldsymbol{r}(\text{rad}\,P(p))$ and study $S(\boldsymbol{s}^*(\text{rad}\,P(p)))$, also we shall get an upper bound of $G(n+1)-G(n)$ for each case that $R_{T_{n+1}}$ has a subtranslation quiver classified there in next section.

§ 3. The Classification of $r(\text{rad}\,P(p))$ and an Upper Bound of $G(n+1)-G(n)$.

As stated before, in this section, we classify $\boldsymbol{s}(\text{rad}\,P(p))$ and $\boldsymbol{r}(\text{rad}\,P(p))$ such that the support of the starting function $s_{\text{rad}\,P(p)}$ is of finite type as a partially ordered set and the value of the function is not exceeding 1. Also we give an upper bound of $G(n+1)-G(n)$ when $R_{T_{n+1}}$ has $\boldsymbol{r}(\text{rad}\,P(p))$ in each case.

rad $P(p)$ has at most three direct summands, otherwise a partially ordered set [1, 1, 1, 1] appears in $S_{x_1}^{T^1}\amalg\cdots\amalg S_{x_r}^{T^r}$.

I. Suppose rad $P(p)$ is indecomposable.

We put $a_0=\text{rad}\,P(p)$. The slice $\boldsymbol{s}(a_0)$ is one of the following four forms.

The case (i)

$$a_0\longrightarrow a_1\longrightarrow\cdots\longrightarrow a_k \qquad 0\leqq k,$$

The case (ii)

$$\begin{array}{ccccccc} & & d_1 & \longrightarrow\cdots\longrightarrow & d_s & & \\ & & \uparrow & & & & \\ a_0\longrightarrow\cdots\longrightarrow & & a_k & \longrightarrow c_1\longrightarrow\cdots\longrightarrow & c_i & & 1\leqq j\leqq i\leqq s,\ 0\leqq k, \\ & & \downarrow & & & & \\ & & b_1 & \longrightarrow\cdots\longrightarrow & b_j & & \end{array}$$

The case (iii)

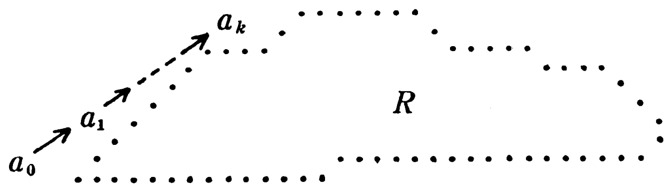
$$\begin{array}{ccccccc} & & & & d_1\longrightarrow\cdots\longrightarrow d_t & & \\ & & & & \uparrow & & \\ & & c_1 & \longrightarrow\cdots\longrightarrow & c_i\longrightarrow e_1\longrightarrow\cdots\longrightarrow e_s & & 1\leqq s\leqq t,\ 1\leqq i,\ j,\ 0\leqq k, \\ & & \uparrow & & & & \\ a_0\longrightarrow\cdots\longrightarrow & & a_k & \longrightarrow b_1\longrightarrow\cdots\longrightarrow b_j & & & \end{array}$$

Then case (iv)

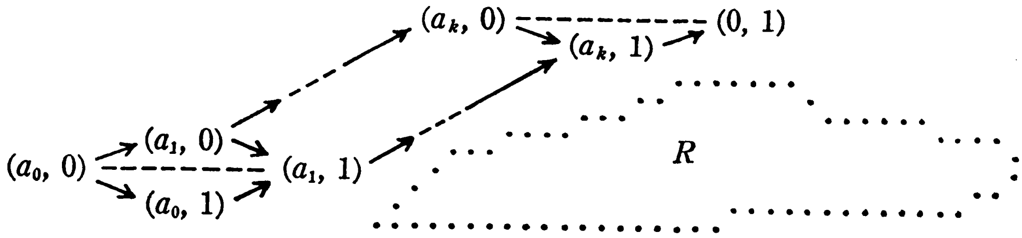
$$\begin{array}{ccccc} & & c_1 & \longrightarrow\cdots\longrightarrow c_i & \\ & & \uparrow & & \\ a_0\longrightarrow\cdots\longrightarrow & & a_k & \longrightarrow b_1\longrightarrow\cdots\longrightarrow b_j & 1\leqq j\leqq i,\ 0\leqq k. \end{array}$$

To avoid the lengthy explanation, the reason of the fact (for example, it is injective or non-injective, etc.) will be shown shortly in parenthesis except that we need to explain particularly.

The case (i). In this case, $r(a_0)$ is as follows. Hence if (T_n^*, g_n^*) has this slice $s(a_0)$, then $G(n+1)-G(n)\leqq 2$.



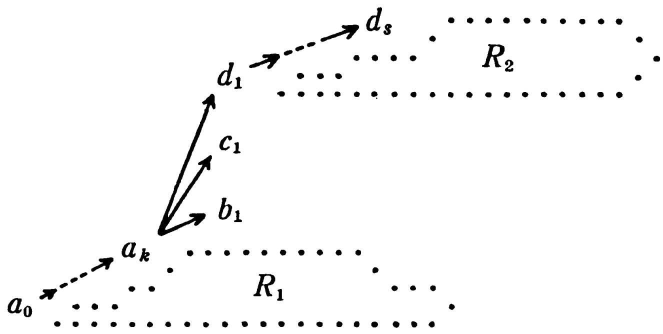
Then $r^*(a_0)$ is as follows.



The case (ii). In this case, a_k is injective. Otherwise $s_{a_0}(\tau^{-1}a_k)=2$. Also $j=1$ and $i=1$ or 2, otherwise it appears

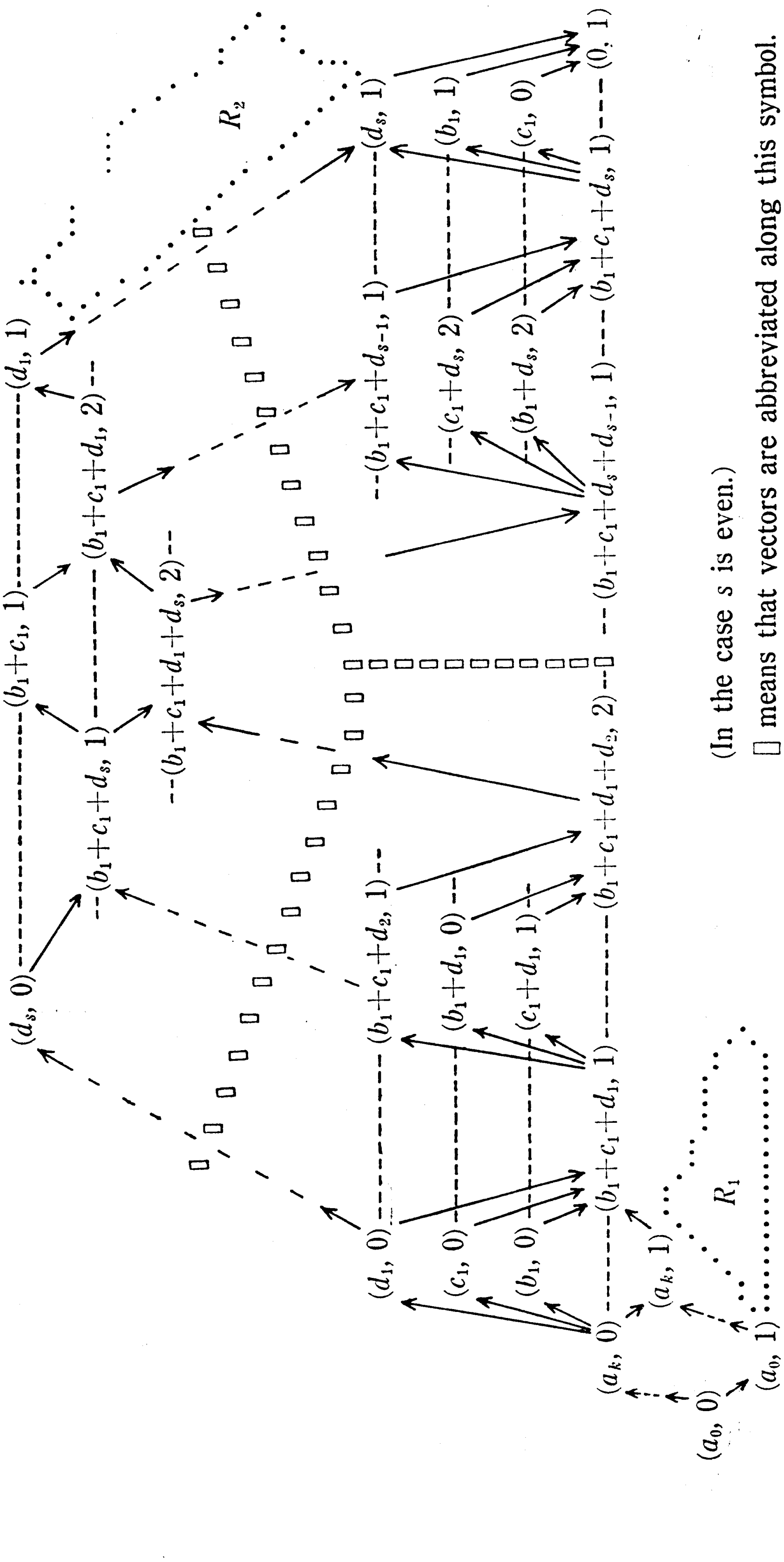
$$\left[\begin{array}{ccc} b_2 & c_2 & d_2 \\ \uparrow & \uparrow & \uparrow \\ b_1 & c_1 & d_1 \end{array}\right]=[2, 2, 2] \quad \text{and} \quad \left[\begin{array}{ccc} & c_3 & d_3 \\ & \uparrow & \uparrow \\ b_1 & c_2 & d_2 \\ & \uparrow & \uparrow \\ & c_1 & d_1 \end{array}\right]=[1, 3, 3].$$

If $i=1$, then $r(a_0)$ and $r^*(a_0)$ are as follows. Hence $G(n+1)-G(n)\leqq n$.



If $i=2$, then $s=2$, 3 or 4, otherwise it appears [1, 2, 5]. By looking over the quiver $S(r^*(\mathrm{rad}(a_0)))$ same as before, if $R_{T_{n+1}}$ has one of these quivers, then

$$G(n+1)-G(n)\leqq\begin{cases} 10 & (n=7) \\ 13 & (n=8) \\ 26 & (n\geqq 9) \end{cases}$$



(In the case s is even.)

□ means that vectors are abbreviated along this symbol.

The case (iii). Let r and m be maximal numbers through all these numbers r' and m' respectively such that $\tau^{-r'}a_{k+1-r'}$ and $\tau^{-m'}c_{i+1-m'}$ exist. In this case, $j=1$ or $s=1$, otherwise [2, 2, 2] appears. If $j\geqq 2$ and $s=1$, then $t=2$, 3 or 4 and a_k is injective since otherwise it appears [1, 2, 5] and [1, 1, 1, 1]. Further if a_k is non-injective, then c_1 is injective or $r=1$. If a_k and c_1 are non-injective, then some c_v is injective, otherwise $s_{a_0}(\tau^{-1}c_i)=2$. In any way, we can find a minimal number v such that c_v is injective if a_k is non-injective. Hence we have the following classification list in this case.

- $s=t=1$
  - a_k is non-injective. $j=1$. ([1, 1, 1, 1]) ……
    - c_1 is injective. ……(1)
    - c_1 is non-injective. $r=1$. ([1, 1, 1, 1]) ……(2)
  - a_k is injective. ……(3)
- $s\geqq 1$, $t\geqq 2$ …
  - $j=1$ …
    - a_k is non-injective.
      - c_1 is injective.—Continue to (*1).
      - c_1 is non-injective. $s=1$ ([2, 2, 2]), $r=1$ ([1, 1, 1, 1]) $t=2$ ([1, 3, 3])—Continue to (*2)
    - a_k is injective. ……
      - $s=1$.—Continue to (*3).
      - $s=2$. $t=2$, 3 or 4 ([1, 2, 5]), c_i is injective. ([1, 1, 1, 1])……(4)
  - $s=1$, $j\geqq 2$ …
    - $t=2$. $j=2$, 3 or 4. ([1, 2, 5])
      - $j=2$. $0\leqq m\leqq 3$.
        - $0\leqq m\leqq 2$……(5)
        - $m=3$. d_1 is injective. ([2, 2, 2])……(6)
      - $j=3$. $m=0$ or 1.……(7)
      - $j=4$. $m=0$ ([N, 4])……(8)
    - $t=3$, 4. $j=2$ ([1, 3, 3]), $m=0$ ([2, 2, 2])……(9)

If $R_{T_{n+1}}$ has one of the cases from (1) to (9) as $r(\operatorname{rad} a_0)$, then we get

$$G(n+1)-G(n)\leqq\begin{cases} 14 & (n=7)\\ 28 & (n=8)\\ 30 & (9\leqq n\leqq 17)\\ 2n-5 & (n\geqq 18).\end{cases}$$

(*1) In this case, $s=1$, $0\leqq m\leqq 3$ and $t\leqq 4$ by [2, 2, 2], [1, 2, 5] and [1, 2, 5] respectively. Assume $m=0$. If $t=2$, then $1\leqq r\leqq 3$ by [1, 2, 5] and if $t=3$ or 4, then $r=1$ by [1, 3, 3]. We have following four cases. (1) $m=0$, $r=1$ and $t=2$, 3 or 4. (2) $m=0$, $r=2$ or 3 and $t=2$. (3) $m=1$, $t=2$ and $r=1$ or 2 by [2, 2, 2] and [N, 4]. (4) $m=2$ or 3, $t=2$ and $r=1$ by [1, 3, 3].

In this case, we get

$$G(n+1)-G(n)\leqq\begin{cases} 10 & (n=7) \\ 16 & (n=8) \\ 28 & (n\geqq 9). \end{cases}$$

(*2) The following two cases are possible; (1) c_i is injective, $v=2$ or 3 by [1, 2, 5]: (2) c_i is non-injective, $v=2$, $m=1$ by [N, 4] and [1, 3, 3].

In these cases, we get

$$G(n+1)-G(n)\leqq\begin{cases} 16 & (n=8) \\ 30 & (n\geqq 9). \end{cases}$$

(*3) In this case, we have six cases by the same method as above and we get

$$G(n+1)-G(n)\leqq\begin{cases} 14 & (n=7) \\ 28 & (n=8) \\ 36 & (9\leqq n\leqq 35) \\ n+1 & (n\geqq 36). \end{cases}$$

The case (iv). This case is most complicated. But we can classify and calculate by similar method discussed above. So we shall only give the result.

$$G(n+1)-G(n)\leqq\begin{cases} 15 & (n=7) \\ 27 & (n=8) \\ 60 & (n\geqq 9). \end{cases}$$

II. Suppose that $\operatorname{rad} P(p)=a_0\oplus b_0$. Then $\boldsymbol{s}(\operatorname{rad} P(p))$ has the following form.

$$\begin{array}{ccccccc} & & d_1 \longrightarrow \cdots \longrightarrow d_t & & & & \\ & & \uparrow & & & & \\ a_0 \longrightarrow \cdots \longrightarrow & & a_i \longrightarrow e_1 \longrightarrow \cdots \longrightarrow e_s & & & & b_0 \longrightarrow \cdots \longrightarrow b_j \end{array}$$

$$0\leqq i,\ j,\ s,\ t \quad \text{and} \quad s\leqq t.$$

Clearly it is impossible that the case $t=s=0$ occurs by calculating vectors in $S(\boldsymbol{s}^*(\operatorname{rad} P(p))$. So assume $s\geqq 1$, then $s=1$ or $j=0$, otherwise it appears [2, 2, 2]. Furthermore if $s\geqq 2$ and $j=0$, then $s=2$ by [1, 3, 3]. We shall left the concrete classification to the readers. We get in these cases

$$G(n+1)-G(n)\leqq\begin{cases} 9 & (n=7) \\ 23 & (n=8) \\ 36 & (9\leqq n\leqq 35) \\ n+1 & (n\geqq 36). \end{cases}$$

III. In the case $\operatorname{rad} P(p)=a_0\oplus b_0\oplus c_0$. Then $\boldsymbol{s}(\operatorname{rad} P(p))$ has the form below, otherwise it appears [1, 1, 1, 1].

$$a_0 \longrightarrow \cdots \longrightarrow a_i \qquad b_0 \longrightarrow \cdots \longrightarrow b_j \qquad c_0 \longrightarrow \cdots \longrightarrow c_k$$

$$0\leqq k\leqq j\leqq i.$$

Here it must be $k=0$, otherwise it appears [2, 2, 2]. Further $j=0$ or 1 by [1, 3, 3]. So there are only two cases. (1) $j=0$ and (2) $j=1$, $i\leqq 3$ by [1, 2, 5]. Hence we get

$$G(n+1)-G(n)\leqq\begin{cases} 5 & (n=5) \\ 11 & (n=6) \\ 20 & (7\leqq n\leqq 22) \\ n-2 & (n\geqq 23). \end{cases}$$

This completes the classifications and the calculation of the possible values of $G(n+1)-G(n)$.

From the above values, we get a result stated in the introduction.

THEOREM 8. *Let n be a natural number and let $G(n)$ and $F(n)$ maximal numbers of all the values of gradings and lengthes of Auslander-Reiten quivers of simply connected algebras with n sinple modules respectively. Then it holds that*

$G(2)=1$, $G(3)=3$, $G(4)=5$, $G(5)=7$, $G(6)=11$, $G(7)=15$, $G(8)\leqq 41$ *and*

$$G(n)\leqq\begin{cases} 60n-469 & (9\leqq n\leqq 32) \\ n^2-4n+615 & (n\geqq 33). \end{cases}$$

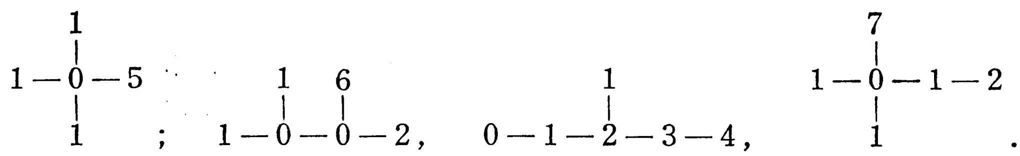
Also for $F(n)=G(n+1)-1$, we have

$F(2)=2$, $F(3)=4$, $F(4)=6$, $F(5)=10$, $F(6)=14$, $F(7)\leqq 40$ *and*

$$F(n)\leqq\begin{cases} 60n-410 & (8\leqq n\leqq 31) \\ n^2-2n+611 & (n\geqq 32). \end{cases}$$

[REMARK] The graded trees which gives $F(5)$ and $F(6)$ are as following.

$F(5)=10$ $F(6)=14$



For the number of indecomposable modules over simply connected algebras, we get the following corollary. (cf. [1])

COROLLARY 9. *The number of indecomposable modules over a simply connected algebra with n simple modules for a natural number n is smaller than* $\dfrac{(n-1)F(n)}{2}+4\left[\dfrac{n+2}{3}\right]$. *Here $[m]$ means a maximal natural number not exceeding m.*

PROOF. The number of vertices whose grading is 0 is smaller than $2\left[\frac{n+2}{3}\right]$. By duality, the number of injective module whose length is maximal is smaller than $2\left[\frac{n+2}{3}\right]$. So we get the above inequality.

References

[1] Bongartz, K., Treue einfach zusamenhangende Algebren I. Commen. Math. Helv., **2** (1982), 282-330.

[2] ——— and Gabriel, P., Covering Spaces in Representation Theory. Invent. Math., **65** (1982), 331-378.

[3] Nazarova, L. A. and Roiter, A. V., Representations of partially ordered set. Zap. Naucn. Sem. LOMI **28** (1972), 5-31.

[4] Riedtman, Chr., Algebren, Darstellungsköcher und zurück. Comm. Math. Helv., **55** (1980), 199-224.

Institute of Mathematics,
Yamanashi University
Kōfu, 400 Japan