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POLYNOMIAL INVARIANTS OF EUCLIDEAN LIE ALGEBRAS

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1. Introduction.

Let Δ be a root system in the sense of Bourbaki [1] and W be its Weyl group. Δ spans an l dimensional real vector space V on which W acts as a finite linear group. By extension of the transpose action, W acts on the symmetric algebra $S(V)$ of the dual space V^*. There is a well known theorem of Chevalley [2], that is, the ring of W-invariant elements of $S(V)$ is generated by l algebraically independent homogeneous polynomials.

How will the situation change when Δ is an infinite root system and W is an infinite group acting in V defined by a generalized Cartan matrix of non-finite type? With regard to this, there is a work of Moody [4], that the indefinite quadratic form by itself generates the entire ring of invariants, when Δ is defined by a generalized Cartan matrix of hyperbolic type.

In this paper, we study the ring of polynomial invariants of the Weyl group of a Euclidean Lie algebra. In this case, V is not isomorphic to V^* as W-module. So we have to consider both the ring of invariants of $S(V)$ and of $S(V^*)$, the symmetric algebra of V^*.

It becomes clear that the unique invariant vector, called null root, by itself generates the entire ring of invariants of $S(V^*)$ (*Theorem* 1) while the ring of invariants of $S(V)$ is isomorphic to that of corresponding finite type, which is generated by algebraically independent homogeneous elements (*Theorem* 2). This indicates that the polynomial invariants of Euclidean Lie algebras are critically situated between those of finite types and those of hyperbolic types.

When Δ is a root system in the sense of Bourbaki [1], this subject has some relation with classical Harish-Chandra's theorem, which states that the center of the universal enveloping algebra of corresponding finite dimensional complex simple Lie algebra isomorphic to the ring of W-invariants of $S(V^*)$. This theorem cannot be extended when Δ is an infinite root system. For example, we cannot use a number of propositions with respect to even the Casimir operator in this case. We hope to discuss this case in near feature.

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2. Preliminary.

A generalized Cartan matrix $A=(a_{ij})$ is a square matrix of integers satisfying $a_{ii}=2$, $a_{ij}\leqq 0$ if $i\neq j$, $a_{ij}=0$ if and only if $a_{ji}=0$. For any generalized Cartan matrix $A=(a_{ij})$ of size $l\times l$ and for any field $\boldsymbol{F}$ of characteristic zero, $\mathfrak{G}=\mathfrak{G}_{F}(A)$ denotes the Lie algebra over $\boldsymbol{F}$ generated by $3l$ generators $e_1, \cdots, e_l, h_1, \cdots, h_l, f_1, \cdots, f_l$ with the defining relations $[h_i, h_j]=0$, $[e_i, f_j]=\delta_{ij}h_i$, $[h, e_j]=a_{ij}e_j$, $[h_i, f_j]=-a_{ij}f_j$ for all i, j, and $(ad\ e_i)^{-a_{ij}+1}e_j=0$, $(ad\ f_i)^{-a_{ij}+1}f_j=0$ for distinct i, j. We call this algebra $\mathfrak{G}$ the Kac-Moody Lie algebra over $\boldsymbol{F}$ associated with A. Let Γ be a free $\boldsymbol{Z}$-module of rank l, and choose a free basis $\{\alpha_1, \cdots, \alpha_l\}$ of Γ. By defining $\deg(e_i)=\alpha_i$, $\deg(h_i)=0$, $\deg(f_i)=-\alpha_i$ for all i, we can view $\mathfrak{G}$ as a Γ-graded Lie algebra $\mathfrak{G}=\bigoplus_{\alpha\in\Gamma}\mathfrak{G}_\alpha$, where $\mathfrak{G}_\alpha$ is the subspace of $\mathfrak{G}$ corresponding to α. Put $\Delta=\}\alpha\in\Gamma|\mathfrak{G}_\alpha\neq 0\}$, called *the root system* of $\mathfrak{G}$. We call $\Pi=\{\alpha_1, \cdots, \alpha_l\}$ *a fundamental root system* of Δ. Let w_i be a $\boldsymbol{Z}$-module automorphism of Γ defined by $w_i(\alpha_j)=\alpha_j-a_{ij}\alpha_i$, and let W be the subgroup of $GL(\Gamma)$ generated by w_i for all i. We call W the *Weyl groop* of A. Then Δ is W-stable. Set $\Delta^{re}=\{w(\alpha_i)\,|\,w\in W$, for all $i\}$ *real roots*, and $\Delta^{im}=\Delta-\Delta^{re}$, *imaginary roots*.

A generalized Cartan matrix A is called *of finite type* if $\mathfrak{G}_F(A)$ is of finite dimension. A generalized Cartan matrix A is called *of Euclidean type* if A is singular and possesses the property that removal of any row and the corresponding column leaves a Cartan matrix of finite type, in which case $\mathfrak{G}_F(A)$ is called *a Euclidean Lie algebra*. Generalized Cartan matrices of Euclidean type are classified (cf. [5]).

A generalized Cartan matrix A is called *symmetrizable* if there is a positive rational diagonal matrix D such that $B=(b_{ij})=DA$ is a symmetric matrix. Generalized Cartan matrices of finite type and of Euclidean type are symmetrizable. Usually D is normalized by the properties that b_{ij} is a half-integer for distinct i, j, and that b_{ii} is an integer and the greatest common divisor of $\{b_{ii}\,|$ for all $i\}$ is 1 (cf. [3]). Then we can define a symmetric bilinear form on Γ by $(\alpha_i, \alpha_j)=b_{ij}$, and we can see easily that this form is W-invariant and $2(\alpha_i, \alpha_j)/(\alpha_i, \alpha_i)=a_{ij}$.

3. Ring of invariants of $S(V^*)$.

From now on we assume that $A=(a_{ij})$ is a generalized Cartan matrix of

Euclidean type of size $(l+1)\times(l+1)$. According to the classification (cf. [3]), A is expressed as $\begin{pmatrix} A_0 & * \\ * & 2 \end{pmatrix}$, where A_0 is of type X_l (resp. B_l, C_l, F_4, G_2, B_l) where A is of type $X_l^{(1)}$ (resp. $D_{l+1}^{(2)}, A_{2l-1}^{(2)}, E_6^{(2)}, D_4^{(3)}, A_2^{(2)}$). We call A_0 *the corresponding finite type of* A. The set Δ^{im} is a free $\boldsymbol{Z}$-module of rank 1 and a nonzero element of Δ^{im} is called *a null root*. Let ν be the generator of Δ^{im} with the coefficient of $\alpha_{l+1}=1$ and V be the $\boldsymbol{R}$-span of Δ, then we obtain:

THEOREM 1. *The ring of W-invariants of the symmetric algebra $S(V^*)$ of V is generated by ν.*

PRROF. Let V_0 be the subspace of V generated by $\alpha_1, \cdots, \alpha_l$. Then $V=V_0+\boldsymbol{R}\nu$ and $\Delta_0=\Delta\cap V_0$ is the root system of the corresponding finite type of A. The root ν is expressed as $\alpha_{l+1}+\phi$, where ϕ is the highest long root of Δ_0 when A is of type $X_l^{(1)}$, highest short root when A is of type $D_{l+1}^{(2)}, A_{2l-1}^{(2)}, E_6^{(2)}$ and $D_4^{(3)}$ and $2\times$highest short root when A is of type $A_2^{(2)}$. The restriction of $(\ ,\ )$ to V_0 is equal to the Killing form on $\mathfrak{G}_C(A_0)$. Considering that ν is W-invariant and $2(\alpha_{l+1}, \alpha_j)/(\alpha_{l+1}, \alpha_{l+1})=a_{l+1\,j}$, we deduce $w_{l+1}(\alpha_j)=w_\phi(\alpha_j)-a_{l+1\,j}\nu$, $w_{l+1}(\nu)=\nu$, where w_ϕ is the reflection with respect to ϕ in V_0 and $j=1, \cdots, l$. We can see easily that there exists an element w of W such that $w(\alpha_j)=\alpha_j-a_{l+1\,j}$, $w(\nu)=\nu$ $(j=1, \cdots, l)$.

Let f be a W-invariant of $S(V^*)$, expressed by a polynomial of $\alpha_1, \cdots \alpha_l, \nu$, then $f(\alpha_1, \cdots, \alpha_l, \nu)=f(\alpha_1-a_{l+1\,1}\nu, \cdots, \alpha_l-a_{l+1}\nu, \nu)$. Differentiating the both sides with respect to ν, we deduce $\sum_{j=1}^{l} a_{l+1\,j}(\partial f/\partial\alpha_j)(\alpha_1, \cdots, \alpha_l, 0)=0$. If $A\neq A_l^{(1)}$, using the classification of Euclidean type, we can see easily that there exists a unique j $(1\leqq j\leqq l)$ such that $a_{l+1\,j}\neq 0$. This indicates $(\partial f/\alpha x_j)(\alpha_1, \cdots, \alpha_l, 0)=0$. As f is W-invariant, we can prove $(\partial f/\partial\alpha_j)(\alpha_1, \cdots, \alpha_l, 0)=0$ for all j $(1\leqq j\leqq l)$. This indicates $f(\alpha_1, \cdots, \alpha_l, 0)$ is constant and $f\in\boldsymbol{R}[\nu]$.

If $A=A_l^{(1)}$ $(l\geqq 3)$, we can easily see

$$(\partial f/\partial\alpha_1)(\alpha_1, \cdots, \alpha_l, 0)+(\partial f/\partial\alpha_l)(\alpha_1, \cdots, \alpha_l, 0)=0$$

when A is expressed as

$$\begin{pmatrix} 2 & -1 & & & -1 \\ -1 & 2 & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & 2 & -1 \\ -1 & & & -1 & 2 \end{pmatrix}$$

As f is W-invariant, we can prove

$$-(\partial f/\partial \alpha_1)(\alpha_1, \cdots, \alpha_l, 0)+(\partial f/\partial \alpha_l)(\alpha_1, \cdots, \alpha_l, 0)=0 .$$

This indicates $(\partial f/\partial \alpha_1)(\alpha_1, \cdots, \alpha_l, 0)=0$ and $f(\alpha_1, \cdots, \alpha_l, 0)$ is constant.

If $A=A_2^{(1)}$, we can deduce

$$(\partial f/\partial \alpha_1)(\alpha_1, \alpha_2, 0)+(\partial f/\partial \alpha_2)(\alpha_1, \alpha_2, 0)=0 ,$$

$$-(\partial f/\partial \alpha_1((\alpha_1, \alpha_2, 0)+(\partial f/\partial(\alpha_1+\alpha_2))(\alpha_1, \alpha_2, 0)=0$$

and

$$(\partial f/\partial(\alpha_1+\alpha_2))(\alpha_1, \alpha_2, 0)-(\partial f/\partial \alpha_2)(\alpha_1, \alpha_2, 0)=0 .$$

This indicates $(\partial f/\partial \alpha_1)(\alpha_1, \alpha_2, 0)=(\partial f/\partial \alpha_2)(\alpha_1, \alpha_2, 0)=0.$ G. E. D.

4. Ring of invariants of $S(V)$.

Let W_0 be the subgroup of W generated by $w_1, \cdots, w_l$, then W_0 acts on V_0 as the Weyl group of Δ_0. This induces the decomposition of V as W_0-module, $V=V_0+\boldsymbol{R}\nu$ and $S(V_0)$ can be considered as a subalgebra of $S(V)$ by using the inclusion from V_0^* to V^* with respect to this decomposition. Then we obtain:

THEOREM 2. *The ring of W-invariants of $S(V)$ is equal to the ring of W_0-invariants of $S(V_0)$. Consequently the ring of W_0-invariants of $S(V)$ is generated by l algebraically independent homogeneous polynomials.*

PROOF. Let $\{\alpha_1^*, \cdots, \alpha_l^*, \nu^*\}$ be the dual basis of $\{\alpha_1, \cdots, \alpha_l, \nu\}$. Easy calculations show that $w_{l+1}(\alpha_j^*)=w_\phi(\alpha_j^*)$ $(j=1, \cdots, l)$, $w_{l+1}(\nu^*)=\nu^*-\sum_{j=1}^{l} a_{l+1\,j}\alpha_j^*$. It is clear that $S(V)^{W_0}$, the ring of W_0-invariants of $S(V_0)$, is a subalgebra of the ring of W-invariants of $S(V)$ and the ring of W_0-invariants of $S(V)$ is $S(V_0)^{W_0}[\nu^*]$, the polynomial ring of the commutative algebra $S(V_0)^{W_0}$ with the indeterminate ν^*.

Clearly the ring of W-invariants of $S(V)$ is a subalgebra of the ring of W_0-invariants of $S(V)$. Let f be any element of the ring of W-invariants of $S(V)$. Then f can be expressed by a polynomial of ν^* with coefficients in $S(V_0)^{W_0}$ and satisfying the equation $f(\nu^*)=f\left(\nu^*-\sum_{j=1}^{l} a_{l+1\,j}\alpha_j^*\right)$, where $\sum_{j=1}^{l} a_{l+1\,j}\alpha_j^*$ is non-zero from the classification. Then the following lemma completes the proof of the theorem. Q. E. D.

LEMMA. *Let R be a commutative algebra over a field of characteristic zero and $f(X)$ be any element of $R[X]$. If $f(X)$ satisfies the equation $f(X)=f(X+c$ where c is not a zero divisor of R, then $\deg f=0$.*

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