TSUKUBA J. MATH.
Vol. 8 No. 2 (1984). 209—218

AN ACCESSIBILITY PROOF OF ORDINAL DIAGRAMS IN INTUITIONISTIC THEORIES FOR ITERATED INDUCTIVE DEFINITIONS

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Let $(I, \prec)$ be a non-empty well-ordered system with the least element 0, and $\tilde{I}$ be $I \cup \{\infty\}$ with the largest element ∞. Let A be a non-empty well-ordered set. Then $O(I, A)$ denotes the system of ordinal diagrams (o.d.'s) based on I and A. (cf. [9, §26].) The accessibility proof for $O(I, A)$ in [9, pp. 298-309] shows that *every* o.d. from $O(I, A)$ is accessible with respect to $<_i$ for *every* i in $\tilde{I}$.

The central notions in this proof are i-fans and i-accessibility for i in $\tilde{I}$. Roughly speaking, an o.d. μ is an i-fan if for every $j \prec i$ and every j-section ν of μ, ν is j-accessible, and an o.d. is i-accessible if it is accessible in i-fans with respect to $<_i$.

Consider the case when the order type of $(I, \prec)$ is a successor ordinal $\xi+1$. If we formalize this accessibility proof for $O(\xi+1, 1)$ $(=O(I, 1))$ naturally, then this proof can be done in the intuitionistic theory $\mathrm{ID}^i_{\xi+1}$ for $\xi+1$-times iterated inductive definitions.

The purpose of this paper is to show the following fact: the accessibility of *each* o.d. from $O(\xi+1, 1)$ with respect to $<_0$ is derivable in ID^i_ξ. (Theorem)

In the case when ξ equals ω, this theorem will complement the consistency proof in [1] in the following sense. We will give in [1] a consistency proof for the subsystem $(\Pi^1_1-\mathrm{CA})+(\mathrm{BI})$ of classical analysis by the accessibility of $O(\omega+1, 1)$ with respect to $<_0$. It follows from the well-known equivalence between the classical version ID_ω of ID^i_ω and $(\Pi^1_1-\mathrm{CA})+(\mathrm{BI})$ that this consistency proof is optimal.

The author is indebted to Dr. T. Yukami for the seminar under the guidance of him during the preparation of this paper. The author wishes to express his heart-felt thanks to Prof. N. Motohashi for reading this paper in manuscript and suggesting a number of linguistic improvements.

Received October 21, 1983.

Let $\prec$ be a primitive recursive well-ordering with the least element 0 and the largest element ξ, and I be the primitive recursive domain of $\prec$. Let $\lambda x.\ x\oplus 1$ and $\lambda x.\ x\ominus 1$ be primitive recursive successor and predecessor function with respect to $\prec$, respectively. And we will assume throughout this paper that the above facts except the well-orderedness of $\prec$ are all derivable in the primitive recursive arithmetic PRA, that is to say, we will assume that the following formulae are all derivable in PRA:

$$x\prec y \longrightarrow I(x)\wedge I(y)\,,$$

$$I(x) \longrightarrow \neg(x\prec x)\,,$$

$$x\prec y\wedge y\prec z \longrightarrow x\prec z\,,$$

$$I(x)\wedge I(y) \longrightarrow x\prec y\vee x=y\vee y\prec x\,,$$

$$I(0),\ I(x) \longrightarrow 0\leqq x\,, \qquad (x\leqq y :=x\prec y\vee x=y)$$

$$I(x) \longrightarrow x\leqq\xi\,,$$

$$I(x) \longrightarrow x\leqq x\oplus 1\,,$$

$$x\prec\xi \longrightarrow x\prec x\oplus 1\,,$$

$$y\prec x \longrightarrow y\oplus 1\leqq x\,,$$

$$I(x) \longrightarrow x\ominus 1\leqq x\,,$$

$$x\prec\xi \longrightarrow (x\oplus 1)\ominus 1=x\,,$$

$$x\ominus 1\prec x \longrightarrow x=(x\ominus 1)\oplus 1\,.$$

Then the following formulae are also derivable in PRA:

$$x\prec\xi \longrightarrow (y\prec x\oplus 1 \longleftrightarrow y\leqq x)\,,$$

$$x\prec\xi \longrightarrow (x\leqq y\leqq x\oplus 1 \longrightarrow y=x\vee y=x\oplus 1)\,,$$

$$y\prec\xi \longrightarrow (y\oplus 1=x \longrightarrow x\ominus 1\prec x)\,.$$

Further let *Suc* and *Lim* be unary predicate constants with their defining axioms:

$$Suc(x) \longleftrightarrow x\ominus 1\prec x\,,$$

$$Lim(x) \longleftrightarrow I(x)\wedge x\neq 0\wedge \neg Suc(x)\,.$$

Then the following formulae are also derivable in PRA:

$$I(x) \longrightarrow (x=0\vee Suc(x)\vee Lim(x))\,.$$

$$Lim(x)\wedge y\prec x \longrightarrow y\oplus 1\prec x\,.$$

Next, we will consider the system of o.d.'s $O^*(I, 1)$. $O^*(I, 1)$ is an inessential modification of $O(I, 1)$. In contrast with $O(I, 1)$, $O^*(I, 1)$ has an identity 0 with respect to $\sharp$. For the precise definition of $O^*(I, 1)$, we refer to Levitz [7].

We will assume an arithmetization of the o.d.'s in $O^*(I, 1)$. Thus we have the following predicate constants for primitive recursive predicates:

'$*$ is an o.d.', '$*_1$ is a component of $*_2$',
$*_1 \equiv *_2$ for '$*_1$, $*_2$ are o.d.'s and $*_1$ is equal to $*_2$.',
$*_1 \subset_{*_3} *_2$ for '$*_1$, $*_2$ are o.d.'s, $*_3 \leqq \xi$ and $*_1$ is a $*_3$-section of $*_2$.',
$*_1 <_{*_3} *_2$ for '$*_1$, $*_2$ are o.d.'s, $*_3 \leqq \xi$ and $*_1$ is smaller than $*_2$ with respect to $<_{*_3}$.'.

In the following, we will employ the following syntactical variables:

i, j, k vary through the elements in I,
μ, ν, ρ, λ vary through o.d.'s.

Following Kreisel [6], we will define the notion of i-accessibility for $i \prec \xi$ in $\mathrm{ID}^i_\xi(\mathfrak{A})$ for some positive operator form $\mathfrak{A}$. Let $\mathfrak{A}(X, Y, i, \mu)$ be the following positive operator form:

$$\mathfrak{F}(i, \mu, Y) \wedge \forall \nu <_i \mu (\mathfrak{F}(i, \nu, Y) \longrightarrow X(\nu))$$

where $\mathfrak{F}(i, \mu, Y)$ is the formula $\forall k \prec i \, \forall \rho \subset_k \mu \, Y(k, \rho)$.

Let Prog $[X, R, Y]$ be the formula

$$\forall \mu (X(\mu) \wedge \forall \nu (R(\nu, \mu) \wedge X(\nu) \longrightarrow Y(\nu)) \longrightarrow Y(\mu)) .$$

If we write A for the set constant $P^{\mathfrak{A}}$, and $F_i(\mu)$ for $\forall j \prec i \, \forall \nu \subset_j \mu \, A_j(\nu)$, then the axioms $(P^{\mathfrak{A}}.1)_\xi$ and $(P^{\mathfrak{A}}.2)_\xi$ in [4, p. 307] become the following $(A.1)_\xi$ and $(A.2)_\xi$, respectively:

$(A.1)_\xi$ $\forall i \prec \xi \, \mathrm{Prog}\,[F_i, <_i, A_i]$,
$(A.2)_\xi$ $\forall i \prec \xi \, (\mathrm{Prog}\,[F_i, <_i, Q] \longrightarrow A_i \subseteqq Q)$,

for each formula Q in $\mathrm{ID}^i_\xi(\mathfrak{A})$.

And further $\mathrm{ID}^i_\xi(\mathfrak{A})$ has the following $(\mathrm{TI})_\xi$ going beyond the Heyting's arithmetic:

$$(\mathrm{TI})_\xi \quad \forall i \prec \xi \, (\forall j \prec i Q(j) \longrightarrow Q(i)) \longrightarrow \forall i \prec \xi Q(i)$$

for each formula Q in $\mathrm{ID}^i_\xi(\mathfrak{A})$.

The intended meanings of $A_i(\mu)$ and $F_j(\nu)$ are that μ is i-accessible and ν is a j-fan in the sense of introduction.

The following proposition is easily verified:

PROPOSITION 1. *The following formulae are all derivable in* $\mathrm{ID}^{i}_{\xi}(\mathfrak{A})$:

1.1. $\forall i \prec \xi\, (A_i \subseteqq F_i)$;
1.2. $\forall i \prec \xi\, \forall \mu\, (A_i(\mu) \to \forall \nu <_i \mu (F_i(\nu) \to A_i(\nu)))$;
1.3. $\forall i \preceq \xi\, \forall \mu\, \forall \nu\, (\mu \equiv \nu \wedge F_i(\mu) \to F_i(\nu))$;
1.4. $\forall i \prec \xi\, \forall \mu\, \forall \nu\, (\mu \equiv \nu \wedge A_i(\mu) \to A_i(\nu))$;
1.5. $\forall i \prec \xi\, \forall \mu\, (\forall \nu (\text{'}\nu$ *is a component of* $\mu\text{'} \to A_i(\nu)) \to A_i(\mu))$.

LEMMA 2. *Let* $\cap_{k<i} A_k(\mu)$ *be the formula* $\forall k \prec i\, A_k(\mu)$. *Then* $\forall i \preceq \xi\, (\forall j \prec i (A_j \subseteqq \cap_{k<j} A_k) \to \mathrm{Prog}\,[F_i, <_i, \cap_{k<i} A_k])$ *is derivable in* $\mathrm{ID}^{i}_{\xi}(\mathfrak{A})$.

PROOF.

2.1. The case $i=0$. Trivial.

2.2. The case $Suc(i)$.

Put

$$i_0 = i \ominus 1\,,$$

then

$$i = i_0 \oplus 1 \quad \text{and} \quad i_0 \prec i$$

Assume that

$$\forall j \prec i\, (A_j \subseteqq \cap_{k<j} A_k)\,,$$

then we have

$$\cap_{k<i} A_k = A_{i_0}\,.$$

Now we have to show

$$\mathrm{Prog}\,[F_{i_0 \oplus 1}, <_{i_0 \oplus 1}, A_{i_0}]\,.$$

But the proof of lemma 26.32 in [9] can be regarded as the proof of $\mathrm{Prog}\,[F_{i_0 \oplus 1}, <_{i_0 \oplus 1}, A_{i_0}]$ in $\mathrm{ID}^{i}_{\xi}(\mathfrak{A})$.

2.3. The case $Lim(i)$.

We can read the proof of lemma 26.33 in [9] as the proof of this case in $\mathrm{ID}^{i}_{\xi}(\mathfrak{A})$.

LEMMA 3. *Let* $\bar{A}$ *be* $\cap_{i<\xi} A_i$. *Then* $\mathrm{Prog}\,[F_\xi, <_\xi, \bar{A}]$ *is derivable in* $\mathrm{ID}^{i}_{\xi}(\mathfrak{A})$.

PROOF.

From $(A.2)_\xi$ we have

$$\forall j \prec \xi\, (\mathrm{Prog}\,[F_j, <_j, \cap_{k<j} A_k] \longrightarrow A_j \subseteqq \cap_{k<j} A_k)\,.$$

Hence it follows from lemma 2 that

$$\forall i \preceq \xi\, (\forall j \prec i\, \mathrm{Prog}\,[F_j, <_j, \cap_{k<j} A_k] \longrightarrow \mathrm{Prog}\,[F_i, <_i, \cap_{k<i} A_k])\,.$$

It follows from this and $(\mathrm{TI})_\xi$ that

$$\forall i \prec \xi\, \mathrm{Prog}\,[F_i, <_i, \cap_{k<i} A_k]\,,$$

and

$$\forall i \prec \xi \, \mathrm{Prog}\,[F_i, <_i, \cap_{k<i} A_k] \longrightarrow \mathrm{Prog}\,[F_\xi, <_\xi, \bar{A}] .$$

Therefore the assertion follows.

LEMMA 4. $\forall \mu <_\xi (\xi, 0)(F_\xi(\mu) \to \bar{A}(\mu))$ *is derivable in* $\mathrm{ID}^1_\xi(\mathfrak{A})$.

PROOF.

Let $R_i(\nu)$ be the formula:

$$\forall \mu <_\xi (i, \nu)(F_\xi(\mu) \longrightarrow \bar{A}(\mu)) .$$

Firstly we will prove the following 4.1.:

4.1. $\forall i \prec \xi \, (R_i(0) \longrightarrow \mathrm{Prog}\,[F_i, <_i, R_i]) .$

For this, suppose that $i \prec \xi$, $R_i(0)$, $F_i(\rho)$, $\forall \nu <_i \rho (F_i(\nu) \to R_i(\nu))$, $\mu <_\xi (i, \rho)$ and $F_\xi(\mu)$.

Now we want to show that $\bar{A}(\mu)$. We may assume μ is connected by proposition 1.5.

Furthermore we may assume

$$(i, 0) \leqq_\xi \mu <_\xi (i, \rho)$$

by the assumptions $R_i(0)$ and $\mu <_\xi (i, \rho)$. Therefore μ must be of the form (i, μ'). $i \prec \xi$ and $(i, \mu') <_\xi (i, \rho)$ imply $\mu' <_i \rho$. $F_\xi((i, \mu'))$ implies $A_i(\mu')$. It follows from proposition 1.1. that $F_i(\mu')$. It follows from these and the assumption $\forall \nu <_i \rho (F_i(\nu) \to R_i(\nu))$ that $R_i(\mu')$, i.e.,

$$\forall \lambda <_\xi \mu \, (F_\xi(\lambda) \longrightarrow \bar{A}(\lambda)) .$$

It follows from this and lemma 3 that $\bar{A}(\mu)$.

4.1. and $(A.2)_\xi$ imply that

$$\forall i \prec \xi \, (R_i(0) \longrightarrow A_i \subseteqq R_i) .$$

Since for some primitive recursive function f, we have:

$$\forall i \prec \xi \, \forall \mu \, (\mu <_\xi (i \oplus 1, 0) \wedge F_\xi(\mu) \longrightarrow \mu <_\xi (i, f(i, \mu)) \wedge A_i(f(i, \mu)))$$

we have the following 4.2.:

4.2. $\forall i \prec \xi \, (R_i(0) \longrightarrow R_{i \oplus 1}(0)) .$

On the other hand, $R_0(0)$ and $\forall i \prec \xi \, (Lim(i) \wedge \forall j \prec i \; R_j(0) \to R_i(0))$ clearly hold. Hence from $(\mathrm{TI})_\xi$ we have:

4.3. $\forall i \prec \xi \; R_i(0) .$

If $Lim(\xi)$ holds, then the assertion follows from 4.3. Assume that $Suc(\xi)$, i.e.,

$\xi=(\xi\ominus 1)\oplus 1$. By 4.3. and 4.2. we have $R_{\xi\ominus 1}(0)$, $R_{\xi\ominus 1}(0)\to R_\xi(0)$, hence also $R_\xi(0)$.

TI $[X, R, Y, \mu]$ abbreviates the formula:

$$X(\mu)\wedge(\text{Prog}\,[X, R, Y]\longrightarrow \forall\nu(R(\nu, \mu)\wedge X(\nu)\longrightarrow Y(\nu))$$

and TI $[X, R, \mu]$ denotes the schema $\{\text{TI}\,[X, R, Q, \mu]\}_Q$. Namely, 'TI $[X, R, \mu]$ is derivable in $\text{ID}^i_\xi(\mathfrak{A})$' means that TI $[X, R, Q, \mu]$ is derivable in $\text{ID}^i_\xi(\mathfrak{A})$ for every formula Q in $\text{ID}^i_\xi(\mathfrak{A})$.

LEMMA 5. TI $[F_\xi, <_\xi, (\xi, 0)]$ *is derivable in* $\text{ID}^i_\xi(\mathfrak{A})$.

PROOF.

5.1. The case $Lim(\xi)$.

For each formula Q, let $Q_i(\mu)$ be the formula:

$$\mu<_\xi(i, 0)\longrightarrow Q(\mu)\,.$$

Since $\mu<_\xi(i, 0)$ implies that μ has no j-section for all $j\geqq i$, the following is easily verified:

$$\mu<_\xi(i, 0)\longrightarrow(\nu<_i\mu\wedge F_i(\nu)\wedge\nu<_\xi(i, 0)\longleftrightarrow\nu<_\xi\mu\wedge F_\xi(\nu))\,.$$

It follows from this that:

$$\text{Prog}\,[F_\xi, <_\xi, Q]\longrightarrow\forall i\prec\xi\,\text{Prog}\,[F_i, <_i, Q_i]\,.$$

This and $(A.2)_\xi$ imply that:

$$\text{Prog}\,[F_\xi, <_\xi, Q]\longrightarrow\forall i\prec\xi\,(A_i\subseteqq Q_i)\,.$$

That is,

$$\text{Prog}\,[F_\xi, <_\xi, Q]\longrightarrow\forall i\prec\xi\,\forall\mu<_\xi(i, 0)(A_i(\mu)\longrightarrow Q(\mu$$

Thus by lemma 4 we have the assertion.

5.2. The case $Suc(\xi)$.

We have easily the following 5.2.1.:

5.2.1. $\forall\mu\,\forall\nu(\nu<_\xi\mu<_\xi(\xi, 0)\longrightarrow\nu<_{\xi\ominus 1}\mu)\,.$

Put

$$R(\mu):=\mu<_\xi(\xi, 0)\longrightarrow Q(\mu)\,,$$

then we have the following 5.2.2. by 5.2.1.:

5.2.2. $\text{Prog}\,[F_\xi, <_\xi, Q]\longrightarrow\text{Prog}\,[F_{\xi\ominus 1}, <_{\xi\ominus 1}, R]\,.$

It follows from 5.2.2. and $(A.2)_\xi$ that:

$$\text{Prog}\,[F_\xi, <_\xi, Q]\longrightarrow\forall\mu<_\xi(\xi, 0)(A_{\xi\ominus 1}(\mu)\longrightarrow Q(\mu))\,.$$

Thus by lemma 4 we have the assertion.

Let $\bar{n}$ be the numeral corresponding to n for each natural number n. Let $\lambda x.\ \xi(x, 0)$ be the primitive recursive function defined by:

$$\xi(0, 0)=0, \qquad \xi(x+1, 0)=(\xi, \xi(x, 0)).$$

Next, we will show that $\mathrm{TI}[F_\xi, <_\xi, \xi(\bar{n}, 0)]$ implies $\mathrm{TI}[F_\xi, <_\xi, \xi(\overline{n+1}, 0)]$ for $n \geqq 1$, following Gentzen [5].

Let $\lambda\mu\nu.\ \mu+^\xi\nu$ be a primitive recursive function such that:

$$\mu\equiv 0 \longrightarrow \mu+^\xi\nu=\nu+^\xi\mu=\nu.$$

Suppose $\mu\not\equiv 0$, $\nu\not\equiv 0$ and

$$\mu\equiv\mu_1\#\cdots\#\mu_m, \qquad \mu_1 {}_\xi\!\geqq \cdots {}_\xi\!\geqq \mu_m \not\equiv 0,$$

$$\nu\equiv\nu_1\#\cdots\#\nu_n, \qquad \nu_1 {}_\xi\!\geqq \cdots {}_\xi\!\geqq \nu_n \not\equiv 0.$$

Let l be the number such that

$$0\leqq l\leqq m \quad \text{and} \quad \mu_l {}_\xi\!\geqq \nu_1 {}_\xi\!> \mu_{l+1}.$$

Then

$$\mu+^\xi\nu=\mu_1\#\cdots\#\mu_l\#\nu_1\#\cdots\#\nu_n.$$

LEMMA 6. *For each formula Q, let $t[Q](\mu)$ be the formula $\forall\rho(F_\xi(\rho)\rightarrow(\forall\nu<_\xi\rho(F_\xi(\nu)\rightarrow Q(\nu))\rightarrow\forall\nu<_\xi\rho+^\xi\mu(F_\xi(\nu)\rightarrow Q(\nu))))$. Then $\mathrm{Prog}[F_\xi, <_\xi, Q]\rightarrow\mathrm{Prog}[F_\xi, <_\xi, t[Q]]$ is derivable in $\mathrm{ID}_\xi^1(\mathfrak{A})$.*

PROOF.
Obvious.

LEMMA 7. *For each formula Q, let $s[Q](\mu)$ be the formula $t[Q]((\xi, \mu))$, i.e.,*

$$\forall\rho(F_\xi(\rho)\longrightarrow(\forall\nu<_\xi\rho(F_\xi(\nu)\longrightarrow Q(\nu))\longrightarrow\forall\nu<_\xi\rho+^\xi(\xi, \mu)(F_\xi(\nu)\longrightarrow Q(\nu)))).$$

Then

$$\mathrm{Prog}[F_\xi, <_\xi, Q]\longrightarrow\mathrm{Prog}[F_\xi, <_\xi, s[Q]]$$

is derivable in $\mathrm{ID}_\xi^1(\mathfrak{A})$.

PROOF.
By induction on x, we have:

7.1. $F_\xi(\lambda)\wedge s[Q](\lambda)\wedge F_\xi(\rho)\wedge\forall\nu<_\xi\rho(F_\xi(\nu)\longrightarrow Q(\nu))\longrightarrow$

$$\longrightarrow\forall x\,\forall\nu<_\xi\rho+^\xi(\xi, \lambda)\cdot x(F_\xi(\nu)\longrightarrow Q(\nu))$$

where $\mu\cdot x=\mu\#\cdots\#\mu(x$ times).

Since we can define primitive recursive functions f and g such that:

$$\mu \not\equiv 0 \wedge \nu <_{\xi} \rho +^{\xi}(\xi, \mu) \wedge F_{\xi}(\nu) \longrightarrow F_{\xi}(f(\nu, \rho, \mu)) \wedge f(\nu, \rho, \mu) <_{\xi} \mu \wedge$$
$$\wedge \nu <_{\xi} \rho +^{\xi}(\xi, f(\nu, \rho, \mu)) \cdot g(\nu, \rho, \mu),$$

it follows from 7.1. that:

7.2. $\mu \not\equiv 0 \wedge F_{\xi}(\mu) \wedge \forall \lambda <_{\xi} \mu (F_{\xi}(\lambda) \longrightarrow \mathrm{s}[Q](\lambda)) \longrightarrow \mathrm{s}[Q](\mu)$.

By lemmata 5 and 6, we have:

7.3. $\mathrm{Prog}\,[F_{\xi}, <_{\xi}, Q] \longrightarrow \mathrm{s}[Q](0)$.

7.2. and 7.3. imply that:

$$\mathrm{Prog}\,[F_{\xi}, <_{\xi}, Q] \longrightarrow \mathrm{Prog}\,[F_{\xi}, <_{\xi}, \mathrm{s}[Q]].$$

From lemmata 5 and 7, we have the following lemma by metainduction on n.

LEMMA 8. $\mathrm{TI}\,[F_{\xi}, <_{\xi}, \xi(\bar{n}, 0)]$ *is derivable in* $\mathrm{ID}^{\mathrm{i}}_{\xi}(\mathfrak{A})$ *for each natural number* n.

THEOREM. $A_0(\ulcorner\mu\urcorner)$ *is derivable in* $\mathrm{ID}^{\mathrm{i}}_{\xi}(\mathfrak{A})$ *for each o.d.* μ *from* $O^*(I, 1)$, *where* $\ulcorner\mu\urcorner$ *is the gödelnumber of* μ.

PROOF.

For some primitive recursive function f, we have in PRA $\nu \leqq_0 \xi(f(\nu), 0)$. By lemmata 3 and 8 we have $\bar{A}(\xi(f(\ulcorner\mu\urcorner), 0))$ in $\mathrm{ID}^{\mathrm{i}}_{\xi}(\mathfrak{A})$. In particular $A_0(\xi(f(\ulcorner\mu\urcorner), 0))$. Hence from proposition 1.2. $A_0(\ulcorner\mu\urcorner)$ is derivable in $\mathrm{ID}^{\mathrm{i}}_{\xi}(\mathfrak{A})$.

REMARKS.

1. Let T^{i} be the theory $\mathrm{ID}^{\mathrm{i}}_{\xi}(\mathfrak{A})$ and $Prov_{\mathrm{T}^{\mathrm{i}}}$ be a canonical proof-predicate for T^{i}. Then we have constructed a primitive recursive function p such that:

PRA *proves that 'x is an o.d. from* $O^*(I, 1)$' $\longrightarrow Prov_{\mathrm{T}^{\mathrm{i}}}(p(x), \ulcorner A_0(\dot{x})\urcorner)$,

where $\ulcorner A_0(\dot{x})\urcorner$ is a term whose value is the gödelnumber of $A_0(\bar{n})$ when the numeral $\bar{n}$ is substituted for the variable x.

2. Let the order type of $\prec$ be 2 or $\omega+1$, T be the classical version of T^{i} and T* be the subsystem (BI) or $(\Pi^1_1\text{—CA})+(\mathrm{BI})$ of classical analysis, respectively. Then by the well-known translation * (cf. [4].), we have

$$\mathrm{T} \vdash A_0(\mu) \quad \text{implies} \quad \mathrm{T}^* \vdash A_0^*(\mu)$$

and also

$$\mathrm{T}^* \vdash A_0^*(\mu) \longrightarrow \mathrm{TI}_{<_0}[\mu]$$

where $\mathrm{TI}_{<_0}[\mu]$ is the formula

$$\forall X(\forall \nu <_0 \mu(\forall \rho <_0 \nu X(\rho) \longrightarrow X(\nu)) \longrightarrow \forall \nu <_0 \mu X(\nu)) .$$

Hence from the remark 1, we have:

PRA *proves that 'x is an o.d. from* $O^*(I, 1)$' $\longrightarrow$ $Prov_{T^*}(p^*(x), TI_{<_0}[\dot{x}])$

for some primitive recursive function p^*.

On the other hand, we will prove in [1] the consistency of (BI), $(\Pi^1_1-CA)+$ (BI) by the accessibility of $O(2, 1)$, $O(\omega+1, 1)$ with respect to $<_0$, respectively.

3. From the remark 2, we have

$$|ID^i_\omega| = |ID_\omega| = |(\Pi^1_1-CA)+(BI)| = |O(\omega+1, 1)|_{<_0}$$

where $|ID^i_\omega|$ denotes the order type of the least unprovable recursive well-ordering in ID^i_ω, etc., and $|O(\omega+1, 1)|_{<_0}$ denotes the order type of the system $O(\omega+1, 1)$ with respect to $<_0$.

Following Buchholz and Pohlers [2], and Pohlers [8] the common ordinal equals to $\Theta\varepsilon_{\Omega_\omega+1}0$. Thus we have indirectly:

$$|O(\omega+1, 1)|_{<_0} = \Theta\varepsilon_{\Omega_\omega+1}0 .$$

This is an analogue to the fact:

$$|O(n+1, 1)|_{<_0} = \Theta\varepsilon_{\Omega_n+1}0 \qquad \text{for every } n \text{ such that } 1 \leqq n < \omega .$$

But note that the latter was established directly in Levitz [7] and Buchholz and Schütte [3].

4. By [2] and [8]

$$|ID^i_\xi| = |ID_\xi| = \Theta\varepsilon_{\Omega_\xi+1}0 \qquad \text{for } \xi < \Theta\Omega_{\Omega_1}0 ,$$

and

$$|ID^i_{<\xi}| = |ID_{<\xi}| = \Theta\Omega_\xi 0 = \sup_{\zeta<\xi}\Theta\varepsilon_{\Omega_\zeta+1}0 \qquad \text{for limit } \xi \leqq \Theta\Omega_{\Omega_1}0 .$$

On the other hand, for limit ξ and $\zeta < \xi$, the subsystem $\{\mu \in O(\xi, 1) : \mu <_0 (\zeta+1, 0)\}$ of $O(\xi, 1)$ is nothing but $O(\zeta+1, 1)$,

Hence we have:

$$|O(\xi, 1)|_{<_0} = \sup_{\zeta<\xi}|O(\zeta+1, 1)|_{<_0} \qquad \text{for limit } \xi .$$

So one may conjecture that

$$|O(\xi+1, 1)|_{<_0} = \Theta\varepsilon_{\Omega_\xi+1}0 ,$$

$$|O(\xi, 1)|_{<_0} = \Theta\Omega_\xi 0 , \qquad \xi ; \text{limit},$$

for appropriately small ξ.

But we have not verified this conjecture in any way.

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