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ON THE WEIGHT OF HIGHER ORDER WEIERSTRASS POINTS

By

Masaaki HOMMA and Shoji OHMORI

Introduction. Let C be a complete nonsingular curve of genus $g\geqq 2$ over an algebraically closed field k of characteristic zero and D a divisor on C with $\dim|D|\geqq 0$. Then we may define the notion of D-Weierstrass points (see e. g. [3]).

Let P be a point on C and $l=\dim|D|+1$. If ν is a positive integer such that $\dim L(D-(\nu-1)P)>\dim L(D-\nu P)$, we call this integer ν a "D-gap" at P. There are exactly l D-gaps and the sequence of D-gaps $\nu_1(P),\cdots,\nu_l(P)$ at P, $\nu_1(P)<\cdots<\nu_l(P)$, is called the D-gap sequence at P. The multiplicty of the Wronskian of D at a point P can be computed as $\sum_{i=1}^{l}(\nu_i(P)-i)$. This integer is called the D-weight at P and denoted by $w_D(P)$. When $w_D(P)$ is positive, we call the point P a D-Weierstrass point. It is well known that for the canonical divisor K,

$$w_K(P)\leqq\frac{1}{2}g(g-1)$$

and equality occurs if and only if C is hyperelliptic and P is a K-Weierstrass point. Furthermore, T. Kato [2] showed that if C is nonhyperelliptic, then $w_K(P)\leqq k(g)$, where

$$k(g)=\begin{cases}\dfrac{1}{3}g(g-1) & \text{if } g=3,4,6,7,9\\[2ex] \dfrac{1}{2}(g^2-5g+10) & \text{if } g=5,8 \text{ or } g\geqq 10,\end{cases}$$

and this maximum is achieved for every $g\geqq 3$.

Our purpose is to give such good bounds on $w_D(P)$ for a divisor D of degree $>2g-2$.

THEOREM I. *Let D be a divisor of degree $d>2g-2$ on C. Then*

$$w_D(P)\leqq\frac{1}{2}g(g+1).$$

Furthermore, equality occurs if and only if C is hyperelliptic, P is a K-Weiersrass

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point and D is linearly equivalent to $K+(d-2g+2)P$.

THEOREM II. *Let D be a divisor of degree $d>2g-2$ on C. If C is nonhyperelliptic, then*

$$w_D(P)\leqq k(g)+g.$$

Furthermor, the maximum is achived for every $g\geqq 3$ and every $d>2g-2$

THEOREM III. *Let P be a point on a nonhyperelliptic curve C and D a divisor of degree $d>2g-2$ on C. If $w_D(P)=k(g)+g$, then $w_K(P)=k(g)$.*

In his paper [1], A. Duma posed the conjecture: if C is nonhyperelliptic of genus g and if $P\in C$ is a K-Weierstrass point, then $w_{qK}(P)\leqq w_K(P)+g$ for every $q\geqq 2$. Unfortunately, there is a counterexample of this conjecture (see §4 below). However, our theorems show that the conjecture is true for a certain limited case.

Notation. Let x be a function or a differential on C. The divisor of zeros of x is denoted by $(x)_0$ and the divisor of poles of x is denoted by $(x)_\infty$. The divisor div x means $(x)_0-(x)_\infty$. Let E be a divisor on C. We denote by $\mathcal{L}(E)$ the the k-vector space of all functions x on C such that div $x+E$ is effective and by $h^0(E)$ the dimension of $\mathcal{L}(E)$ over k. The dimension of the k-space of all holomorphic differentials ω with $(\omega)_0\succ E$ is denoted by $h^1(E)$. The degree of E is denoted by deg E. If two divisors E and E' are linearly equivalent, we denote it by $E\sim E'$. The complete linear system of all effective divisors E' with $E'\sim E$ is denoted by $|E|$.

§1. $w_D(P)\leqq\frac{1}{2}g(g+1)$

Let C be a complete nonsingular curve of genus $g\geqq 2$ over k and D a divisor of degree $d>2g-2$ on C. The dimension $h^0(D)$ of the k-space $\mathcal{L}(D)$ is always denoted by l. Note that $l=d+1-g$ by the Riemann-Roch theorem. Let $P\in C$. We denote by $\nu_1(P)<\cdots<\nu_l(P)$ the D-gap sequence at P. Then we have

$$\nu_i(P)=i \text{ for } 1\leqq i\leqq d-2g+1$$

by the Riemann-Roch theorem, and may denote by

$$\nu_i(P)=d-2g+1+\mu_{i-(d-2g+1)}(P) \text{ for } d-2g+2\leqq i\leqq l,$$

where $\mu_1(P)<\cdots<\mu_g(P)$ are positive integers. Hence we have

$$w_D(P)=\sum_{i=1}^{g}(\mu_i(P)-i).$$

THEOREM I. *We have*

$$w_D(P) \leqq \frac{1}{2} g(g+1).$$

Furthermore, equality occurs if and only if C is hyperelliptic, P is a K-Weierstrass point and $D \sim K+(d-2g+2)P$.

PROOF. By the definition of gap sequence, we have

(1) $h^0(D-(d-2g+\mu_j)P)=g-j+1.$

Since

(2) $\deg(D-(d-2g+\mu_j)P)=2g-\mu_j,$

we have $g-j \leqq \frac{1}{2}(2g-\mu_j)$ by Clifford's theorem. Hence $\mu_j \leqq 2j$ and therefore we have

$$w_D(P)=\sum_{j=1}^{g}(\mu_j-j) \leqq \frac{1}{2} g(g+1)$$

If equality occurs, then $\mu_j=2j$ for $j=1,\cdots,g$. In particular, putting $j=1$ we have $\deg(D-(d-2g+2)P)=2g-2$ and $h^0(D-(d-2g+2)P)=g$. This means $D-(d-2g+2)P \sim K$. Putting $j=2$ and appealing to Clifford's theorem, we have that C is hyperelliptic and $|D-(d-2g+4)P|=(g-2)g_2^1$, where g_2^1 is the linear system of dimension 1 and degree 2 on C. Hence we have $|2P|=g_2^1$, which means that P is a K-Weierstrass point.

Conversely, it is obvious that if C is hyperelliptic, $D \sim K+(d-2g+2)P$ and P is a K-Weierstrass point, then the D-gap sequence at P is

$$\{1, 2, \cdots, d-2g+1, d-2g+3, d-2g+5, \cdots, d+1\}.$$

Hence we have $w_D(P)=\frac{1}{2} g(g+1)$.

§ 2. Nonhyperelliptic case (1)

From now on, we assume that C is nonhyperelliptic. The following theorem, which is essentially due to H. H. Martens [4], plays an important role in our estimate of a bound on $w_D(P)$.

THEOREM 2.1 (Martens). *Assume that C is nonhyperelliptic of genus $g \geqq 4$. Let E be a divisor of degree e with $0 \leqq e \leqq 2g-1$. If $E \nsim 0$ nor K, then*

$$2(h^0(E)-1) \leqq e-1.$$

Furthermore, equality holds if and only if one of the following occurs:

(i) $e=1$ *and* $E\sim Q$, *where* Q *is a point*;

(ii) C *is trigonal*, $e=3$ *and* $|E|=g_3^1$, *where* g_3^1 *is a linear system of dimension* 1 *and degree* 3;

(iii) C *is plane quintic*, $e=5$ *and* E *is a line section*;

(vi) C *is trigonal*, $e=2g-5$ *and* $|K-E|=g_3^1$;

(v) $e=2g-3$ *and* $K-E\sim Q$, *where* Q *is a point*;

(vi) $e=2g-1$.

PROOF. The first assertion follows from Clifford's theorem. The "if" part of the second assertion is obvious and the "only if" part is an immediate consequence of the following lemma. (Note that if $2(h^0(E)-1)=e-1$, then $2(h^0(K-E)-1)=\deg(K-E)-1$.)

LEMMA 2.2. *Let* E *be a divisor of degree* e *on a nonhyperelliptic curve of genus* $g\geqq 4$. *If* $2(h^0(E)-1)=e-1$ *and* $0\leqq e\leqq g-1$, *then* $h^0(E)\leqq 2$ *except that the case* (iii) *in Theorem 2.1 occurs.*

For the proof, see [4], 2.5.1.

THEOREM II. *Let* D *be a divisor of degree* $d>2g-2$ *on a nonhyperelliptic curve* C *of genus* g. *Then we have*

$$w_D(P)\leqq k(g)+g$$

for any $P\in C$, *where* $k(g)$ *is Kato's bound on* $w_K(P)$.

PROOF. We prove this by several steps.

Step 1. First we estimate μ_i's by applying Clifford's theorem to (1) and (2). Since C is nonhyperelliptic, we have:

$\mu_1\leqq 2$ and equality occurs if and only if $D\sim K+(d-2g+2)P$;

$\mu_i\leqq 2i-1$ if $i=2,\cdots,g-1$;

$\mu_g\leqq 2g$ and equality occurs if and only if $D\sim dP$.

Step 2. If $\mu_1=2$, *then the K-gap sequence at* P *coincides with* $\mu_1-1, \mu_2-1,\cdots, \mu_g-1$. Indeed, if $\mu_1=2$, then $D-(d-2g+2)P\sim K$ by *Step 1.* Hence we have

$h^0(K-(\mu_i-2)P)=h^0(D-(d-2g+\mu_i)P)>h^0(D-(d-2g+\mu_i+1)P)=h^0(K-(\mu_i-1)P)$.

This means that $\mu_1-1,\cdots,\mu_g-1$ is the K-gap sequence at P.

This fact implies that

$$w_D(P)=w_K(P)+g \text{ if } \mu_1(P)=2.$$

In particular, our inequality holds if $\mu_1(P)=2$. So we may assume that $\mu_1(P)=1$.

Step 3. Assume that $g=3$. Using *Step 1*, we have

$$w_D(P)=\sum_{i=1}^{3}(\mu_i-i)\leqq(3-2)+(6-3)=4 \text{ if } \mu_1=1.$$

On the other hand, $k(3)+3=5$. Therefore our theorem holds when $g=3$.

Next assume that $g=4$. Then we have $w_D(P)\leqq 7$ if $\mu_1=1$. On the other hand, $k(4)+4=8$. Thus our theorem holds when $g=4$.

Step 4. From now on, we assume that $g\geqq 5$. By virtue of Martens' theorem, the μ_i's can be estimated as follows:

$\mu_2\leqq 3$ and equality occurs if and only if there is a point Q such that $K-D+(d-2g+3)P\sim Q$;

$\mu_3\leqq 5$ and equality occurs if and only if C is trigonal and $|K-D+(d-2g+5)P|=g_3^1$;

$\mu_4\leqq 7$ and equality occurs if and only if C is plane quintic ($g=6$) and $D-(d-5)P$ is linearly equivalent to a line section;

$\mu_i\leqq 2i-2$ for $i=5,\cdots,g-2$ if $g\geqq 7$;

$\mu_{g-1}\leqq 2(g-1)-1$ and equality occurs if and only if C is trigonal and $|D-(d-3)P|=g_3^1$;

$\mu_g\leqq 2g$ and equality occurs if and only if $D\sim dP$.

Step 5. In this step we prove the following lemma.

LEMMA 2.3. *If $\mu_1=1$, then at least one of the following holds $\mu_3<5$ or $\mu_{g-1}<2(g-1)-1$ or $\mu_g<2g$.*

PROOF. Suppose that $\mu_3=5$, $\mu_{g-1}=2(g-1)-1$ and $\mu_g=2g$. Then, by *Step 4* we have that $|K-D+(d-2g+5)P|=g_3^1$, $|D-(d-3)P|=g_3^1$ and $D\sim dP$. Since $g\geqq 5$, g_3^1 is unique. Hence $K-D+(d-2g+5)P\sim D-(d-3)P$ and $D-(d-2g+2)P\sim K$. This implies $\mu_1=2$, which is a contradiction.

Step 6. Assume that $g=6$. If $\mu_1=1$, then at least one of the inequalities $\mu_3<5$, $\mu_5<9$, $\mu_6<12$ holds by Lemma 2.3.
Hence

$$w_D(P)\leqq(3-2)+(5-3)+(7-4)+(9-5)+(12-6)-1=15<16=k(6)+6.$$

Therefore the theorem holds when $g=6$.

Step 7. We will establish the theorem in this step. Let $g=5$ or $g\geqq 7$. Using *Step 4* and Lemma 2.3, we have

$$w_D(P) \leqq (3-2)+(5-3)+\sum_{i=4}^{g-2}(i-2)+(g-2)+g-1=\frac{1}{2}(g^2-3g+10),$$

if $\mu_1=1$. On the other hand,

$$k(g)+g=\begin{cases} 21 & \text{if } g=7 \\ 33 & \text{if } g=9 \\ \dfrac{1}{2}(g^2-3g+10) & \text{if } g=5, 8 \text{ or } g \geqq 10 \end{cases}$$

Note that if $g=7$, then

$$\frac{1}{2}(g^2-3g+10)=19<k(7)+7$$

and that if $g=9$, then

$$\frac{1}{2}(g^2-3g+10)=32<k(9)+9$$

Therefore the inequality $w_D(P) \leqq k(g)+g$ holds for all $g \geqq 3$. This complete the proof.

REMARK 2.4. For every fixed couple (g, d) with $d>2g-2 \geqq 4$, there is a triple (C, D, P) such that C is of genus g, D is of degree d and that $w_D(P)=k(g)+g$. Indeed, Kato [2] showed that there is a couple (C, P) such that C is of genus g and $w_K(P)=k(g)$. Letting $D=K+(d-2g+2)P$, (C, D, P) has the required properties.

§ 3. Nonhyperelliptic case (2)

Let E be a divisor on C and let $P \in C$. We denote by $\mathcal{N}(E; P)$ the set of positive integers which are not E-gap at P. Note that $\mathcal{N}(K; P)$ is a semigroup. We need the following lemmas, but their proofs are not difficult.

LEMMA 3.1. *The semigroup $\mathcal{N}(K; P)$ acts on $\mathcal{N}(E; P)$ by a natural way, i.e., if $m \in \mathcal{N}(K; P)$ and $n \in \mathcal{N}(E; P)$, then $m+n \in \mathcal{N}(E; P)$.*

LEMMA 3.2. *Let E be a divisor on C with $h^1(E)>0$. If a point $P \in C$ is not a base point of $|K-E|$, then any E-gap is also a K-gap.*

The aim of this section is to prove the following theorem.

THEOREM III. *Let C be a nonhyperelliptic curve of genus g and D a divisor of degree $d>2g-2$ on C. Let $P \in C$. If $w_D(P)=k(g)+g$, then $w_K(P)=k(g)$.*

PROOF. Note that $w_D(P)=w_K(P)+g$ if $\mu_1(P)=2$, which was shown in *Step 2* of the proof of Theorem II. Hence the assertion holds when $\mu_1(P)=2$.

First we will show that $w_D(P)=k(g)+g$ implies $\mu_1(P)=2$ except for the case $g=5$. If $g=3,4,6,7$ or 9, this was shown in the proof of Theorem II (see *Step 3*, *Step 6* and *Step 7*). So we assume that $g=8$ or $g\geqq 10$. By virtue of *Step 7*, in the inequalities $w_D(P)\leqq k(g)+g$ and $\mu_1(P)\geqq 1$, equality may occur in the three cases:

Case 1. $\mu_1=1, \mu_2=3, \mu_3=5, \mu_i=2_i-2 \quad (i=4,\cdots, g-2)$,
$\mu_{g-1}=2g-3, \mu_g=2g-1$;
Case 2. $\mu_1=1, \mu_2=3, \mu_3=5, \mu_i=2i-2 \quad (i=4,\cdots, g-2)$,
$\mu_{g-1}=2g-4, \mu_g=2g$;
Case 3. $\mu_1=1, \mu_2=3, \mu_3=4, \mu_i=2i-2 \quad (i=4,\cdots, g-2)$,
$\mu_{g-1}=2g-3, \mu_g=2g$.

In every case, since $\mu_2=3$, there is a point Q such that $D-(d-2g+3)P\sim K-Q$ (see *Step 4*). Note that $Q\neq P$. In fact, if $Q=P$, then $D-(d-2g+2)P\sim K$, which implies $\mu_1=2$. Since $K-Q\sim D-(d-2g+3)P$ and $Q\neq P$, the $(K-Q)$-gap sequence at P coincides with $\mu_2-2,\cdots,\mu_g-2$. Hence there is a positive integer α such that the set of all K-gaps at P coincides with $\{\mu_2-2,\cdots,\mu_g-2\}\cup\{\alpha\}$ by Lemma 3.2. Using the above list, we can write down the $(K-Q)$-gap sequence at P according to each case:

Case 1. $1, 3, 4, 6,\cdots, 2g-8, 2g-5, 2g-3$;
Case 2. $1, 3, 4, 6,\cdots, 2g-8, 2g-6, 2g-2$;
Case 3. $1, 2, 4, 6,\cdots, 2g-8, 2g-5, 2g-2$.

Note that since C is nonhyperelliptic, $\alpha=2$ when either *Case 1* or *Case 2* occurs. Suppose that *Case 1* occurs. Since $2g-7$ is a non-K-gap at P and 2 is a non-$(K-Q)$-gap at P, $2g-5$ $(=2g-7+2)$ must be a non-$(K-Q)$-gap at P by Lemma 3.1, which is a contradiction. Next, suppose that *Case 2* occurs. Since 5 is a non-K-gap at P and $2g-7$ is a non-$(K-Q)$-gap at P, $2g-2$ $(=5+2g-7)$ must be a non-$(K-Q)$-gap at P, which is a contradiction. Finally, suppose that *Case 3* occurs. In this case, either 3 or 5 is a non-K-gap at P and 3 and 5 are non-$(K-Q)$-gaps at P. Hence 8 $(=3+5)$ must be a non-$(K-Q)$-gap at P, which is a contradiction. Therefore equality $w_D(P)=k(g)+g$ can not be compatible with $\mu_1(P)=1$ when $g\neq 5$.

Now, we will show the theorem when $g=5$. By an argument similar to the previous case, in the inequalities $w_D(P)\leqq k(5)+5$ and $\mu_1(P)\geqq 1$, equality may occur in the following three cases:

Case i. $\mu_1=1, \mu_2=3, \mu_3=5, \mu_4=7, \mu_5=9$;
Case ii. $\mu_1=1, \mu_2=3, \mu_3=5, \mu_4=6, \mu_5=10$;
Case iii. $\mu_1=1, \mu_2=3, \mu_3=4, \mu_4=7, \mu_5=10$.

In every case there is a point $Q\neq P$ such that the $(K-Q)$-gap sequence at P is $\mu_2-2,\cdots,\mu_5-2$ and there is an integer α such that the set of all K-gaps at P is

$$\{\mu_2-2,\ \mu_3-2,\ \mu_4-2,\ \mu_5-2\}\cup\{\alpha\}.$$

Therefore, we have

(i) If *Case i* occurs, then the K-gap sequence at P coincides with $1, 2, 3, 5, 7$.

(ii) If *Case ii* occurs, then it coincides with $1, 2, 3, 4, 8$.

(iii) If *Case iii* occurs, then it coincides with one of the following:

(iii. 1) $1, 2, 3, 5, 8$;

(iii. 2) $1, 2, 4, 5, 8$;

(iii. 3) $1, 2, 5, 6, 8$;

(iii. 4) $1, 2, 5, 7, 8$;

(iii. 5) $1, 2, 5, 8, 9$.

Suppose that *Case ii* occurs. Since 6 is a non-K-gap at P and 2 is a non-$(K-Q)$-gap at P, 8 $(=6+2)$ must be a non-$(K-Q)$-gap at P, which is a contradiction. Hence *Case ii* can not occur. Since the set of all non-K-gaps forms a semigroup, the cases (iii. 1), (iii. 3), (iii. 4) and (iii. 5) cannot occur. If (iii. 2) occurs, then $w_K(P)=k(5)$, and then the theorem holds. We will show that *Case i* does not occur. Since $h^0(K-Q-2P)=3$, we have $|Q+2P|=g_3^1$. On the other hand $|4P|=g_4^1$. Hence, we have $|2Q+4P|=g_6^3$, which is a contradiction.

The proof of Theorem III shows also the following corollary.

COROLLARY 3.3. *Let notation and assumption be as in Theorem III. Furthermore, assume that* $g\neq 5$. *Then* $w_D(P)=k(g)+g$ *if and only if* $D\sim K+(d-2g+2)P$ *and* $w_K(P)=k(g)$.

§ 4. Examples

First we will show that the conclusion of corollary 3.4 does not hold if $g=5$.

EXAMPLE 4.1. (see [1], Beispiel 2.2). Let C be the normalization of the plane curve C' defined by

$$y^3=x^2(x^5-1).$$

It is easy to check that the normalization $C\xrightarrow{\pi}C'$ is one to one as set theoretic and C is of genus 5. Let $P_0=\pi^{-1}((0:0:1))$ and let $P_\infty=\pi^{-1}((0:1:0))$. Then the K-gap sequence at P_0 is

$$1, 2, 4, 5, 8,$$

and the $(K-P_\infty)$-gap sequence at P_0 is

$$1, 2, 5, 8.$$

Letting

$$D=K-P_\infty+(d-7)P_0,$$

the D-gap sequence at P_0 is

$$1, 2, \cdots, d-9, d-8, d-6, d-5, d-2, d+1.$$

Hence $\mu_1(P_0)=1$ and $w_D(P_0)=10$ $(=k(5)+5)$.

The next is a counterexample of Duma's conjecture.

EXAMPLE 4.2. Let C' be a plane curve defined by

$$y^5=x(x-\lambda_1)^2(x-\lambda_2)^2(x-\lambda_3)^2,$$

where $\lambda_1, \lambda_2, \lambda_3$ are mutually distinct nonzero scalars.

Let $C \xrightarrow{\pi} C'$ be the normalization. Then π is one to one and C is of genus 6. Letting

$$P_i=\pi^{-1}((\lambda_i:0:1)) \quad (i=1, 2, 3)$$
$$P_0=\pi^{-1}((0:0:1))$$
$$P_\infty=\pi^{-1}((0:1:0)),$$

we have

$$\operatorname{div} x=5P_0-5P_\infty$$
$$\operatorname{div} y=P_0+2P_1+2P_2+2P_3-7P_\infty$$
$$\operatorname{div} dx=4P_0+4P_1+4P_2+4P_3-6P_\infty.$$

Hence we have

$$\operatorname{div}\frac{dx}{y}=3P_0+2P_1+2P_2+2P_3+P_\infty$$
$$\operatorname{div}\frac{dx}{y^2}=2P_0+8P_\infty$$
$$\operatorname{div}\frac{x}{y^2}dx=7P_0+3P_\infty$$
$$\operatorname{div}(x-\lambda_1)(x-\lambda_2)(x-\lambda_3)/y^2dx=P_0+3P_1+3P_2+3P_3$$
$$\operatorname{div}(x-\lambda_1)(x-\lambda_2)(x-\lambda_3)/y^3dx=P_1+P_2+P_3+7P_\infty$$
$$\operatorname{div} x(x-\lambda_1)(x-\lambda_2)(x-\lambda_3)/y^4dx=5P_0+P_1+P_2+P_3+2P_\infty.$$

Hence the K-gap sequence at P_0 is

$$1, 2, 3, 4, 6, 8,$$

and 13 integers $1, 2, \cdots, 9, 10, 11, 13, 15$ are $2K$-gaps at P_0. Now,

$$\operatorname{div} x^2/y(x-\lambda_1)(x-\lambda_2)(x-\lambda_3)(dx)^2=17P_0+P_1+P_2+P_3,$$
$$\operatorname{div}\frac{x^3}{y^4}(dx)^2=19P_0+P_\infty,$$

hence the $2K$-gap sequence at P_0 is

$$1, 2, \cdots, 9, 10, 11, 13, 15, 18, 20.$$

Therefore we have

$$w_K(P_0)+g=9<12=w_{2K}(P_0).$$

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M. Homma
Department of Mathematics
Ryukyu University
Okinawa 903-01, Japan

S. Ohmori
Institute of Mathematics
University of Tsukuba
Ibaraki 305, Japan