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APOSYNDESIS AND COHERENCE OF CONTINUA UNDER REFINABLE MAPS

By

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1. Introduction. A continuous map $f: X\rightarrow Y$ between metric spaces is said to be *refinable* if for each $\varepsilon>0$, there is an ε-map g from X onto Y which is ε-near to f. Such a map g is called an *ε-refinement* of f. The main purpose of this paper is to show that refinable maps preserve various aposyndetic and coherent properties of continua.

In the last section, we shall consider the behavior of Wilson's oscillatory sets. In [5], J. Ford and J.W. Rogers proved that for a refinable map $f: X\rightarrow Y$, $f^{-1}(y)$ is contained in a component of $f^{-1}(H)$ for every continuum neighborhood H of y in Y. We show that the oscillatory set about y has the same property as above H. Applying this to irreducible continua, we give a partial answer to H. Kato's question [9].

Throughout this paper a *continuum* is a compact connected metric space and a *map* is a continuous map. We shall fix one refinable map $f: X\rightarrow Y$ between continua and a $(1/n)$-refinement f_n of f for each positive integer n. For each closed subset B of Y, we shall denote B' the *limit of some convergent subsequenec* of $\{f_n^{-1}(B)\}$. Note that B' is not uniquely determined, since it depends on a choise of such a subsequence. If B is degenerate, $B=\{y\}$, we denote y' in place of $\{y\}'$. Also we shall use freely the following theorem:

THEOREM (Ford and Rogers [5]). *For each subcontinum B of Y, B' is a subcontinuum of X satisfying $f(B')=B$ and $f^{-1}(int(B))\subset B'$.*

2. Aposyndesis. In [7], F.B. Jones defined aposyndesis as a dual concept of semi-locally connectedness.

DEFINITION. Let x and y be a pair of distinct points of a continuum M. Then M is said to be *aposyndetic at x with respect to y* if there is a continuum neighborhood of x in M missing y. Furthermore, M is said to be *aposyndetic at x* if for each y, distinct to x, it is aposyndetic at x with respect to y, and is

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said to be *aposyndetic* if it is aposyndetic at each point of M. For each point y of M, the set L_y consists of all points x of M such that $x=y$ or M is not aposyndetic at x with respect to y.

THEOREM 1. (1) *For each point x in X, $f(L_x)\subset L_{f(x)}$.*

(2) *For each point y in Y, there is a point $x\in f^{-1}(y)$ such that $f(L_x)=L_y$. In fact we can choose x to be y'.*

PROOF. (1) Suppose, on the contrary, $f(z)\notin L_{f(x)}$ for some $z\in L_x$. There is a continuum neighborhood H of $f(z)$ in Y missing $f(x)$. Then H' is a subcontinuum of X satisfying $z\in int(H')$ and $x\notin H'$. This contradicts to $z\in L_x$.

(2) Let $\{f_{n_i}\}$ be a subsequence of $\{f_n\}$ such that $\lim f_{n_i}^{-1}(y)=y'$ exists. Since $f(y')=y$, by (1) it is sufficient to show that $f(L_{y'})\supset L_y$. Let $z\in L_y-\{y\}$. We can choose a subsequence $\{f_{m_j}\}$ of $\{f_{n_i}\}$ such that $\lim f_{m_j}^{-1}(z)=z'$ exists. If $z'\notin L_{y'}$, then there is a continuum neighborhood K of z' such that $y'\notin K$. Since $\lim f_{m_j}^{-1}(y)=y'\in X-K$ and $\lim f_{m_j}^{-1}(z)=z'\in int(K)$, there exists an index m_j such that $f_{m_j}^{-1}(y)\subset X-K$ and $f_{m_j}^{-1}(z)\subset int(K)$. Therefore $f_{m_j}(K)$ is a continuum neighborhood of z in Y missing y. This contradicts to $z\in L_y$. Hence $z'\in L_{y'}$ and $z\in f(L_{y'})$.

COROLLARY. *If X is aposyndetic, then so is Y.*

PROOF. Let $y\in Y$. Choose $x\in X$ such that $f(L_x)=L_y$. Since X is aposyndetic, L_x consists of only one point and hence so dose L_y.

REMARK 1. In [4], H. S. Davis, D. P. Stadtlander and P. H. Swingle generalized the concept of L_y as follows (see also [1]). Let A be a closed subset of a continuum M. Then the set $T(A)$ consists of all points of M which have no continuum neighborhood missing A. Then we can generalize Theorem 1 as follows:

THEOREM 1′. (1) *For each closed subset A of X, $f(T(A))\subset T(f(A))$.*

(2) *For each closed subset B of Y, there is a closed subset A of X such that $f(T(A))=T(B)$. We may choose A to be B'.*

REMARK 2. For a point x of a continuum M, F.B. Jones defined K_x to be the set consisting of x and all points y in $M-\{x\}$ such that M is not aposyndetic at x with respect to y. Similarly we can prove the following theorem:

THEOREM 1″. (1) *For each point x in X, $f(K_x)\subset K_{f(x)}$.*

(2) *For each point y in Y, there is a point $x\in f^{-1}(y)$ such that $f(K_x)=K_y$.*

In fact we can choose x to be y'.

DEFINITION. A continuum M is said to be *semi-aposyndetic* if for each pair of distinct points x, y of M, M is aposyndetic at (at least) one of x and y with respect to the other.

THEOREM 2. *If X is semi-aposyndetic, then so is Y.*

PROOF. Let y_1, y_2 be distinct points of Y. There is a subsequence $\{f_{n_i}\}$ of $\{f_n\}$ such that both $\lim f_{n_i}^{-1}(y_1)=y_1'$ and $\lim f_{n_i}^{-1}(y_2)=y_2'$ exist. Since X is semi-aposyndetic and $y_1'\neq y_2'$, we may assume $y_1'\notin L_{y_2'}$. Hence there are a continuum neighborhood K of y_1' missing y_2' and an index n_i such that $f_{n_i}^{-1}(y_1)\subset int(K)$ and $f_{n_i}^{-1}(y_2)\subset X-K$. Therefore $f_{n_i}(K)$ is a continuum neighborhood of y_1 missing y_2, and hence $y_1\notin L_{y_2}$.

DEFINITION. A continuum M is said to be *mutually aposyndetic* if for each pair of distinct points x, y of M, there exist continuum neighborhoods H and K of x and y respectively, such that $H\cap K=\emptyset$. A continuum M is said to be *n-aposyndetic at x* if for each set of distinct n points $\{x_1, \cdots, x_n\}$ of $M-\{x\}$, there exists a continuum neighborhood H of x such that $H\cap\{x_1, \cdots, x_n\}=\emptyset$. If M is n-aposyndetic at each of its point, then it is said to be *n-aposyndetic.*

Note that n-aposyndetic implies m-aposyndetic for every $m\leqq n$. There is an n-aposyndetic continuum which is not $(n+1)$-aposyndetic for every n. Also, there is an aposyndetic (and hence semi-locally connected) continuum which is not mutually aposyndetic.

EXAMPLE. Let A_k be the continuum in E^3 (Euclidean 3-space) consisting of the join of the closure of $\{(0, y, 1/i)\,;\ 0\leqq y\leqq 1,\ i=1, 2, \cdots\}$ with the point $(1, k, 0)$. Then the continuum $M=\bigcup_{k=0}^{n} A_k$ is n-aposyndetic but not $(n+1)$-aposyndetic.

EXAMPLE. Let A be the plane continuum consisting of the join of the closure of $\{(0, 1/i)\,;\ i=1, 2, \cdots\}$ with the point $(1, 0)$ and let B be the plane continuum consisting of the join of the closure of $\{(1, -1/i)\,;\ i=1, 2, \cdots\}$ with the point $(0, 0)$. Then the product $(A\cup B)\times[0, 1]$ is aposyndetic but is not mutually aposyndetic.

As the proof of Theorem 2, we can prove the following theorem.

THEOREM 3. (1) *If X is mutually aposyndetic, then so is Y.*
(2) *If X is n-aposyndetic, then so is Y.*

QUESTION. *Does a refinable map preserve strongly aposyndetic?* Here a continuum M is *strongly aposyndetic* if it is decomposable and whenever H and K are proper subcontinua such that $M=H\cup K$, then H and K are aposyndetic.

3. Unicoherence. If a continuum M is the union of two proper subcontinua A and B, then we denote it by $M=A\oplus B$, and say that $A\oplus B$ is a *decomposition* of M. In [2], D.E. Bennett generalized the concept of unicoherence.

DEFINITION. A decomposable continuum M is said to be *k-coherent* provided that for each decomposition $M=A\oplus B$, the number of components of $A\cap B$ is at most k. A 1-coherent continuum is called *unicoherent*. A unicoherent continuum M is said to be *strongly unicoherent* if for each decomposition $M=A\oplus B$, both A and B are unicoherent. A *hereditarily unicoherent* continuum is a continuum each subcontinuum of which is unicoherent.

Note that a k-coherent continuum is also l-coherent for every $l\geqq k$. The suspention of a discrete space consisting of k points is k-coherent but not $(k-1)$-coherent.

These concepts are all preserved by refinable maps. First we prove two lemmas.

LEMMA 1. *Let $Y_1, \cdots, Y_k$ be closed subsets of Y. Then there is a subsequence $\{f_{n_i}\}$ of $\{f_n\}$ such that $\lim f_{n_i}^{-1}(Y_j)$ exists for each $j=1, \cdots, k$.*

PROOF. We can easily prove this by induction on k.

LEMMA 2. *Let Y_1, Y_2 be subcontinua of Y and let $X_i=\lim f_n^{-1}(Y_i)$, $i=1, 2$. Then the number of components of $X_1\cap X_2$ is not less than that of $Y_1\cap Y_2$.*

PROOF. It is sufficient to show that $f(X_1\cap X_2)=Y_1\cap Y_2$. Since $f(X_i)=Y_i$, $f(X_1\cap X_2)\subset Y_1\cap Y_2$. On the other hand, let $y\in Y_1\cap Y_2$. Then $\limsup f_n^{-1}(y)$ is nonempty and is contained in both X_1 and X_2. But this set is mapped onto $\{y\}$ by f. This completes the proof.

THEOREM 4. *If X is k-coherent, then so is Y.*

PROOF. In [5], Ford and Rogers proved that X is decomposable if and only if Y is decomposable. Therefore Theorem 4 is a direct consequence of Lemma's 1 and 2.

THEOREM 5. *If X is strongly unicoherent, then so is Y.*

PROOF. Let $Y=Y_0\oplus Y_1$ be a decomposition of Y. It is sufficient to prove that Y_0 is unicoherent. Let $Y_0=Y_2\oplus Y_3$ be a decomposition of Y_0. We must show that $Y_2 \cap Y_3$ is connected. By Lemma 1, we may assume that $\{f_n^{-1}(Y_i)\}$ converges to Y_i' for each $i=0, \cdots, 3$. Clearly Y_0' and Y_1' are proper subcontinua of X and $X=Y_0' \cup Y_1'$. Hence Y_0' is unicoherent. It is easy to show that $Y_0'=Y_2' \cup Y_3'$. Therefore $Y_2' \cap Y_3'$ is connected, and so is $Y_2 \cap Y_3$ by Lemma 2. This completes the proof, because Y is unicoherent by Theorem 4.

Similarly we can prove the following theorem.

THEOREM 6. *If X is hereditarily unicoherent, then so is Y.*

4. Wilson's oscillatory sets. In [10], W. A. Wilson defined the oscillatory set of a continuum about x as follows.

DEFINITION. Let $\{V_i\}$ be a sequence of neighborhoods of x in a continuum M such that $V_1 \supset V_2 \supset \cdots$, and $\cap V_i = \{x\}$. There is a subcontinuum B_i of M containing V_i such that every proper subcontinuum of B_i does not contain V_i. Inductively we can choose $\{B_i\}$ so that $B_1 \supset B_2 \supset \cdots$. The set $M[x] = \cap B_i$ is said to be an *oscillatory set* of M *about* x.

Note that the oscillatory set is not uniquely determined. Wilson proved that if M is irreducible, then $M[x]$ is uniquely determined [11].

THEOREM 7. *For each point y of Y and each oscillatory set $Y[y]$, there is a subcontinuum H of X such that $f^{-1}(y) \subset H \subset f^{-1}(Y[y])$.*

PROOF. Let $\{V_i\}$ and $\{B_i\}$ be defining sequences of neighborhoods and continua respectively, for the oscillatory set $Y[y]$. For each i, there is a subsequence $\{f_{ij}\}_{j=1}^{\infty}$ of $\{f_n\}$ such that

(1) $\{f_{i+1,j}\}_{j=1}^{\infty}$ is a subsequence of $\{f_{ij}\}_{j=1}^{\infty}$.

(2) $\lim f_{ij}^{-1}(B_i) = B_i'$ exists.

Then $\{B_i'\}$ is a descending sequence of continua satisfying $f(B_i') = B_i$ and $f^{-1}(V_i) \subset B_i'$. Put $H = \cap B_i'$. Then H is a continuum and $f(H) = f(\cap B_i') \subset \cap f(B_i') = \cap B_i = Y[y]$. Hence $H \subset f^{-1}(Y[y])$. On the other hand, since $f^{-1}(V_i) \subset f^{-1}(int(B_i)) \subset B_i'$, we have $f^{-1}(y) \subset H$. This completes the proof.

In [9], H. Kato proved that *X is irreducible if and only if Y is irreducible.* Let X be an irreducible continuum between a and b. He asked the following question: *Is Y irreducible between $f(a)$ and $f(b)$?* The following theorem is a partial answer to this question.

THEOREM 8. *Let X be an irreducible continuum between a and b. If there is a dense subset F of Y such that Y is semi-locally connected at each point of F, then Y is irreducible between $f(a)$ and $f(b)$.*

To prove this, we need the following lemma.

LEMMA 3. *If M is irreducible, then* $M[x]=K_x$.

PROOF. Recall that $y\in K_x$ if and only if every continuum neighborhood of x contains y. It is clear that $K_x\subset M[x]$. Let $y\in M-K_x$. Then there is a continuum neighborhood H of x in M such that $y\notin H$. Note that the oscillatory set $H[x]$ is also an oscillatory set of M about x. Since M is irreducible, the oscillatory set of M is uniquely determined. Hence $H[x]=M[x]$ and $y\notin M[x]$.

PROOF OF THEOREM 8. Put $A=Y[f(a)]$ and $B=Y[f(b)]$. Suppose, on the contrary, that Y is not irreducible between $f(a)$ and $f(b)$. Then there is a proper subcontinuum I of Y containing $f(a)$ and $f(b)$. Since F is dense, there is a point $y\in F-I$. By the above Lemma, $y\notin A\cup I\cup B$. By Theorem 7, there are continua H, K in $f^{-1}(A)$ and in $f^{-1}(B)$ containing a and b respectively. Thus $H\cup I'\cup K$ is a continuum containing a and b, and hence equal to X. Hence $A\cup I\cup B\supset f(H\cup I'\cup K)=Y$, which contradicts to $y\notin A\cup I\cup B$.

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