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SPECTRAL REPRESENTATIONS AND ASYMPTOTIC WAVE FUNCTIONS FOR LONG-RANGE PERTURBATIONS OF THE D'ALEMBERT EQUATION

By

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Introduction.

In this paper, we shall investigate the asymptotic behaviour for $t\rightarrow\infty$ of the acoustic wave $w(x, t)$ governed by the equation:

$$\partial_t^2 w(x, t)-\sum_{j,k=1}^{n}\partial_j a_{jk}(x)\partial_k w(x, t)=0 \qquad \text{in } \boldsymbol{R}^n\ (n\geqq 2). \tag{0.1}$$

Here $\partial_t=\partial/\partial t$, $\partial_j=\partial/\partial x_j$, and $\sum_{j,k=1}^{n}\partial_j a_{jk}(x)\partial_k$ is assumed to be a long-range perturbation of the Laplacian Δ in $\boldsymbol{R}^n$.

This problem has been studied by Wilcox for the d'Alembert equation (cf., e.g., Wilcox [9]). He has shown that as $t\rightarrow\infty$, each wave behaves asymptotically like a diverging spherical wave, the representation function of which is called the asymptotic wave function in [9]. This result has been extended to symmetric hyperbolic systems of first order with constant coefficents by Kitahara [3] and Wilcox [10]. In short-range problems, namely if $a_{jk}(x)-\delta_{jk}=O(r^{-1-\varepsilon})$ as $r\rightarrow\infty$, where $r=|x|$ and $\varepsilon>0$, then the corresponding asymptotic wave function also forms a diverging spherical wave, but in long-range problems, the asymptotic wave function is no longer a diverging spherical wave.

A long-range problem is investigated recently in Mochizuki [5] for the equation

$$\partial_t^2 w(x, t)-c(x)^2 p(x)\nabla\cdot\left\{\frac{1}{p(x)}\nabla w(x, t)\right\}=0 \tag{0.2}$$

in an exterior domain of $\boldsymbol{R}^n$, where $\nabla=\nabla_x$ is the gradient in $\boldsymbol{R}^n$. On the basis of the spectral representation theory, he has determined the asymptotic wave function $w^{\infty}(x, t)$ as a modified diverging spherical wave:

$$w^{\infty}(x, t)=\frac{1}{\sqrt{2}}\sqrt{c(x)p(x)}\,r^{-(n-1)/2}F(\xi(x)-t, \tilde{x}),$$

where $\xi(x)=\int^{r}c(s\tilde{x})^{-1}ds$, $\tilde{x}=x/r$, and the wave profile $F(s, \tilde{x})(s\in\boldsymbol{R})$ is a gene-

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ralized Radon transform of the initial data. His remedy lies in employing an approximate phase function for the stationary problem associated with (0.2). The radiative derivative of the appoximate phase is obtained as an approximate solution of a Riccati equation. Such an approximation is somewhat rough, and effective under rather strong decay conditions on the angular derivatives of the coefficient $c(x)$, namely, there assumed, e.g.,

$$\tilde{\nabla}c(x)=O(r^{-3/2-\varepsilon}), \qquad \varepsilon>0,$$

where $\tilde{\nabla}=\nabla-\tilde{x}\partial_r$ and $\partial_r=\tilde{x}\cdot\nabla$. In our problem, these conditions correspond to the followings:

$$\partial^p(a_{jk}(x)-\delta_{jk})=O(r^{-p-1/2-\varepsilon}), \qquad \varepsilon>0, \ p=0, 1, 2, \tag{0.3}$$

where ∂^p denotes an arbitrary derivative of p-th order.

The purpose of this paper is to construct the corresponding asymptotic wave function to the wave governed by (0.1) with more general decay conditions on $a_{jk}(x)$ than (0.3). Throughout the present paper, we assume on $a_{jk}(x)$ the followings:

ASSUMPTIONS. The matrix $A(x)=(a_{jk}(x))$ is real symmetric and positive difinite uniformly in $x\in\boldsymbol{R}^n$. Furthermore there exists a constant δ $(0<\delta\leqq 1/2)$ such that $a_{jk}(x)\in C^{j_0+3}(\boldsymbol{R}^n)$ and as $r\to\infty$

$$\partial^p(a_{jk}(x)-\delta_{jk})=O(r^{-p-\delta}) \qquad \text{for} \quad p=0, 1, 2,$$

$$\tilde{\partial}^p a_{jk}(x)=O(r^{-p-\delta}) \qquad \text{for} \quad p=3, 4, \cdots, j_0+3,$$

where j_0 is the smallest integer such that $(j_0+2)\delta\geqq 2$, and

$$\tilde{\partial}^p=\text{any of } \tilde{\partial}_1^{p_1}\cdots\tilde{\partial}_n^{p_n}, \quad \tilde{\partial}_j=\partial_j-\tilde{x}_j\partial_r, \quad \tilde{x}_j=x_j/r,$$

p_j, $j=1, \cdots, n$, are non-negative integers such that $p=\sum_{j=1}^{n} p_j$.

The essential part of our results is to construct an approximate phase function $\rho(x, \sigma)$ $(\sigma\in\boldsymbol{R}\setminus\{0\})$ for the selfadjoint operator $L=-\sum_{j,k=1}^{n}\partial_j a_{jk}(x)\partial_k$ in $\mathfrak{H}=L^2(\boldsymbol{R}^n)$. In Section 1, we determine $\rho(x, \sigma)$ as an approximate solution of the equation

$$\sigma^2+\nabla\rho(x, \sigma)\cdot A(x)\nabla\rho(x, \sigma)-\nabla\cdot A(x)\nabla\rho(x, \sigma)=0.$$

Indeed, $\rho(x, \kappa)$ is given by

$$\rho(x, \sigma)=-i\sigma K(x)+\frac{n-1}{2}\log r-\frac{1}{2}\log\partial_r K(x)$$

with $K(x)$ solving approximately the equation

$$\nabla K(x)\cdot A(x)\nabla K(x)-1=0.$$

We make use of this phase function to define the radiation condition (Definition 1.5). In Section 2, we first justify the limiting absorption principle and then apply it to construct the unitary operators $\mathcal{F}_{\pm}: \mathfrak{H} \to L^2(\boldsymbol{R}_{\pm} \times S^{n-1})$ as follows:

$$(\mathcal{F}_{\pm} f)(\sigma, \tilde{x}) = \sqrt{\frac{2}{\pi}} \sigma \lim_{r \to \infty} e^{\rho(r\tilde{x}, \sigma)} (\mathcal{R}_{\sigma + i0} f)(r\tilde{x}) \qquad \text{for} \quad \sigma \in \boldsymbol{R}_{\pm},$$

where $\mathcal{R}_{\kappa} = (L - \kappa^2)^{-1}$, $\boldsymbol{R}_{+} = (0, \infty)$, $\boldsymbol{R}_{-} = (-\infty, 0)$. $\mathcal{F}_{\pm}$ give the generalized Fourier transforms associated with L. In Section 3, with the aid of the operators $\mathcal{F}_{\pm}$, we can develop the same arguments as in Mochizuki [5] to construct the asymptotic wave function associated with (0.1) (Theorem 3.1).

Finally we list the notation to be often employed in this paper.

$$r = |x| \quad \text{and} \quad \tilde{x} = {}^t(\tilde{x}_1, \cdots, \tilde{x}_n), \ \tilde{x}_j = x_j / r \quad \text{for} \quad x \in \boldsymbol{R}^n \setminus \{0\}.$$

$$A = A(x) = (a_{jk}(x)) \quad \text{and} \quad \Phi = \Phi(x) = \tilde{x} \cdot A(x) \tilde{x}.$$

$$\partial_j = \partial / \partial x_j, \quad \tilde{\partial}_j = \partial_j - \tilde{x}_j \partial_r, \quad \partial_r = \tilde{x} \cdot \nabla,$$

$$\nabla = {}^t(\partial_1, \cdots, \partial_n) \quad \text{and} \quad \tilde{\nabla} = \nabla - \tilde{x} \partial_r.$$

$$S(R) = \{x \in \boldsymbol{R}^n ; |x| = R\} \quad \text{for} \quad R > 0.$$

$$B(R) = \{x \in \boldsymbol{R}^n ; |x| < R\} \quad \text{for} \quad R > 0.$$

$$E(R) = \{x \in \boldsymbol{R}^n ; |x| > R\} \quad \text{for} \quad R > 0,$$

$$B(R, R') = \{x \in \boldsymbol{R}^n ; R < |x| < R'\} \quad \text{for} \quad 0 < R < R'.$$

$L^2_{\nu}(\boldsymbol{G})$ ($\nu \in \boldsymbol{R}$) denotes the Hilbert space of all measurable functions f such that $(1+r)^{\nu} f(x)$ is square integrable over a domain $\boldsymbol{G}$ of $\boldsymbol{R}^n$. The norm is denoted by $\|\cdot\|_{\nu, G}$. When $\nu = 0$ or $\boldsymbol{G} = \boldsymbol{R}^n$, we shall omit the corresponding subscript.

§ 1. Approximate phase function and radiation condition.

The purpose of this section is to define the radiation condition for the stationary problem

$$Lu - \kappa^2 u = f \quad \text{in} \quad \boldsymbol{R}^n, \tag{1.1}$$

which is associated with (0.1). Here L is a positive selfadjoint operator acting in $\mathfrak{H}$ defined by

$$\begin{cases} \mathcal{D}(L), \text{ the domain of } L = H^2(\boldsymbol{R}^n), \\ Lu = -\nabla \cdot A(x) \nabla u = -\sum_{j,k=1}^{n} \partial_j a_{jk}(x) \partial_k u \qquad \text{for} \quad u \in \mathcal{D}(L), \end{cases} \tag{1.2}$$

where $H^2(\boldsymbol{R}^n)$ denotes the Sobolev space of order two (cf., e.g., Mochizuki [4]). κ is a complex parameter which varies in a closed upper half plane and f lies in a suitable weighted L^2-space.

For this purpose we need an approximate phase function $\rho(x, \kappa)$, which is determined as an appropriate appoximate solution near $r=\infty$ of the equation

$$\kappa^2+\nabla\rho(x, \kappa)\cdot A(x)\nabla\rho(x, \kappa)-\nabla\cdot A(x)\nabla\rho(x, \kappa)=0. \tag{1.3}$$

(1.3) is derived from

$$\{-\nabla\cdot A(x)\nabla-\kappa^2\}e^{-\rho(x,\kappa)}=0.$$

When $\delta>1/2$, we can adopt as an approximate phase function, the function $\rho_0(x, \kappa)$ defined by

$$\rho_0(x, \kappa)=-i\kappa\int_0^r \Phi(s\tilde{x})^{-1/2}ds+\frac{n-1}{2}\log r+\frac{1}{4}\log \Phi(x) \tag{1.4}$$

(cf., Mochizuki [5]). $\rho_0(x, \kappa)$ is constructed as a well behaved approximate solution of the equation

$$\left\{-\Delta-\frac{1}{r^2}\Lambda-\kappa^2\Phi(x)^{-1}\right\}e^{-\rho_0(x,\kappa)}=0,$$

where Λ is the negative Laplace-Beltrami operater on the unit sphere S^{n-1} in $\boldsymbol{R}^n$. $\rho_0(x, \kappa)$ satisfies

$$\kappa^2+\nabla\rho_0(x, \kappa)\cdot A(x)\nabla\rho_0(x, \kappa)-\nabla\cdot A(x)\nabla\rho_0(x, \kappa)=O(r^{-2\delta}) \tag{1.5}$$

as $r\to\infty$. In view of (1.5), we see that in our problem, i.e., in case $\delta<1/2$, $\rho_0(x, \kappa)$ is not adequate as an approximate phase to our requirement (cf., (1.29) in Lemma 1.4, mentioned below).

We shall construct $\rho(x, \kappa)$ below. Our tactics is to modify $\rho_0(x, \kappa)$ by the function $Y(x)$, which is suggested by private communication with K. Mochizuki and J. Uchiyama. As will be seen in Lemma 1.4, the modifier $Y(x)$ must be an appropriate approximate solution of the equation

$$\begin{aligned}2\Phi(x)^{-1/2}\tilde{x}\cdot A(x)(-\Psi(x)+\nabla Y(x))\\ =(\Psi(x)-\nabla Y(x))\cdot A(x)(\Psi(x)-\nabla Y(x)).\end{aligned} \tag{1.6}$$

Here and in the sequel, we put

$$\Psi(x)=\int_0^r \nabla\Phi(s\tilde{x})^{-1/2}ds.$$

See Saitō [8] and Isozaki [2], where stationary modifiers are investigated for the Schrödinger operators with long-range potentials. We first construct the modifier $Y(x)$ and then difine $\rho(x, \kappa)$.

LEMMA 1.1. *There exists a real-valued $C^3(\boldsymbol{R}^n)$-function $Y(x)$ having the following properties: For some constant $C>0$,*

(1.7) $$|\partial^p Y(x)| \leqq C(1+r)^{1-p-2\delta} \quad \textit{for} \quad p=0, 1, 2, 3,$$

(1.8) $$|2\Phi(x)^{-1/2}\tilde{x}\cdot A(x)(\Psi(x)-\nabla Y(x)) + (\Psi(x)-\nabla Y(x))\cdot A(x)(\Psi(x)-\nabla Y(x))| \leqq C(1+r)^{-2}.$$

In the proof of this lemma, we shall find it convenient to use some functional spaces and operators. Let k and l be non-negative integers such that $k \leqq l$, and let η be a real number. We denote by $C^l_{k,\eta}$ the space of scalar or vector valued functions defined by

$$C^l_{k,\eta} = \{f \in C^l(\boldsymbol{R}^n \setminus \{0\});$$
$$|\partial^p f(x)| \leqq C(1+r)^{-p+\eta} \quad \text{if} \quad 0 \leqq p \leqq k,$$
$$|\tilde{\partial}^p f(x)| \leqq C(1+r)^{-p+\eta} \quad \text{if} \quad 0 \leqq p \leqq l.\}$$

For $f \in C^l_{k,\eta}$, we define operators J and J' by

$$(Jf)(x) = \int_0^r f(s\tilde{x})ds,$$

$$(J'f)(x) = \int_r^\infty f(s\tilde{x})ds, \text{ only if this integral converges,}$$

respectively.

PROPOSITION 1.2. *Let $C^l_{k,\eta}$, J and J' be as above. Then we have for $k \leqq l-1$*

(i) $$J(C^l_{k,\eta}) \subset \begin{cases} C^l_{k+1,\eta+1} & \textit{if} \quad \eta > -1, \\ C^l_{k+1,0} & \textit{if} \quad \eta < -1, \end{cases}$$

(1.9) $$\nabla \circ J(C^l_{k,\eta}) \subset \begin{cases} C^{l-1}_{k,\eta} & \textit{if} \quad \eta > -1, \\ C^{l-1}_{k,-1} & \textit{if} \quad \eta < -1, \end{cases}$$

(ii) $$J'(C^l_{k,\eta}) \subset C^l_{k+1,\eta+1} \quad \textit{if} \quad \eta < -1,$$
$$\nabla \circ J'(C^l_{k,\eta}) \subset C^{l-1}_{k,\eta} \quad \textit{if} \quad \eta < -1.$$

(iii)
(1.10) $$C^l_{k,\eta} \times C^{l'}_{k',\eta'} \subset C^{\min(l,l')}_{\min(k,k'),\eta+\eta'}.$$

Here we have put

$$J^{(\prime)}(C^l_{k,\eta}) = \{J^{(\prime)}f;\ f \in C^l_{k,\eta}\},$$
$$\nabla \circ J^{(\prime)}(C^l_{k,\eta}) = \{\nabla(J^{(\prime)}f);\ f \in C^l_{k,\eta}\},$$
$$C^l_{k,\eta} \times C^{l'}_{k',\eta'} = \{f \cdot g;\ f \in C^l_{k,\eta},\ g \in C^{l'}_{k',\eta'}\}.$$

PROOF. Noting that

$$\nabla(f(s\tilde{x}))=\frac{s}{r}(\tilde{\nabla}f)(s\tilde{x}),$$

$$\tilde{\nabla}\int_{0(r)}^{r(\infty)} f(s\tilde{x})ds=\int_{0(r)}^{r(\infty)}\nabla(f(s\tilde{x}))ds,$$

we can easily prove the proposition.

We note that by Assumptions and Proposition 1.2,

$$\begin{cases} a_{jk}(x)-\delta_{jk}\in C_{2,-\delta}^{j_0+3}, \\ \Psi(x)\in C_{2,-\delta}^{j_0+2}. \end{cases} \tag{1.11}$$

Now we are ready to prove Lemma 1.1.

PROOF OF LEMMA 1.1. Without loss of generality, we may assume that $1/\delta$ is not an integer. In order to obtain $Y(x)$, we solve (1.6) by the method of successive approximation. Let j_1 be the integer such that $(j_1+1)\delta<1$ and $(j_1+2)\delta>1$.

First Step. Put $Y_0(x)=0$ and define $Y_j(x)$ for $j=1, \cdots, j_1$, successively as follows: For $r>0$,

$$Y_j(x)=J\Big(\frac{1}{2}\Phi^{1/2}G_{j-1}\cdot AG_{j-1}+\tilde{x}\cdot(A-I)G_{j-1}\Big)(x), \tag{1.12}$$

where we have put

$$G_j=G_j(x)=\Psi(x)-\nabla Y_j(x)$$

Then $Y_j(x)$ has the following properties:

$$Y_j(x)\in C_{3,1-2\delta}^{j_0+3-j}, \tag{1.13}$$

$$\nabla Y(x)\in C_{2,-2\delta}^{j_0+2-j}, \tag{1.14}$$

$$\tilde{x}\cdot A(x)G_j(x)+\frac{1}{2}\Phi(x)^{1/2}G_j(x)\cdot A(x)G_j(x)\in C_{2,-(j+2)\delta}^{j_0+2-j}, \tag{1.15}$$

where j_0 is the constant appearing in Assumptions. In fact, using (1.11) and Proposition 1.2, (i), (iii), we can easily verify (1.13) and (1.14) by induction on j. We shall show (1.15). We begin by rewriting (1.15) as follows:

$$\begin{aligned} &\tilde{x}\cdot A(x)G_j(x)+\frac{1}{2}\Phi(x)^{1/2}G_j(x)\cdot A(x)G_j(x) \\ &\quad=\tilde{x}\cdot(A(x)-I)G_j(x)+\frac{1}{2}\Phi(x)^{1/2}G_j(x)\cdot A(x)G_j(x)+\tilde{x}\cdot\Psi(x)-\partial_r Y_j(x) \end{aligned} \tag{1.16}$$

We note that $\tilde{x}\cdot\Psi(x)=0$, and substitute the right-hand side of (1.12) into the last term of (1.16). Then we have

$$\tilde{x}\cdot A(x)G_j(x)+\frac{1}{2}\Phi(x)^{1/2}G_j(x)\cdot A(x)G_j(x)=R_j(x)\,, \tag{1.17}$$

where

$$R_j(x)=B_j(x)\cdot(\nabla Y_j(x)-\nabla Y_{j-1}(x))\,, \tag{1.18}$$

$$B_j(x)=(I-A(x))\tilde{x}+\frac{1}{2}\Phi(x)^{1/2}\{-2A(x)\Psi(x)+A(x)(\nabla Y_j(x)+\nabla Y_{j-1}(x))\}.$$

Using (1.10), (1.11) and (1.14), we have

$$B_j(x)\in\mathcal{C}^{j_0+2-j}_{2,-\delta}\,. \tag{1.19}$$

So it remains to estimate the term $\nabla Y_j(x)-\nabla Y_{j-1}(x)$. Taking account of the relation

$$Y_j(x)-Y_{j-1}(x)=J(B_{j-1}\cdot(\nabla Y_{j-1}-\nabla Y_{j-2}))(x)\,, \tag{1.20}$$

and using (1.19), (1.9) and (1.10), we can verify by induction on j

$$\nabla Y_j(x)-\nabla Y_{j-1}(x)\in\mathcal{C}^{j_0+2-j}_{2,-(j+1)\delta}\,. \tag{1.21}$$

Combining (1.17)~(1.19) and (1.21), we have (1.15).

Second Step. For $j>j_1$, we must improve the definition of $Y_j(x)$. For if we define $Y_j(x)$ for $j>j_1$ by (1.12), then we can also have (1.13) and (1.14), and then (1.19). However, since $(j+2)\delta>1$ for $j>j_1$, (1.20), (1.21) with $j=j_1$, and (1.9) imply $\nabla Y_j(x)-\nabla Y_{j-1}(x)\in\mathcal{C}^{j_0+2-j}_{2,-1}$, so we can merely obtain $R_j(x)\in\mathcal{C}^{j_0+2-j}_{2,-1-\delta}$. Now we construct $Y_j(x)$ so as to satisfy (1.15). To this end, we define $Y_j(x)$ for $j=j_1+1,\cdots,j_0$ by adding the correction term:

$$Y_j(x)=J\Big(\frac{1}{2}\Phi^{1/2}G_{j-1}\cdot AG_{j-1}+\tilde{x}\cdot(A-I)G_{j-1}\Big)(x)+\phi_j(\tilde{x})\,, \tag{1.22}$$

where

$$\phi_j(\tilde{x})=\begin{cases}-\displaystyle\int_0^\infty R_{j_1}(s\tilde{x})ds & \text{if}\quad j=j_1+1\,,\\[2ex] \phi_{j-1}(\tilde{x})-\displaystyle\int_0^\infty R_{j-1}(s\tilde{x})ds & \text{if}\quad j>j_1+1\end{cases} \tag{1.23}$$

with $R_j(x)$ defined by (1.18). Then from (1.22) and (1.23), we obtain the relation

$$\begin{aligned}Y_j(x)-Y_{j-1}(x)&=J(R_{j-1})(x)+\phi_j(\tilde{x})-\phi_{j-1}(\tilde{x})\\&=J'(R_{j-1})(x)\,,\end{aligned} \tag{1.24}$$

replacing (1.20). (1.15) and (1.17) with $j=j_1$ give $\phi_{j_1+1}(\tilde{x})\in\mathcal{C}^{j_0+2-j_1}_{j_0+2-j_1,0}$. This and the properties of $Y_{j_1}(x)$ show (1.13) and (1.14) with $j=j_1+1$, and then (1.19) with $j=j_1+1$, Taking account of (1.24) and Proposition 1.2, (ii), as in the first step, we can prove (1.21) with $j=j_1+1$, and then $R_{j_1+1}(x)\in\mathcal{C}^{j_0+1-j_1}_{2,-(j_1+3)\delta}$, which gives $\phi_{j_1+1}(\tilde{x})\in\mathcal{C}^{j_0+1-j_1}_{j_0+1-j_1,0}$. Iterating this argument, for any $j>j_1$, we can verify (1.13), (1.14) and

$$R_j(x)\in\mathcal{C}^{j_0+2-j}_{2,-(j+2)\delta}\,. \tag{1.25}$$

Since $\partial_r\phi_j(\tilde{x})=0$, we also have (1.17) by (1.16), which together with (1.25) imply (1.15) for $j>j_1$. We now choose a real-valued $C^3(\boldsymbol{R}^n)$-function $\chi_0(x)$ such that $\chi_0(x)=0$ for $r\leqq 1$ and $=1$ for $r\geqq 2$. Let us put

$$Y(x)=\chi_0(x)Y_{j_0}(x)\,.$$

Then by (1.13) and (1.15) with $j=j_0$, $Y(x)$ satisfies all of the assertions of the lemma. Q.E.D.

In view of (1.7), we can choose a large constant $R_0>0$ so that for $r\geqq R_0$,

$$\int_0^r \Phi(s\tilde{x})^{-1/2}ds-Y(x)\geqq 0 \qquad \text{and} \quad \Phi(x)^{-1/2}-\partial_r Y(x)\geqq 1/4\,. \tag{1.26}$$

Let us fix such an R_0 below and put

$$K(x)=\psi_0(r)\left\{\int_0^r \Phi(s\tilde{x})^{-1/2}ds-Y(x)\right\}, \tag{1.27}$$

where $\psi_0(s)$ is a C^3-, monotone increasing function of $s\in[0,\infty)$ such that $\psi_0(s)=0$ for $s\leqq R_0$ and $=1$ for $s\geqq R_0+1$. $K(x)$ depends on the choice of ψ_0, but we do not specify it here. We remark that in virtue of (1.8), we have the following inequality for $K(x)$: For some positive constant C,

$$|\nabla K(x)\cdot A(x)\nabla K(x)-1|\leqq C(1+r)^{-2}\,.$$

Now we introduce the approximate phase function $\rho(x,\kappa)$.

DEFINITION 1.3. Let R_0 and $K(x)$ be as in (1.26) and (1.27), respectively. We define a $C^3(\boldsymbol{R}^n)$-function $\rho(x,\kappa)$ with $\kappa\in\boldsymbol{C}$ by

$$\rho(x,\kappa)=-i\kappa K(x)+\frac{n-1}{2}\log r-\frac{1}{2}\log\partial_r K(x) \qquad \text{for} \quad r\geqq R_0+1\,. \tag{1.28}$$

Then we have the following lemma.

LEMMA 1.4. *Let $K_+(K_-)$ be any compact set of $\{\kappa\in\boldsymbol{C};\operatorname{Re}\kappa>0\ (<0)$ and $\operatorname{Im}\kappa\geqq 0\}$, respectively. Let $\rho(x,\kappa)$ be as in* (1.28). *Then the function $\rho(x,\kappa)$ is a well behaved approximate solution of* (1.3): *As $r\to\infty$,*

$$\kappa^2+\nabla\rho(x,\kappa)\cdot A(x)\nabla\rho(x,\kappa)-\nabla\cdot A(x)\nabla\rho(x,\kappa)=O(r^{-2})+p(x,\kappa) \tag{1.29}$$

uniformly in $\kappa\in K_\pm$, where $p(x,\kappa)$ is a complex-valued function such that

(1.30) $|p(x, \kappa)|\leqq C(1+r)^{-1-\delta}$ *and* $|\nabla p(x, \kappa)|\leqq C(1+r)^{-2-\delta}$

for some $C=C(K_\pm)>0$. *Furthermore* $\rho(x, \kappa)$ *with* $\kappa=\sigma+i\tau\in K_\pm$ *has the following properties: For any* $\varepsilon>0$, *there exists a constant* $R_\varepsilon\geqq R_0+1$ *such that*

(1.31) $$\mathrm{Re}\{\tilde{x}\cdot A(x)\nabla\rho(x, \kappa)\}\geqq\frac{\tau}{2}\Phi(x)^{1/2}+\frac{n-1-\varepsilon}{2r}\Phi(x) \quad \textit{for} \quad r\geqq R_\varepsilon.$$

(1.32) $$\mathrm{Im}\{\tilde{x}\cdot A(x)\nabla\rho(x, \kappa)\}=-\sigma\Phi(x)^{1/2}+O(r^{-\delta}) \quad \textit{as} \quad r\to\infty.$$

PROOF. (1.31) and (1.32) are immediate consequences of (1.7) and (1.28). We shall show (1.29) and (1.30). We rewrite $\rho(x, \kappa)$ as follows:

$$\rho(x, \kappa)=\rho_0(x, \kappa)+i\kappa Y(x)+Z(x),$$

where $\rho_0(x, \kappa)$ is defined by (1.4) and

$$Z(x)=-\frac{1}{2}\log(1-\Phi(x)^{1/2}\partial_r Y(x)) \quad \text{for} \quad r\geqq R_0+1.$$

Then we have for $p=0, 1, 2$,

(1.33) $$|\partial^p Z(x)|\leqq C(1+r)^{-p-2\delta}.$$

For the sake of simplicity, let us put

$$\rho=\rho(x, \kappa), \quad \rho_0=\rho_0(x, \kappa), \quad Y=Y(x) \quad \text{and} \quad Z=Z(x).$$

Noting that

$$\nabla\rho=\tilde{x}\partial_r\rho_0+\tilde{\nabla}\rho_0+i\kappa\nabla Y+\nabla Z,$$

we have by a straightforward calculation,

(1.34) $$\begin{aligned}&\kappa^2+\nabla\rho\cdot A\nabla\rho-\nabla\cdot A\nabla\rho\\&=\Big[\Phi\Big\{\Phi^{-1}\kappa^2+(\partial_r\rho_0)^2-\partial_r^2\rho_0-\frac{n-1}{r}\partial_r\rho_0\Big\}-{}^t\nabla A\cdot\nabla\rho_0\\&\quad-\sum_{j,k}a_{jk}\tilde{\partial}_j\tilde{\partial}_k\rho_0-\frac{1}{r}(\mathrm{Trace}(A)-n\Phi)\partial_r\rho_0-2\tilde{x}\cdot A\partial_r\tilde{\nabla}\rho_0-\frac{1}{r}\tilde{x}\cdot A\tilde{\nabla}\rho_0\Big]\\&\quad+\{2\partial_r\rho_0\tilde{x}\cdot A(\tilde{\nabla}\rho_0+i\kappa\nabla Y)+(\tilde{\nabla}\rho_0+i\kappa\nabla Y)\cdot A(\tilde{\nabla}\rho_0+i\kappa\nabla Y)\}\\&\quad+\{2\nabla Z\cdot A(\nabla\rho_0+i\kappa\nabla Y)+\nabla Z\cdot A\nabla Z-\nabla\cdot A\nabla(Z+i\kappa Y)\}\\&=T_1+T_2+T_3.\end{aligned}$$

Here we have put

(1.35) $$\begin{aligned}T_1=&-\frac{(n-1)(n-3)}{4r^2}\Phi+\frac{5}{16}\Phi^{-1}(\partial_r\Phi)^2-\frac{1}{4}\partial_r^2\Phi-\frac{1}{4}\Phi^{-1}\sum_{j,k}a_{jk}\tilde{\partial}_j\tilde{\partial}_k\Phi\\&+\frac{1}{4}\Phi^{-2}\tilde{\nabla}\Phi\cdot A\tilde{\nabla}\Phi-\frac{1}{2}\tilde{x}\cdot A\partial_r(\Phi^{-1}\tilde{\nabla}\Phi)\end{aligned}$$

$$-\frac{1}{r}(\text{Trace}\,(A)-n\Phi)\Big(\frac{n-1}{2r}+\frac{1}{4}\Phi^{-1}\partial_r\Phi\Big)-\frac{1}{4r}\Phi^{-1}\tilde{x}\cdot A\tilde{\nabla}\Phi$$

$$+{}^t\nabla A\cdot\Big(\tilde{x}\frac{n-1}{2r}+\frac{1}{4}\Phi^{-1}\nabla\Phi\Big)$$

$$+i\kappa\Big\{-\Phi^{3/2}(\Phi\partial_r\Phi+\tilde{x}\cdot A\tilde{\nabla}\Phi)+\frac{2}{r}\tilde{x}\cdot A\Psi+{}^t\nabla A\cdot(\tilde{x}\Phi^{-1/2}+\Psi)$$

$$+\sum_{j,k}a_{jk}\int_0^r\partial_j\partial_k\Phi^{-1/2}ds+\frac{1}{r}(\text{Trace}\,(A)-n\Phi)\Phi^{-1/2}\Big\},$$

(1.36) $$T_2=\kappa^2\{2\Phi^{-1/2}\tilde{x}\cdot A(-\Psi+\nabla Y)-(\Psi-\nabla Y)\cdot A(\Psi-\nabla Y)\}$$

$$+\frac{1}{4}\Phi^{-1}\Big\{\Big(\frac{n-1}{r}+\frac{1}{2}\Phi^{-1}\partial_r\Phi\Big)\tilde{x}+\frac{1}{4}\Phi^{-1}\tilde{\nabla}\Phi\Big\}\cdot A\tilde{\nabla}\Phi$$

$$+i\kappa\Big\{-\frac{1}{2}\Phi^{-3/2}\tilde{x}\cdot A\tilde{\nabla}\Phi+\Big(\frac{n-1}{r}\tilde{x}+\frac{1}{2}\Phi^{-1}\nabla\Phi\Big)\cdot A(-\Psi+\nabla Y)\Big\},$$

(1.37) $$T_3=\nabla Z\cdot A\Big(\tilde{x}\frac{n-1}{r}+\frac{1}{2}\Phi^{-1}\nabla\Phi+\nabla Z\Big)-\nabla\cdot A\nabla Z$$

$$+i\kappa\{2\nabla Z\cdot A(-\tilde{x}\Phi^{-1/2}-\Psi+\nabla Y)-\nabla\cdot A\nabla Y\}.$$

Combining (1.34)~(1.37), we have by Assumptions, Lemma 1.1 and (1.33),

(1.38) $$\kappa^2+\nabla\rho\cdot A\nabla\rho-\nabla\cdot A\nabla\rho=O(r^{-2})+p(x,\kappa),$$

where for $r\geqq R_0+1$,

(1.39) $$p(x,\kappa)=-i\kappa\Big\{\Phi^{-3/2}\Big(\Phi\partial_r\Phi+\frac{3}{2}\tilde{x}\cdot A\tilde{\nabla}\Phi\Big)-\frac{2}{r}\tilde{x}\cdot A\Psi-{}^t\nabla A\cdot(\tilde{x}\Phi^{-1/2}+\Psi)$$

$$+\sum_{j,k}a_{jk}\int_0^r\partial_j\partial_k\Phi^{-1/2}ds-\frac{1}{r}(\text{Trace}\,(A)-n\Phi)\Phi^{-1/2}$$

$$+\Big(\frac{n-1}{r}\tilde{x}+\frac{1}{2}\Phi^{-1}\nabla\Phi+2\nabla Z\Big)\cdot A(\Psi-\nabla Y)$$

$$+2\Phi^{-1/2}\tilde{x}\cdot A\nabla Z-\nabla\cdot A\nabla Y\Big\}.$$

(1.38) and (1.39) imply (1.29) and (1.30). Q. E. D.

Now we are in a position to state the radiation condition for (1.1).

DEFINITION 1.5. A solution of (1.1) with $\kappa\in K_+(K_-)$ is said to satisfy the radiation condition if

(1.40) $$u\in L^2_{(-1-\alpha)/2}\quad\text{and}\quad \tilde{x}\cdot A(x)(\nabla+\nabla\rho(x,\kappa))u\in L^2_{(-1+\beta)/2}(E(R_0)),$$

where R_0 is the same constant as in Definition 1.3 and α, β are positive constants satisfying

$$\alpha \leqq \beta \quad \text{and} \quad \alpha+\beta \leqq 2 .$$

§ 2. Limiting absorption principle and spectral representations.

First we prove the limiting absorption principle. Our improved approximate phase function $\rho(x, \kappa)$ and definition of the radiation condition contribute essentially to the following lemma which plays a crucial role in this section.

LEMMA 2.1. *Let* $K_{\pm}$ *be as in Lemma* 1.4 *and* α, β *be positive constants with* $\alpha+\beta<2$ *and* $\alpha \leqq \beta$. *Let* u *be a solution of* (1.1) *with* $\kappa=\sigma+i\tau \in K_{\pm}$ *and* $f \in L^2_{(1+\beta)/2}$, *satisfying the radiation condition. Then there exist constants* $C>0$ *and* $R_1 \geqq R_0+1$, *independent of* κ, f *and* u *such that*

$$\|(\nabla+\nabla\rho(\cdot, \kappa))u\|^2_{(-1+\beta)/2, E(R_1)} \leqq C\{\|u\|^2_{(-1-\alpha)/2}+\|f\|^2_{(1+\beta)/2}\}. \tag{2.1}$$

PROOF. Putting

$$\theta=\theta(x, \kappa)=(\nabla+\nabla\rho(x, \kappa))u ,$$

we rewrite (1.1) as follows:

$$-\nabla\cdot A\theta+\nabla\rho\cdot A\theta=f+(\kappa^2+\nabla\rho\cdot A\nabla\rho-\nabla\cdot A\nabla\rho)u . \tag{2.2}$$

Let $\eta=\eta(x, \kappa)$ be defined in $E(R_0+1)$ by

$$\eta(x, \kappa)=\operatorname{Im}\nabla\rho(x, \kappa)/\operatorname{Im}\partial_r\rho(x, \kappa)=\tilde{x}+\tilde{\eta} ,$$

where $\tilde{\eta}=\tilde{\eta}(x, \kappa)=\operatorname{Im}\tilde{\nabla}\rho(x, \kappa)/\operatorname{Im}\partial_r\rho(x, \kappa)$. Choose a sufficiently large constant $R_1 \geqq R_0+1$ and a C^1-, monotone increasing function $\phi(s)$ of $s \in [0, \infty)$ such that $\phi(s)=0$ for $s \leqq R_1$ and $=1$ for $s \geqq R_1+1$.

First Step. We have the following two identities: For $R \geqq R_1+1$,

$$\begin{aligned}
(2.3) \quad & -\int_{S(R)} r^\beta\Big[\operatorname{Re}\{\tilde{x}\cdot A\theta(\eta\cdot A\bar{\theta})\}-\frac{1}{2}\tilde{x}\cdot A\eta(\theta\cdot A\bar{\theta})\Big]dS \\
& +\int_{B(R_1, R)} \phi(r)r^\beta\Big[\Big\{\operatorname{Re}(\eta\cdot A\nabla\rho)-\frac{1}{2}(\nabla\cdot A\eta)-\frac{\beta}{2r}\tilde{x}\cdot A\eta\Big\}\theta\cdot A\bar{\theta} \\
& +\frac{1}{r}\{|A\theta|^2+(\beta-1)|\tilde{x}\cdot A\theta|^2+\operatorname{Re}[\beta\tilde{x}\cdot A\theta(\tilde{\eta}\cdot A\bar{\theta})]\} \\
& +\operatorname{Re}\{((\theta\cdot A\nabla)\tilde{\eta})\cdot A\bar{\theta}\}-\frac{1}{2}\theta\cdot((\eta\cdot A\nabla)A)\bar{\theta} \\
& +\operatorname{Re}\{\sum_{j,k,l,m} a_{jm}(\partial_j a_{kl})\eta_l\theta_m\bar{\theta}_k\}\Big]dx \\
& +\int_{B(R_1, R_1+1)} \phi'(r)r^\beta\Big[\operatorname{Re}\{\tilde{x}\cdot A\theta(\eta\cdot A\bar{\theta})\}-\frac{1}{2}\tilde{x}\cdot A\eta(\theta\cdot A\bar{\theta})\Big]dx
\end{aligned}$$

$$=\int_{B(R_1,R)}\phi(r)r^{\beta}\,\mathrm{Re}\,[f(\eta\cdot A\bar{\theta})+(\kappa^2+\nabla\rho\cdot A\nabla\rho$$
$$-\nabla\cdot A\nabla\rho)u(\eta\cdot A\bar{\theta})]dx\,,$$

(2.4)
$$\int_{B(R_1,R)}\phi(r)r^{\beta}\mathrm{Re}\,\{p(x,\kappa)u(\eta\cdot A\bar{\theta})\}\,dx$$
$$=\frac{1}{2}\int_{S(R)}\mathrm{Im}\,\{\tilde{x}\cdot A\theta\phi(r)r^{\beta}(\mathrm{Im}\,\partial_r\rho)^{-1}\overline{p(x,\kappa)u}\}\,dS$$
$$-\frac{1}{2}\int_{B(R_1,R_1+1)}\phi'(r)r^{\beta}(\mathrm{Im}\,\partial_r\rho)^{-1}\,\mathrm{Im}\,\{\tilde{x}\cdot A\theta\overline{p(x,\kappa)u}\}\,dx$$
$$-\frac{1}{2}\int_{B(R_1,R)}\phi(r)r^{\beta}(\mathrm{Im}\,\partial_r\rho)^{-1}\,\mathrm{Im}\Big[\bar{u}\Big\{\frac{\beta}{r}\overline{p(x,\kappa)}\tilde{x}+\nabla\overline{p(x,\kappa)}$$
$$-\overline{p(x,\kappa)}(\mathrm{Im}\,\partial_r\rho)^{-1}\,\mathrm{Im}\,\nabla\partial_r\rho\Big\}\cdot A\theta$$
$$+\overline{p(x,\kappa)}\{\theta\cdot A\bar{\theta}-f\bar{u}-(\kappa^2+\nabla\rho\cdot A\nabla\rho-\nabla\cdot A\nabla\rho)|u|^2\}\Big]dx\,.$$

Here $p(x,\kappa)$ is the function defined by (1.39). In fact, multiplying the both sides of (2.2) by $\phi(r)r^{\beta}\eta\cdot A\bar{\theta}$ and taking the real part, we have (2.3) by integrating by parts over $B(R_1, R)$. And also, multiplying (2.2) by $-\frac{i}{2}\phi(r)r^{\beta}$ $\cdot(\mathrm{Im}\,\partial_r\rho)^{-1}\overline{p(x,\kappa)u}$ and taking the real part, we have (2.4) by integrating by parts over $B(R_1, R)$.

Second Step. Using two identities (2.3) and (2.4), we shall show (2.1). Let us put

$I=$the integrand in the second integral term in the left-hand side of (2.3).

If $1<\beta<2$, by (1.7), (1.28), (1.31) and Assumptions, we see that there exist positive constants C_1 and $\varepsilon=\varepsilon(R_1)$ such that in $E(R_1+1)$,

$$I\geqq r^{\beta}\Big[\mathrm{Re}\,\{\tilde{x}\cdot A\nabla\rho\}-C_1\tau r^{-\delta}-\frac{n-3+\beta}{2r}\Phi-C_1r^{-1-\delta}\Big]\theta\cdot A\bar{\theta}$$
$$\geqq r^{\beta}\Big(\frac{\tau}{2}\Phi^{1/2}+\frac{n-1-\varepsilon}{2r}\Phi-C_1\tau r^{-\delta}-\frac{n-1-\varepsilon}{2r}\Phi+\frac{2-\beta+\varepsilon}{2r}\Phi-C_1r^{-1-\delta}\Big)\theta\cdot A\bar{\theta}\,.$$

Since $2>\beta$ and R_1 is sufficiently large, we have for some $C_2>0$,

(2.5)
$$I\geqq C_2r^{-1+\beta}|\theta|^2\quad\text{in}\quad E(R_1+1)\,.$$

If $0<\beta\leqq1$, we have for some positive constants C_3 and C_4,

(2.7)
$$I\geqq r^{\beta}\Big[\mathrm{Re}\,\{\tilde{x}\cdot A\nabla\rho\}-C_3\tau r^{-\delta}-\frac{n-1-\beta}{2r}\Phi-C_3r^{-1-\delta}\Big]\theta\cdot A\bar{\theta}$$

$$+(1-\beta)r^{-1+\beta}(|A\theta|^2-|\tilde{x}\cdot A\theta|^2)\geqq C_4r^{-1+\beta}|\theta|^2 \quad \text{in} \quad E(R_1+1).$$

In fact, if we take R_1 sufficiently large, we can choose a small constant ε in (1.31) so that $\varepsilon<\beta/2$. Noting that

$$\frac{1}{2}\tilde{x}\cdot A\eta(\theta\cdot A\bar{\theta})-\mathrm{Re}\{\tilde{x}\cdot A\theta(\tilde{\eta}\cdot A\bar{\theta})\}\geqq 0 \quad \text{in} \quad E(R_1+1), \tag{2.8}$$

and combining (2.4)~(2.8), we see by (1.29) and the condition $\alpha+\beta<2$,

$$\begin{aligned}\int_{E(R_1+1)} r^{-1+\beta}|\theta|^2dx \\ \leqq C\Big[\Big|\int_{B(R_1,R)}\phi(r)r^\beta\,\mathrm{Re}\{p(x,\kappa)u(\eta\cdot A\bar{\theta})\}\,dx\Big|+\|u\|^2_{(-1-\alpha)/2} \\ +\|f\|^2_{(1+\beta)/2}+\int_{B(R_1,R_1+1)}|\theta|^2dx+\int_{S(R)}(r^\beta|\tilde{x}\cdot A\theta|^2+r^{-\alpha}|u|^2)dS\Big].\end{aligned} \tag{2.9}$$

Using (2.4), we can estimate the first term in the right-hand side of (2.9) as follows: For any $\varepsilon>0$,

$$\begin{aligned}\Big|\int_{B(R_1,R)}\phi(r)r^\beta\,\mathrm{Re}\{p(x,\kappa)u(\eta\cdot A\bar{\theta})\}\,dx \\ \leqq\int_{B(R_1+1,R)}(\varepsilon+Cr^{-\delta})r^{-1+\beta}|\theta|^2dx+C_\varepsilon\|u\|^2_{(-1-\alpha)/2} \\ +C\Big[\|f\|^2_{(1+\beta)/2}+\int_{B(R_1,R_1+1)}|\theta|^2dx+\int_{S(R)}(r^\beta|\tilde{x}\cdot A\theta|^2+r^{-\alpha}|u|^2)dS\Big].\end{aligned} \tag{2.10}$$

Combining (2.9) and (2.10), we have for some $C>0$,

$$\begin{aligned}\int_{E(R_1+1)} r^{-1+\beta}|\theta|^2dx\leqq C\Big\{\|u\|^2_{(-1-\alpha)/2}+\|f\|^2_{(1+\beta)/2} \\ +\int_{S(R)}(r^\beta|\tilde{x}\cdot A\theta|^2+r^{-\alpha}|u|^2)dS\Big\}.\end{aligned} \tag{2.11}$$

Thus the radiation condition allows us to let $R\to\infty$ in (2.11), which completes the proof of the lemma.

With the aid of Lemma 2.1, we can follow the same line as in the proof of Mochizuki-Uchiyama [6] to verify the following theorem, the limiting absorption principle (cf., § 2 and the proofs of Theorems 1~5 of [6]).

THEOREM 2.2. *Let L be defined by* (1.2) *and $K_\pm$ be as in Lemma* 1.4. *Let α, β be constants satisfying* $0<\alpha\leqq\beta$ *and* $\alpha+\beta<2$. *For $\kappa\in K_\pm\backslash\boldsymbol{R}$, let us put $\mathfrak{R}_\kappa=(L-\kappa^2)^{-1}$. Then the following assertions hold.*

(i) *There exists a constant $C=C(K_\pm)>0$ such that for any $f\in L^2_{(1+\beta)/2}$ and $\kappa\in K_\pm$,*

$u(\kappa, f)=\mathcal{R}_{\kappa}f$ satisfies

$$\|u(\kappa, f)\|_{(-1-\alpha)/2}\leqq C\|f\|_{(1+\beta)/2}, \tag{2.12}$$

$$\|(\nabla+\nabla\rho(\cdot, \kappa))u(\kappa, f)\|_{(-1+\beta)/2, E(R_0)}\leqq C\|f\|_{(1+\beta)/2}. \tag{2.13}$$

(ii) *$u(\kappa, f)$ is continuous in $L^2_{(-1-\alpha)/2}$ with respect to $\kappa\in K_{\pm}\backslash\boldsymbol{R}$ and $f\in L^2_{(1+\beta)/2}$, and as a function of κ can be extended in $L^2_{(-1-\alpha)/2}$ to $K_{\pm}$. The extended function $u(\sigma+i0, f)$ satisfies* (2.12) *and* (2.13) *with $\kappa=\sigma+i0$.*

(iii) *$u(\sigma+i0, f)$ is a unique solution of* (1.1) *with $\kappa=\sigma+i0\in K_{\pm}$ and $f\in L^2_{(1+\beta)/2}$, satisfying the radiation condition.*

(iv) *L is absolutely continuous.*

Now we shall establish the spectral representations for L. For $\sigma\in\boldsymbol{R}\backslash\{0\}$ and $f\in L^2_{(1+\beta)/2}$ with $0<\beta<2$, let us put

$$[\mathcal{F}(\sigma, r)f](\tilde{x})=\sqrt{\frac{2}{\pi}}\sigma e^{\rho(r\tilde{x}, \sigma)}(\mathcal{R}_{\sigma}f)(r\tilde{x}) \tag{2.14}$$

Then we have

PROPOSITION 2.3. *Let α, β satisfy $0<\alpha\leqq\beta$, $\alpha+\beta<2$ and $\alpha<\delta$. For any $\sigma\in\boldsymbol{R}\backslash\{0\}$ and $f\in L^2_{(1+\beta)/2}$, there exists a sequence $\{r_m\}$ tending to infinity such that*

$$\lim_{m\to\infty}\int_{S(r_m)}(r^{-\alpha}|u|^2+r^{\beta}|(\nabla+\nabla\rho(x, \sigma))u|^2)dS=0, \tag{2.15}$$

where $u=\mathcal{R}_{\sigma}f$, and

$$\frac{\sigma}{\pi i}(\mathcal{R}_{\sigma}f-\mathcal{R}_{-\sigma}f, f)=\lim_{m\to\infty}\|\mathcal{F}(\sigma, r_m)f\|^2_{L^2(S^{n-1})}. \tag{2.16}$$

PROOF. (2.14) is an immediate consequence of (2.12) and (2.13) with $\kappa=\sigma+i0$. (2.16) follows from (1.28) and (1.32) and (2.14) through the Green formula.

Q. E. D.

We shall show the strong convergence of $\{\mathcal{F}(\sigma, r_m)f\}$ in $L^2(S^{n-1})$.

PROPOSITION 2.4. *Let α, β satisfy $0<\alpha\leqq\beta$, $\alpha+\beta<2$, $\alpha<\delta$ and $\alpha+2(1-\delta)<\beta$, and let $\sigma\in\boldsymbol{R}\backslash\{0\}$ and $f\in L^2_{(1+\beta)/2}$. Let $\{r_m\}$ be any sequence satisfying* (2.15) *with α, β, above. Then $\{\mathcal{F}(\sigma, r_m)f\}$ defined by* (2.14) *converges strongly in $L^2(S^{n-1})$.*

In order to prove this proposition, we need the following lemma.

LEMMA 2.5. *Let $\phi\in\mathcal{D}(\Lambda^{1/2})$, where Λ is the negative Laplace-Beltrami operator on S^{n-1}. Then under the same assumption as in Proposition* 2.4, *we have* *for* $r_k > r_m > R_0+1$,

(2.17) $$|(\mathcal{F}(\sigma, r_k)f - \mathcal{F}(\sigma, r_m)f, \phi)_{L^2(S^{n-1})}| \leqq C(m) r_m^{-(\delta-\alpha/2)} (\|\phi\|_{L^2(S^{n-1})} + \|\Lambda^{1/2}\phi\|_{L^2(S^{n-1})}),$$

(2.18) $$\|\Lambda^{1/2}\mathcal{F}(\sigma, r_m)f\|_{L^2(S^{n-1})} \leqq C(m) r_m^{1-\beta/2},$$

where

(2.19) $$C(m) = C\Big[\sup_{l \geqq m}\Big[\int_{S(r_l)} \{r^{-\alpha}|u|^2 + r^{\beta}|(\nabla + \nabla\rho(x, \sigma))u|^2\} dS\Big]^{1/2} + \|u\|_{(-1-\alpha)/2, E(r_m)} + \|(\nabla + \nabla\rho(\cdot, \sigma))u\|_{(-1+\beta)/2, E(r_m)} + \|f\|_{(1+\beta)/2, E(r_m)}\Big]$$

with $u = \mathcal{R}_\sigma f$.

SKETCH OF THE PROOF OF LEMMA 2.5. Let ψ be a C^∞-function of $s = [0, \infty)$ such that $\psi(s) = 1$ for $s \geqq R_0+1$ and $=0$ for $s \leqq R_0$. For a smooth function $\phi = \phi(\tilde{x})$, let us put

$$v_\phi = v_\phi(x, \sigma) = \frac{1}{\sqrt{2\pi}} e^{-\rho(x, \sigma)} \phi(\tilde{x}) \psi(r).$$

Then we have as $r \to \infty$,

(2.20) $$v_\phi = O(r^{-(n-1)/2}),$$

(2.21) $$\tilde{x} \cdot A(x)(\nabla + \nabla\rho(x, \sigma)) v_\phi = O(r^{-1-\delta-(n-1)/2}),$$

(2.22) $$g_\phi = g_\phi(x, \sigma) \equiv (-\nabla \cdot A(x)\nabla - \sigma^2) v_\phi(x, \sigma) = O(r^{-1-\delta-(n-1)/2}).$$

Using (2.20)~(2.22), by the Green formula, we have for $\phi \in \mathcal{D}(\Lambda^{1/2})$, $r_m \geqq R_0+1$,

(2.23) $$i(\mathcal{F}(\sigma, r_m)f, \phi)_{L^2(S^{n-1})} = \int_{B(r_m)} (u\bar{g}_\phi - f\bar{v}_\phi) dx - \int_{S(r_m)} \{\tilde{x} \cdot A(\nabla + \nabla\rho) u\bar{v}_\phi - u\tilde{x} \cdot A\overline{(\nabla + \nabla\rho) v_\phi} + O(r^{-\delta-(n-1)/2}) u\bar{\phi}\} dS,$$

(2.24) $$\int_{B(r_m)} u\bar{g}_\phi dx = -\frac{1}{\sqrt{2\pi}} \int_{S(r_m)} e^{-\rho} \tilde{x} \cdot A\nabla\bar{\phi} u \, dS - \int_{B(r_m)} e^{-\bar{\rho}} \times \{u(\sigma^2 + \overline{\nabla\rho \cdot A\nabla\rho} - \overline{\nabla \cdot A\nabla\rho})\overline{\phi\psi} - u\overline{\nabla\rho \cdot A\nabla(\phi\psi)} - \nabla u \cdot A\overline{\nabla(\psi\phi)}\} dx.$$

Combining (2.23) and (2.24), we obtain (2.17). (2.18) can be seen by a direct calculation from (2.14). Q.E.D.

PROOF OF PROPOSITION 2.4. We see by Lemma 2.5 and its proof that the weak limit F of $\{\mathcal{F}(\sigma, r_m)f\}$ exists in $L^2(S^{n-1})$ and F does not depend on the choice of the sequence $\{r_m\}$ specified in Proposttion 2.3. By Lemma 2.5 we have

$$(2.25)\quad |(F-\mathcal{F}(\sigma, r_m)f, \mathcal{F}(\sigma, r_m)f)_{L^2(S^{n-1})}| \leqq C(m)\{C+C(m)r_m^{-(\delta-\alpha/2)+1-\beta/2}\},$$

where C is a positive constant independent of m. We see by (2.19) that $C(m)\rightarrow 0$ as $m\rightarrow\infty$. Letting $m\rightarrow\infty$ in (2.25), we have

$$(2.26)\quad \lim_{m\rightarrow\infty} \|\mathcal{F}(\sigma, r_m)f\|_{L^2(S^{n-1})} = \|F\|_{L^2(S^{n-1})}.$$

(2.26) and the existence of the weak limit imply the strong convergence of $\{\mathcal{F}(\sigma, r_m)f\}$ in $L^2(S^{n-1})$. Q.E.D.

DEFINITION 2.6. Let α, β be as in Proposition 2.4. For $\sigma\in\boldsymbol{R}\setminus\{0\}$ and $f\in L^2_{(1+\beta)/2}$, let $\mathcal{F}(\sigma)f\in L^2(S^{n-1})$ be defined by

$$\mathcal{F}(\sigma)f = \text{strong}\lim_{m\rightarrow\infty} \mathcal{F}(\sigma, r_m)f,$$

where $\mathcal{F}(\sigma, r)f$ is defined by (2.14) and $\{r_m\}$ is any sequence satisfying (2.15) with these α, β.

Let $\tilde{\beta}$ satisfy $0<\tilde{\beta}<2\delta$. Then as in Lemma 3.2 of Ikebe [1] or Lemma 3.2 of Mochizuki-Uchiyama [7], the operator $\mathcal{F}(\sigma)$ can be extended to a bounded linear operator from $L^2_{(1+\tilde{\beta})/2}$ to $L^2(S^{n-1})$, which will be denoted by $\mathcal{F}(\sigma)$ also.

Making use of Propositions 2,3 and 2.4, and the above fact, we have the following spectral representation theorem for L. Since the theorem can be verified in the same way as in Ikebe [1] or Mochizuki-Uchiyama [7], we may omit the proof.

THEOREM 2.7. (i) *Let $\tilde{\beta}$ satisfy $0<\tilde{\beta}<2\delta$ and let the operator $\mathcal{F}_\pm$ from $L^2_{(1+\tilde{\beta})/2}$ to $\mathfrak{H}_\pm = L^2(\boldsymbol{R}_\pm\times S^{n-1})$ ($\boldsymbol{R}_+=(0, \infty)$ and $\boldsymbol{R}_-=(-\infty, 0)$) be defined by*

$$(\mathcal{F}_\pm f)(\sigma, \tilde{x}) = [\mathcal{F}(\sigma)f](\tilde{x}) \qquad \textit{for} \quad (\sigma, \tilde{x})\in\boldsymbol{R}_\pm\times S^{n-1}.$$

Then $\mathcal{F}_\pm$ can be extended to a unitary operator from $\mathfrak{H}$ to $\mathfrak{H}_\pm$, which will be also denoted by $\mathcal{F}_\pm$.

(ii) *For any bounded Borel function $\alpha(\lambda)$ on $\boldsymbol{R}$ and any $f\in\mathfrak{H}$, we have*

$$\begin{aligned}\alpha(L)f &= \mathcal{F}_\pm^*\alpha(\sigma^2)\mathcal{F}_\pm f \\ &= \text{strong}\lim_{N\rightarrow\infty}\int_{e_{\pm N}} \mathcal{F}(\sigma)^*\alpha(\sigma^2)(\mathcal{F}_\pm f)(\sigma, \cdot)d\sigma \quad \textit{in} \quad \mathfrak{H},\end{aligned}$$

where $e_{+N}=(1/N, N)$ and $e_{-N}=(-N, -1/N)$.

§ 3. Asymptotic wave functions.

We consider the Cauchy problem

$$(3.1)\qquad \begin{cases} \partial_t^2 w(x, t)-\nabla\cdot A(x)\nabla w(x, t)=0 & \text{for } (x, t)\in \boldsymbol{R}^n\times \boldsymbol{R}, \\ w(x, 0)=f_1(x) \quad \text{and} \quad \partial_t w(x, 0)=f_2(x) & \text{for } x\in \boldsymbol{R}^n, \end{cases}$$

where $\partial_t=\partial/\partial t$. Let H be the positive square root of L, $H=\sqrt{L}>0$ and let $\overline{\mathcal{D}}(H^{-1})$ denote the closure of $\mathcal{D}(H^{-1})$, the domain of H^{-1} in the norm $\|H^{-1}\cdot\|$. Then H^{-1} can be extended to a unitary operator from $\overline{\mathcal{D}}(H^{-1})$ onto $\mathfrak{H}$, which will be denoted by $\overline{H^{-1}}$. For $\{f_1, f_2\}\in\mathfrak{H}\times\overline{\mathcal{D}}(H^{-1})$ and $t\in\boldsymbol{R}$, we define the weak solution $w(\cdot, t)\in\mathfrak{H}$ of (3.1) as follows:

$$(3.2)\qquad w(\cdot, t)=\frac{1}{2}e^{-itH}(f_1+i\overline{H^{-1}}f_2)+\frac{1}{2}e^{itH}(f_1-i\overline{H^{-1}}f_2).$$

We shall determine the asymptotic wave function as $t\to\infty$, associated with the solution (3.2). With the aid of the generalized Fourier transforms $\mathcal{F}_\pm$ associated with L, constructed in Section 2, we can develop the same arguments as in Mochizuki [5] to construct the asymptotic wave function $w^\infty(x, t)$. So we state only the results without any proof.

For $f=\{f_1, f_2\}\in\mathfrak{H}\times\overline{\mathcal{D}}(H^{-1})$, let us define the *wave profile* $F(s, \tilde{x})$ $(s\in\boldsymbol{R})$ as follows:

$$(3.3)\qquad F(s, \tilde{x})=\frac{-i}{2\sqrt{\pi}}\int_{R_+}e^{i\sigma s}[\mathcal{F}_+(f_1+i\overline{H^{-1}}f_2)](\sigma, \tilde{x})d\sigma$$
$$+\frac{-i}{2\sqrt{\pi}}\int_{R_-}e^{i\sigma s}[\mathcal{F}_-(f_1-i\overline{H^{-1}}f_2)](\sigma, \tilde{x})d\sigma\,.$$

Then the correspondence $\mathfrak{H}\times\overline{\mathcal{D}}(H^{-1})\ni f\to F(s, \tilde{x})\in L^2(\boldsymbol{R}\times S^{n-1})$ is unitary.

Let $\psi(s)$ be a C^∞-, non-decreasing function of $s\geqq 0$ such that $\psi(s)=1$ for $s\geqq K_0+1/4$ and $=0$ for $s\leqq K_0$, where $K_0=\sup_{\omega\in S^{n-1}}K((R_0+1)\omega)$, and the function $K(x)$ is defined by (1.19). Let us define the *asymptotic wave function* $w^\infty(x, t)$ corresponding to (3.2) by the following modified diverging spherical wave:

$$(3.4)\qquad w^\infty(x, t)=\frac{1}{\sqrt{2}}\psi(K(x))(\partial_r K(x))^{1/2}r^{-(n-1)/2}F(K(x)-t, \tilde{x}),$$

Then we have

THEOREM 3.1. *For arbitrary Cauchy data* $\{f_1, f_2\}\in\mathfrak{H}\times\overline{\mathcal{D}}(H^{-1})$, *let* $w(\cdot, t)$ *be the solution of* (3.1) *defined by* (3.2) *and let* $w^\infty(\cdot, t)$ *be the corresponding asymptotic wave fuuction defined by* (3.4). *Then we have*

$$\lim_{t\to\infty}\|w(\cdot, t)-w^\infty(\cdot, t)\|=0,$$

and $w^{\infty}(\cdot, t)$ *has the following properties*: $w^{\infty}(\cdot, t)\in\mathfrak{H}$ *is continuous in all* $t\in\boldsymbol{R}$ *and* $\|w^{\infty}(\cdot, t)\|$ *is monotone increasing in t. Furthermore*

$$\lim_{t\to\infty}\|w^{\infty}(\cdot, t)\|=\frac{1}{\sqrt{2}}(\|f_1\|^2+\|\overline{H^{-1}}f_2\|^2)^{1/2},$$

$$\lim_{t\to-\infty}\|w^{\infty}(\cdot, t\|)=0.$$

Finally, we remark that we can also calculate the asymptotic distribution of the wave energy for $t\to\infty$ as in Wilcox [9] and Mochizuki [5].

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