

RETRACTION OF SPACES WITH A σ -ALMOST LOCALLY FINITE BASE

By

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1. Introduction.

In [3], M. Itō and K. Tamano introduced the notion of almost local finiteness and the class of all spaces with a σ -almost locally finite base. Hereafter, $\sigma\text{-}\mathcal{ALF}$ denotes this class and we call a space of $\sigma\text{-}\mathcal{ALF}$ a $\sigma\text{-}ALF$ -space. $\sigma\text{-}\mathcal{ALF}$ is countably productive, hereditary and the closed image of a $\sigma\text{-}ALF$ -space is M_1 (see [3]). But it is not known whether there exists an M_1 -space which is not a $\sigma\text{-}ALF$ -space. If M_1 -spaces are $\sigma\text{-}ALF$ -spaces, Ceder's long-standing unsolved question will be affirmatively answered; that is, every stratifiable space is M_1 .

In this paper, we shall prove that a $\sigma\text{-}ALF$ -space X can be imbedded in a $\sigma\text{-}ALF$ -space $Z(X)$ as a closed subspace in such a way that X is an $AR(\sigma\text{-}\mathcal{ALF})$ (resp. $ANR(\sigma\text{-}\mathcal{ALF})$) if and only if X is a retract (resp. neighborhood retract) of $Z(X)$. Moreover, by using this theorem we shall prove that a space is an $AR(\sigma\text{-}\mathcal{ALF})$ (resp. $ANR(\sigma\text{-}\mathcal{ALF})$) if and only if it is an $AE(\sigma\text{-}\mathcal{ALF})$ (resp. $ANE(\sigma\text{-}\mathcal{ALF})$). In proofs of these theorems, the construction of $Z(X)$, which is constructed by R. Cauty [1], plays an important role. For the analogous results in the class of stratifiable spaces or M_1 -spaces, see [1] or [4], respectively.

Throughout this paper, all spaces are assumed to be Hausdorff topological spaces and all maps to be continuous. N denotes the set of all natural numbers. Let \mathcal{Q} be a class of spaces. For the definitions of $AR(\mathcal{Q})$, $ANR(\mathcal{Q})$, $AE(\mathcal{Q})$ and $ANE(\mathcal{Q})$, see [2].

2. Auxiliary lemma.

For the definitions of uniformly approaching anti-cover and D -space, see [6]. The following lemma essentially proved in the proof of [5, Lemma 3.3].

LEMMA 2.1. *Let X be a D -space, F a closed subset of X and f a map from F into a space Y . Let Y also denote the natural imbedding of Y in the adjunction space $X \cup_f Y = Z$. If $\mathcal{U} = \{U_\alpha : \alpha \in A\}$ is an almost locally finite open family in Y , then for each $\alpha \in A$ there is a family $\{U'_\beta : \beta \in B_\alpha\}$ of open subsets in Z satis-*

fying the following three conditions:

- (C1) $\mathcal{U}' = \{U'_\beta : \beta \in B_\alpha, \alpha \in A\}$ is almost locally finite at each point of Z ,
 (C2) for each $\beta \in B_\alpha$, $U'_\beta \cap Y = U_\alpha$, and for every open subset V in Z with $V \cap Y = U_\alpha$ there is $\beta \in B_\alpha$ such that $U_\alpha \subset U'_\beta \subset V$, and
 (C3) for every open subset W in Y , there is an open subset W' of Z such that $W' \cap Y = W$ and $W' \cap U'_\beta = \emptyset$ whenever $\beta \in B_\alpha$ and $W \cap U_\alpha = \emptyset$.

PROOF. Let p be the projection from the free union $X \cup Y$ to Z . Since X is monotonically normal, let G be a monotone normality operator for X satisfying the property $G(H, K) \cap G(K, H) = \emptyset$ for a disjoint closed pair (H, K) . Since X is a D -space, F has a uniformly approaching anti-cover $\mathcal{C} = \{V_\lambda : \lambda \in A\}$ in X such that \mathcal{C} is locally finite in $X - F$. For each $U_\alpha \in \mathcal{U}$, let $U'_\alpha = \cup \{G(x, F - p^{-1}(U_\alpha)) : x \in p^{-1}(U_\alpha)\}$. Then U'_α is obviously open in X . For each $\alpha \in A$, let $B_\alpha = \{\gamma(\alpha) \subset A : p^{-1}(U'_{\gamma(\alpha)}) \text{ is open in } U'_\alpha\}$, where $U'_{\gamma(\alpha)} = U_\alpha \cup p(\cup \{V_\lambda : \lambda \in \gamma(\alpha)\})$. Let $B = \cup \{B_\alpha : \alpha \in A\}$, and $\mathcal{U}' = \{U'_\beta : \beta \in B\}$. Then it is easy to see that the conditions (C2) and (C3) are satisfied by \mathcal{U}' .

Finally to prove (C1), first we consider the case $x \in Z - Y$. In this case, it is easily verified by the local finiteness of \mathcal{C} in $X - F$. Next, we consider the case $x \in Y$. Then there exist an open neighborhood V of x in Y and open finite subsets $\{H_1, \dots, H_n\}$ of Y such that

$$\mathcal{U}|V = \{U \cap V : U \in \mathcal{U}\} \subset \{H_i \cap W : i=1, \dots, n\}$$

and W is a neighborhood of x in Y .

Without loss of generality, we assume that

$$H_i \supset \cup \{U_\alpha \in \mathcal{U} : U_\alpha \cap V = H_i \cap W \text{ for some neighborhood } W \text{ of } x\}.$$

Let $V' = V \cup p(\cup \{G(y, F - p^{-1}(V)) : y \in p^{-1}(V) \cap X\})$ and

$$H'_i = H_i \cup p(\cup \{G(y, F - p^{-1}(H_i)) : y \in p^{-1}(H_i) \cap X\})$$

for each $i=1, \dots, n$. Then it is easy to see that V' is a neighborhood of x in Z and

$$\mathcal{U}'|V' \subset \{H'_i \cap W : i=1, \dots, n \text{ and } W \text{ is a neighborhood of } x \text{ in } Z\}.$$

Thus \mathcal{U}' is almost locally finite at x . This completes the proof.

3. Main theorems.

CONSTRUCTION 3.1. Let X be a space. $M(X)$ denotes the full simplicial complex which has all points of X as the set of vertices. Then there is a canonical bijection i from the 0-skeleton M^0 of $M(X)$ onto X . Let $Z' = M(X) \cup_i X$

be the adjunction space and $p' : M(X) \cup X \rightarrow Z'$ the projection. By the aid of p' , we identify X with $p'(X) \subset Z'$. Since the restriction of p' to $M(X)$ is a bijection from $M(X)$ onto Z' , by the abuse of language, a simplex σ of $M(X)$ is said to be contained in a subset U of Z' if $p'(\sigma)$ is contained in U . $Z(X)$ denotes the space such that Z' is the underlying set of $Z(X)$ and the topology of $Z(X)$ has a base which consists of a collection of sets U , which is open in Z' , satisfying the following condition:

(C) If σ is a simplex of $M(X)$ such that all vertices of σ are contained in $U \cap X$, then σ is contained in U .

Let $p : M(X) \cup X \rightarrow Z(X)$ be the projection. Then p is obviously continuous. Let M^n be the n -skeleton of $M(X)$ and $Z^n = p(M^n \cup X)$.

LEMMA 3.2. *If X is a σ -ALF-space, then $Z(X)$ is also a σ -ALF-space.*

PROOF. For each $n \in \mathbb{N}$, let Y be the free union of all $(n+1)$ -simplexes of $M(X)$, F the boundary of Y and $f : F \rightarrow Z^n$ the map defined by $f(x) = p(x)$ for $x \in F$. Then the set $Y \cup_f Z^n$ is equal to the set Z^{n+1} . Let $\{U_\alpha : \alpha \in A\}$ be an almost locally finite open family in Z^n . Since Y is a metric space, Y is a D -space. Therefore the technique of proof of Lemma 2.1 yields that, for each $\alpha \in A$, there is a family $\{U'_\beta : \beta \in B_\alpha\}$ of open subsets in Z^{n+1} satisfying (C1)-(C3). (Note that this proof is slightly different from that of Lemma 2.1; i.e. if σ is $(n+1)$ -simplex and U_α contains all vertices of σ , then σ is contained in U'_β , $\beta \in B_\alpha$.)

Now, let $\mathcal{U}_0 = \{U(\alpha_0) : \alpha_0 \in A\}$ be an almost locally finite open family in $Z^0 (= X)$. Then for fixed point $x \in Z^0$ there exist an open neighborhood V_0 of x in Z^0 and open subsets $\{H_1(0), \dots, H_n(0)\}$ of Z^0 such that

$$\mathcal{U}_0|V_0 \subset \{H_i(0) \cap W : i=1, \dots, n \text{ and } W \text{ is a neighborhood of } x \text{ in } Z^0\}.$$

From the preceding paragraph, we get that every $U(\alpha_0)$ can be extended to open subsets $\{U(\alpha_0, \alpha_1) : \alpha_1 \in A(\alpha_0)\}$ in Z^1 in such a way that the family

$$\mathcal{U}_1 = \{U(\alpha_0, \alpha_1) : \alpha_0 \in A, \alpha_1 \in A(\alpha_0)\}$$

satisfies (C1)-(C3). In particular, we may assume that the method of extensions is the same one of Lemma 2.1. Therefore there exist an open neighborhood V_1 of x in Z^1 and open subsets $\{H_1(1), \dots, H_n(1)\}$ of Z^1 such that

$$\mathcal{U}_1|V_1 \subset \{H_i(1) \cap W : i=1, \dots, n \text{ and } W \text{ is a neighborhood of } x \text{ in } Z^1\},$$

$V_1 \cap Z^0 = V_0$ and $H_i(1) \cap Z^0 = H_i(0)$ for $i=1, \dots, n$. Repeating this process we get for each $k \in \mathbb{N}$ an almost locally finite open family

$$\mathcal{U}_k = \{U(\alpha_0, \alpha_1, \dots, \alpha_k) : \alpha_0 \in A, \alpha_1 \in A(\alpha_0), \dots, \alpha_k \in A(\alpha_0, \dots, \alpha_{k-1})\}$$

in Z^k , an open neighborhood V_k of x in Z^k and open subsets $\{H_1(k), \dots, H_n(k)\}$ of Z^k such that

$$\mathcal{U}_k|V_k \subset \{H_i(k) \cap W : i=1, \dots, n \text{ and } W \text{ is a neighborhood of } x \text{ in } Z^k\},$$

$V_k \cap Z^{k-1} = V_{k-1}$ and $H_i(k) \cap Z^{k-1} = H_i(k-1)$ for $i=1, \dots, n$. Let

$$\Sigma = \{(\alpha_0, \alpha_1, \alpha_2, \dots) : \alpha_0 \in A, \alpha_1 \in A(\alpha_0), \alpha_2 \in A(\alpha_0, \alpha_1), \dots\}.$$

For each $(\alpha_0, \alpha_1, \alpha_2, \dots) \in \Sigma$, let

$$U(\alpha_0, \alpha_1, \alpha_2, \dots) = \cup \{U(\alpha_0, \alpha_1, \dots, \alpha_k) : k \in N\}.$$

Then $U(\alpha_0, \alpha_1, \alpha_2, \dots)$ is an open set of $Z(X)$, because for each $k \in N$,

$$U(\alpha_0, \alpha_1, \alpha_2, \dots) \cap Z^k = U(\alpha_0, \alpha_1, \dots, \alpha_k)$$

is open in Z^k and $U(\alpha_0, \alpha_1, \alpha_2, \dots)$ satisfies (C) by the construction of $U(\alpha_0, \alpha_1, \dots, \alpha_k)$. Next, we claim that

$$\mathcal{U} = \{U(\alpha_0, \alpha_1, \alpha_2, \dots) : (\alpha_0, \alpha_1, \alpha_2, \dots) \in \Sigma\}$$

is almost locally finite in $Z(X)$. Let $V = \cup \{V_k : k=0, 1, 2, \dots\}$ and

$$H_i = \cup \{H_i(k) : k=0, 1, 2, \dots\} \quad \text{for } i=1, \dots, n.$$

Then it is easily verified that V is an open neighborhood of x in $Z(X)$ and H_i open in $Z(X)$ satisfying

$$\mathcal{U}|V \subset \{H_i \cap W : i=1, \dots, n \text{ and } W \text{ is a neighborhood of } x \text{ in } Z(X)\}.$$

Thus \mathcal{U} is almost locally finite at $x \in Z^0$. By the same method, at each point $y \in Z^k$ for any $k \in N$, \mathcal{U} is almost locally finite.

Finally, we shall show that $Z(X)$ has a σ -almost locally finite base. Let $\{\mathcal{U}_n\}$ is a σ -almost locally finite base for Z^0 . Then it is easily seen that the extensions $\{\mathcal{U}'_n\}$ of $\{\mathcal{U}_n\}$ to $Z(X)$, by the same method above, is a σ -almost locally finite local base at each point of Z^0 . Furthermore, since $M(X)$ has a σ -almost locally finite base by [5, Theorem 4.1] and the open subspace $Z(X) - Z^0$ is homeomorphic to an open subspace of $M(X)$, there is a σ -almost locally finite (in $Z(X)$) local base $\{\mathcal{V}_n\}$ at each point of $Z(X) - Z^0$. Thus $\{\mathcal{U}'_n\} \cup \{\mathcal{V}_n\}$ is a σ -almost locally finite base for $Z(X)$. This completes the proof.

The following lemma was proved in [1, Lemma 1.2].

LEMMA 3.3. *Let X be a space. If Y is a stratifiable space, A a closed subset of Y and $f : A \rightarrow X$ a map, then there is a map $F : Y \rightarrow Z(X)$ with $F|A = f$.*

The following theorem is an immediate consequence of Lemma 3.2 and 3.3.

THEOREM 3.4. *A σ -ALF-space X is an $AR(\sigma\text{-}\mathcal{ALF})$ (resp. $ANR(\sigma\text{-}\mathcal{ALF})$) if and only if X is a retract (resp. neighborhood retract) of $Z(X)$.*

The following theorem is a direct consequence of Theorem 3.4 and Lemma 3.3.

THEOREM 3.5. *A space is an $AR(\sigma\text{-}\mathcal{ALF})$ (resp. $ANR(\sigma\text{-}\mathcal{ALF})$) if and only if it is an $AE(\sigma\text{-}\mathcal{ALF})$ (resp. $ANE(\sigma\text{-}\mathcal{ALF})$).*

REMARK 3.6. The analogous facts of [4, Section 4] can be proved by the same method in [4].

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