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METRIZABILITY OF PIXLEY-ROY HYPERSPACES

By

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§ 1. Introduction.

Throughout this paper, all spaces are assumed to be T_1-spaces. The *Pixley-Roy hyperspace* $\mathcal{F}[X]$ over a space X, defined by C. Pixley and P. Roy in [30], is the set of all non-empty finite subsets of X with the topology generated by the sets of the form $[F, U]=\{G\in\mathcal{F}[X] : F\subset G\subset U\}$, where $F\in\mathcal{F}[X]$ and U is an open subset in X containing F. In [14], it was pointed out that for any space X, $\mathcal{F}[X]$ is a zero-dimensional hereditarily metacompact space.

Generalized notions of metrizability were introduced by several authors. They still retain many of the desirable features of metric spaces (see [1], [11], [16], [21], [26] and [29]).

Our main purpose of this paper is to discuss these notions and to investigate metrizability in Pixley-Roy hyperspaces. More precisely, we shall establish the following theorems.

THEOREM 1.1. *For a space X, the following conditions are equivalent.*
(a) *$\mathcal{F}[X]$ is metrizable,*
(b) *$\mathcal{F}[X]$ is a Lašnev space,*
(c) *$\mathcal{F}[X]$ is a paracompact perfectly normal quasi-k-space.*

THEOREM 1.2. *For a space X, the following conditions are equivalent.*
(a) *$\mathcal{F}[X]$ is an M_1-space,*
(b) *$\mathcal{F}[X]$ is a stratifiable space,*
(c) *$\mathcal{F}[X]$ is a paracompact σ-space,*
(d) *$\mathcal{F}[X]$ is a paracompact perfectly normal space.*

THEOREM 1.3. *For a space X, the following conditions are equivalent.*
(a) *$\mathcal{F}[X]$ is metrizable,*
(b) *$\mathcal{F}[X]$ is a paracompact p-space.*

Secondly, we study weakly separated spaces and partially separated spaces

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in the sense of M.G. Tkačenko [35] and M.G. Bell [4]. Weakly (partially) separated spaces play the fundamental role for the paracompactness of Pixley-Roy hyperspaces (see [4], [7], [31] and [33]). Several results concerning partially separated spaces are given, one of which asserts that every Pixley-Roy hyperspace is the open finite-to-one image of a paracompact Hausdorff space by using the fact that every Pixley-Roy hyperspace is partially separated. This may be of interest in connection with the following H.J.K. Junnila's problem in [22]: Is every metacompact space the pseudo-open compact image of a paracompact Hausdorff space? Furthermore, we also prove that for generalized ordered spaces, semi-stratifiable spaces or locally Čech complete Tychonoff spaces, weak separatedness is equivalent to partial separatedness.

Our undefined terminology follows [10], [15] and [19]. For Pixley-Roy hyperspaces, the reader is refered to [14], [31], [33] and [34].

§ 2. Proofs of Theorems 1.1 and 1.2.

DEFINITION 2.1. A space X is said to be a *quasi-k-space* [28] if, given $A \subset X$, A is closed whenever $A \cap K$ is relatively closed in K for every countably compact $K \subset X$.

Since for any space X, $\mathcal{F}[X]$ is hereditarily metacompact (see E.K. van Douwen [14]), $\mathcal{F}[X]$ is a quasi-k-space if and only if $\mathcal{F}[X]$ is a k-space. Closed images of metric spaces were characterized internally by N.S. Lašnev in [23] (and thus, such spaces bear his name) and K. Morita and T. Rishel obtained

LEMMA 2.2 ([27]). *A regular space X is a Lašnev space if and only if the following conditions are satisfied:*

(a) *X is a quasi-k-space,*

(b) *there is a sequence $\{\mathfrak{E}_n : n \in N\}$ of hereditarily closure preserving closed covers of X with the properties below:*

(i) *for any point $x \in X$, any sequence $\{A_n : n \in N\}$ of sets, such that $A_n \in \mathfrak{E}_n$ and $x \in A_n$ for all $n \in N$, is either hereditarily closure preserving or forms a network at x.*

(ii) *for any point $x \in X$, there is a network $\{A_n : n \in N\}$ at x such that $A_n \in \mathfrak{E}_n$ for $n \in N$.*

A space is said to be *σ-discrete* if it is the union of countably many closed discrete subspaces.

LEMMA 2.3 (Lutzer [25]). *For a space X, the following conditions are equivalent.*

(a) *$\mathcal{F}[X]$ is perfect,*

(b) *$\mathcal{F}[X]$ is semi-stratifiable,*

(c) *$\mathcal{F}[X]$ is a σ-space,*

(d) *$\mathcal{F}[X]$ is σ-discrete,*

(e) *every point of X is a G_δ-point in X.*

In order to prove the implication (c)→(b) of Theorem 1.1, we show

LEMMA 2.4. *If X is σ-discrete and paracompact Hausdorff, then there is a sequence $\{\mathfrak{U}_n : n\in N\}$ of open covers of pairwise disjoint open subsets of X satisfying the following condition:*

$$(*)\quad \begin{cases} \textit{if for each } x\in X \textit{ and } n\in N,\ A_n \textit{ is the unique element} \\ \textit{of } \mathfrak{U}_n \textit{ such that } x\in A_n, \textit{ then } \{A_n : n\in N\} \textit{ is a} \\ \textit{hereditarily closure preserving closed collection of } X. \end{cases}$$

PROOF. Let $X=\cup\{X_n : n\in N\}$, where each X_n is a closed discrete subspace of X. We may assume $X_n\cap X_m=\emptyset$ for $n, m\in N$ and $n\neq m$. Since X is a zero-dimensional paracompact Hausdorff space, there is a discrete collection $\mathfrak{U}_1'=\{U_1(x) : x\in X_1\}$ of open-and-closed subsets of X such that $x\in U_1(x)$ for each $x\in X_1$. Then $U_1=X-\cup\{U(x) : x\in X_1\}$ is an open-and-closed subset of X. Let $\mathfrak{U}_1=\mathfrak{U}_1'\cup\{U_1\}$. Inductively, we obtain a sequence $\{\mathfrak{U}_n : n\in N\}$ of open covers of pairwise disjoint open subsets of X such that:

(1) for $n\in N$ and $x\in\cup\{X_i : i\leqq n\}$, there is an element $U_n(x)$ of $\mathfrak{U}_n$ such that $x\in U_n(x)$ and if $x, y\in\cup\{X_i : i\leqq n\}$ and $x\neq y$, then $U_n(x)\cap U_n(y)=\emptyset$.

(2) each $\mathfrak{U}_{n+1}$ is a refinement of $\mathfrak{U}_n$.

If $x\in X_n$, then, using (1) and (2), $U_m(x)\subset U_n(x)$ for $n\leqq m$. Pick an arbitrary element $x\in X$ and let A_n be the unique open subset of $\mathfrak{U}_n$ containing x for each $n\in N$. Since each $\mathfrak{U}_n$ is pairwise disjoint in X, each A_n is an open-and-closed subset of X. Let E_n be a closed subset of X such that $E_n\subset A_n$ for each $n\in N$ and let $y\notin\cup\{E_n : n\in N\}$. Then there are $n, m\in N$ such that $x\in X_n$ and $y\in X_m$. Let $s=\max\{n, m\}$. By using (1), (2) and the fact that $A_i=U_i(x)$ for $n\leqq i$, we have $U_s(y)\cap(\cup\{E_i : s\leqq i\})=\emptyset$. Since $\{E_i : i<s\}$ is a finite collection of closed subsets of X, there is an open neighborhood W of y such that $W\cap(\cup\{E_i : i<s\})=\emptyset$. Thus we have $(U_s(y)\cap W)\cap(\cup\{E_i : i\in N\})=\emptyset$. Hence $\cup\{E_i : i\in N\}$ is a closed subset of X. Thus $cl(\cup\{E_i : i\in N\})=\cup\{E_i : i\in N\}$. It follows that the collection $\{\mathfrak{U}_n : n\in N\}$ satisfies the condition $(*)$. The proof is completed.

LEMMA 2.5. *If X is σ-discrete and a paracompact Hausdorff quasi-k-space, then X is a Lašnev space.*

PROOF. Let $\{\mathfrak{U}_n : n \in N\}$ be a sequence of open covers of pairwise disjoint open subsets of X constructed in Lemma 2.4 and let $\{X_n : n \in N\}$ be a countable cover of closed discrete subsets of X which were used to construct $\{\mathfrak{U}_n : n \in N\}$. For each $n \in N$, let $\mathfrak{S}_n = \mathfrak{U}_n \cup \{\{x\} : x \in X_n\}$. Then each $\mathfrak{S}_n$ is a hereditarily closure preserving closed cover of X. By Lemma 2.4, the sequence $\{\mathfrak{S}_n : n \in N\}$ satisfies the condition (b) of Lemma 2.2. Thus X is a Lašnev space, which completes the proof.

PROOF OF THEOREM 1.1. The implication (a)→(c) is obvious and the implication (c)→(b) follows from Lemmas 2.3 and 2.5.

(b)→(a). By the well known Morita-Hanai-Stone's theorem, it suffices to prove that $\mathcal{F}[X]$ is first countable. We modify the proof of a theorem in Hyman [20]. Let $f : M \to \mathcal{F}[X]$ be a closed mapping from a metric space M with a compatible metric d onto $\mathcal{F}[X]$ and let F be a non-isolated point of $\mathcal{F}[X]$. Since $\mathcal{F}[X]$ is a Fréchet space, there is a sequence $\{F_n : n \in N\}$ of points of $\mathcal{F}[X]$ converging to F such that $\{F_n : n \in N\} \subset \mathcal{F}[X] - \{F\}$. Without loss of generality, we may assume that $F \subset F_n$ for each $n \in N$. Let $E = f^{-1}(F)$ and $E_n = f^{-1}(F_n)$ for each $n \in N$. Put

$$U_n = \cup \left\{ S\left(x, \frac{1}{2} d(x, E)\right) : x \in E_n \right\}$$

for each $n \in N$, where $S(x, \varepsilon) = \{y \in M : d(x, y) < \varepsilon\}$. Then each $\mathcal{O}_n = \mathcal{F}[X] - f(M - U_n)$ is an open neighborhood of F_n. Then there is an open neighborhood V_n of F_n in X such that $[F_n, V_n] \subset \mathcal{O}_n$ for $n \in N$. Since $F \subset F_n$ for $n \in N$, we have $F \subset V_n$. We shall prove that $\{[F, V_n] : n \in N\}$ is a countable neighborhood base at F in $\mathcal{F}[X]$. To see this, let W be an open neighborhood of F in X and let $G = f^{-1}([F, W])$. Then G is an open neighborhood of E. Put

$$G' = \cup \left\{ S\left(x, \frac{1}{2} d(x, M - G)\right) : x \in E \right\}$$

and let $\mathcal{O}' = \mathcal{F}[X] - f(M - G')$. Since $\mathcal{O}'$ is an open neighborhood of F, there is an $n \in N$ such that $F_n \in \mathcal{O}'$. Pick an arbitrary $H \in \mathcal{O}_n$ and let $x \in f^{-1}(H)$. Since $H \in \mathcal{O}_n$, there is an element $y \in E_n$ such that $d(x, y) < (1/2)d(y, E)$. Since $E_n = f^{-1}(F_n) \subset G'$, there is an element $z \in E$ such that $d(y, z) < (1/2)d(z, M - G)$. Thus $d(y, E) < (1/2)d(z, M - G)$. Hence

$$d(x, z) \leqq d(x, y)+d(y, z)$$
$$< \frac{1}{4} d(z, M-G)+\frac{1}{2} d(z, M-G)=\frac{3}{4} d(z, M-G).$$

Thus $x \in G$ and hence, $f^{-1}(H) \subset G$. Since H is an arbitrary element of $\mathcal{O}_n$, we have $\mathcal{O}_n \subset [F, W]$. Thus $[F_n, V_n] \subset [F, W]$. It follows that $[F, V_n] \subset [F, W]$. The proof is completed.

LEMMA 2.6 (Przymusiński [31]). *For a space* X,

(i) *the following conditions are equivalent.*

(a) $\mathcal{F}[X]$ *is paracompact,*

(b) *for every non-empty finite subset* F *of* X, *one can choose an open neighborhood* $U(F)$ *so that the inclusions* $F \subset U(H)$ *and* $H \subset U(F)$ *imply* $F \cap H \neq \emptyset$.

(ii) *the following conditions are equivalent.*

(a) $\mathcal{F}[X]$ *is hereditarily paracompact,*

(b) *for every non-empty finite subset* F *of* X, *one can choose an open neighborhood* $U(F)$ *so that the inclusions* $F \subset U(H)$ *and* $H \subset U(F)$ *imply* $F \subset H$ *or* $H \subset F$.

PROOF OF THEOREM 1.2. The implications (a)$\rightleftarrows$(b)$\rightarrow$(c)$\rightleftarrows$(d) follows from Lemma 2.3 and G. Gruenhage [17] or [18].

(c)$\rightarrow$(b). Assume that $\mathcal{F}[X]$ is a paracompact σ-space. Then $\mathcal{F}[X]$ is hereditarily paracompact. For each $F \in \mathcal{F}[X]$, let $U(F)$ be an open neighborhood of F in X satisfying the condition (b) of Lemma 2.6 (ii). By Lemma 2.3, every point of X is a G_δ-point in X. Thus for each $x \in X$, there is a decreasing sequence $\{V_n(x): n \in N\}$ of open neighborhoods of x such that $\cap \{V_n(x): n \in N\} = \{x\}$. Put

$$\mathcal{G}(n, F)=[F, (\cup \{V_n(x): x \in F\}) \cap U(F)]$$

for each $n \in N$ and $F \in \mathcal{F}[X]$. Thus for each $F \in \mathcal{F}[X]$, a sequence $\{\mathcal{G}(n, F): n \in N\}$ of open neighborhoods of F is given. By Borges [8], it suffices to prove that for every closed subset $\mathcal{E}$ of $\mathcal{F}[X]$, $\mathcal{E}=\cap \{cl_{\mathcal{F}[X]}(\cup \{\mathcal{G}(n, F): F \in \mathcal{E}\}): n \in N\}$. Let $\mathcal{E}$ be a closed subset of $\mathcal{F}[X]$ and let $F \notin \mathcal{E}$. Then there is an open neighborhood W of F in X such that (1) $[F, W] \cap \mathcal{E}=\emptyset$, and (2) $W \subset U(F)$. Thus for each $E \in \mathcal{E}$, either $F \not\subset E$ or $E \not\subset W$. It is clear that if $E \in \mathcal{E}$ and $E \not\subset W$, then $[F, W] \cap \mathcal{G}(1, E)=\emptyset$. By using the condition (b) of Lemma 2.6 (ii), it follows that if $[F, W] \cap \mathcal{G}(1, E) \neq \emptyset$ for some $E \in \mathcal{E}$, then E is a proper subset of F. Thus $\mathcal{E}'=\{E \in \mathcal{E}:]F, W] \cap \mathcal{G}(1, E) \neq \emptyset\}$ is finite. Then there is an $n \in N$ such that if $E \in \mathcal{E}'$, then $F-\cup \{V_n(x): x \in E\} \neq \emptyset$. Then $[F, W] \cap (\cup \{\mathcal{G}(n, E): E \in \mathcal{E}\})$

$=\emptyset$. Thus $F\notin cl_{\mathcal{F}[X]}(\cup\{\mathcal{G}(n, E): E\in\mathcal{E}\})$. It follows that $\mathcal{E}=\cap\{cl_{\mathcal{F}[X]}(\cup\{\mathcal{G}(n, E): E\in\mathcal{E}\}): n\in N\}$. The proof is completed.

§3. Proof of Theorem 1.3.

In [9], it was pointed out that if a Tychonoff space X is a p-space, then the following condition is satisfied:

$$(\ddagger)\quad \begin{cases} \text{there is a sequence } \{\mathfrak{U}_n: n\in N\} \text{ of open covers of } X \\ \text{such that if } x\in U_n\in\mathfrak{U}_n \text{ for } n\in N, \text{ then } \cap\{cl\,U_n: n\in N\} \\ \text{is a compact subset of } X \end{cases}$$

LEMMA 3.1. *Let X be a space. If $\mathcal{F}[X]$ satisfies the condition ($\ddagger$), then every point of X is a G_δ-point in X. Hence $\mathcal{F}[X]$ is semi-stratifiable.*

PROOF. Let $\{\mathfrak{U}_n: n\in N\}$ be a sequence of open covers of $\mathcal{F}[X]$ satisfying the condition ($\ddagger$). Assume that x is not a G_δ-point in X and let $\mathcal{V}_n$ be an element of $\mathfrak{U}_n$ containing $\{x\}$ for $n\in N$. Then $\cap\{cl_{\mathcal{F}[X]}\mathcal{V}_n: n\in N\}$ is a compact subset of $\mathcal{F}[X]$. For $n\in N$, let V_n be an open neighborhood of x such that:

(1) $[\{x\}, V_n]\subset\mathcal{V}_n$ for each $n\in N$, and

(2) $V_{n+1}\subset V_n$ for each $n\in N$.

Then, using (1), $\cap\{[\{x\}, V_n]: n\in N\}=[\{x\}, \cap\{V_n: n\in N\}]$ is a compact subset of $\mathcal{F}[X]$. Since x is not a G_δ-point in X, $\cap\{V_n: n\in N\}$ contains uncountably many points. Let $\{y_n: n\in N\}$ be a countable distinct points of $\cap\{V_n: n\in N\}$ and put

$$\mathcal{E}_n=[\{x, y_1, \cdots, y_n\}, V_n]$$

for each $n\in N$. Using (2), $\{\mathcal{E}_n: n\in N\}$ is a closed collection which satisfies the finite intersection property. Since each $\mathcal{E}_n$ meets $[\{x\}, \cap\{V_n: n\ni N\}]$, $\cap\{\mathcal{E}_n: n\in N\}$ must be non-empty. However, it is clear that $\cap\{\mathcal{E}_n: n\in N\}=\emptyset$, which is a contradiction. Thus every point of X is a G_δ-point in X. By Lemma 2.3, $\mathcal{F}[X]$ is semi-stratifiable. The proof is completed.

Since a paracompact Hausdorff space is metrizable if and only if it is a Moore space (see [15]), Theorem 1.3 follows from the next theorem.

THEOREM 3.2. *For a space X, the following conditions are equivalent.*

(a) *$\mathcal{F}[X]$ is a p-space,*

(b) *$\mathcal{F}[X]$ is a Moore space,*

(c) *X is first countable.*

PROOF. The equivalence of (b) and (c) was proved by E. K. van Douwen [14] and the implication (b)→(a) is well known. The implication (a)→(b) immediately follows from Lemma 3.1 and G. Creede [13].

THEOREM 3.3. *For a space X, the following conditions are equivalent.*

(a) $\mathcal{F}[X]$ *is metrizable,*

(b) $\mathcal{F}[X]$ *is an M-space.*

PROOF. Since a metacompact M-space is a paracompact M-space (=a paracompact p-space), this follows from Theorem 1.3.

For each $n \in N$, let $\mathcal{F}_n[X]=\{F\in\mathcal{F}[X] : |F|\leqq n\}$, where $|F|$ stands for the cardinality of F. Notice that for each $n\in N$, $\mathcal{F}_n[X]$ is a closed subspace of $\mathcal{F}[X]$ and, in particular, $\mathcal{F}_1[X]$ is a closed discrete subspace of $\mathcal{F}[X]$.

We consider compact subsets in Pixley-Roy hyperspaces. Let $A(\omega_1)$ be the one-point compactification of a discrete space of cardinality ω_1, where ω_1 is the first uncountable ordinal and let ∞ be the non-isolated point of $A(\omega_1)$.

THEOREM 3.4. *For a space X, we consider the following conditions:*

(a) $\mathcal{F}[X]$ *contains an uncountable compact subset,*

(b) $\mathcal{F}[X]$ *contains a copy of $A(\omega_1)$, and*

(c) X *contains a copy of $A(\omega_1)$.*

Then (c)→(a)⇄(b). *Furthermore, in case X is Hausdorff, all of the above conditions are equivalent.*

PROOF. (b)→(a). Obvious.

(c)→(b). Assume that X contains a copy of $A(\omega_1)$. We denote it by $A(\omega_1)$, too. Let $\mathcal{A}=\{\{x, \infty\} : x\in A(\omega_1)-\{\infty\}\}\cup\{\infty\}$. Then it is clear that $\mathcal{A}$ is a copy of $A(\omega_1)$.

(a)→(b). Let $\mathcal{K}$ be a compact subset of $\mathcal{F}[X]$. Since $\mathcal{F}_1[X]$ is a closed discrete subspace of $\mathcal{F}[X]$, $\mathcal{K}_1=\mathcal{K}\cap\mathcal{F}_1[X]$ is finite. Let $\mathcal{K}_1=\{\{x_i\} : i\leqq n_1\}$ and let $U(x_i)$ be an open subset in X containing x_i for $i\leqq n_1$ such that $\{[\{x_i\}, U(x_i)] : i\leqq n_1\}$ is pairwise disjoint in $\mathcal{F}[X]$. Let $\mathcal{K}_2'=\mathcal{K}-\cup\{[\{x_i\}, U(x_i)] : i\leqq n_1\}$. Then $\mathcal{K}_2'$ is a compact subset of $\mathcal{F}[X]$. If $\mathcal{K}_2=\mathcal{K}_2'\cap\mathcal{F}_2[X]$, then, since $\mathcal{K}_2$ is a closed discrete subspace of $\mathcal{K}_2'$, $\mathcal{K}_2$ is finite. Let $\mathcal{K}_2=\{\{y_i, z_i\} : i\leqq n_2\}$ and let $U(y_i, z_i)$ be an open set in X containing $\{y_i, z_i\}$ for each $i\leqq n_2$ such that:

(1) $[\{y_i, z_i\}, U(y_i, z_i)]\cap(\cup\{[\{x_j\}, U(x_j)] : j\leqq n_1\})=\emptyset$ for each $i\leqq n_2$, and

(2) $\{[\{y_i, z_i\}, U(y_i, z_i)] : i\leqq n_2\}$ is pairwise disjoint in $\mathcal{F}[X]$.

Let $\mathcal{K}_3'=\mathcal{K}_2'-\cup\{[\{y_i, z_i\}, U(y_i, z_i)] : i\leqq n_2\}$ and let $\mathcal{K}_3=\mathcal{K}_3'\cap\mathcal{F}_3[X]$. Then $\mathcal{K}_3$ is finite. Since $\mathcal{K}$ is a compact subset of $\mathcal{F}[X]$, this process is finished by finitely many times. Thus there is a finite subset $\{E_i : i \leqq n\}$ of $\mathcal{K}$ and a finite family $\{U_i : i \leqq n\}$ of open subsets of X, where each E_i is contained in U_i, such that:

(3) $\{[E_i, U_i] : i \leqq n\}$ covers $\mathcal{K}$, and

(4) $\{[E_i, U_i] : i \leqq n\}$ is pairwise disjoint in $\mathcal{F}[X]$.

Let $\mathcal{K}$ be an uncountable compact subset of $\mathcal{F}[X]$. By the above consideration, we may assume that $\mathcal{K}$ is contained in a $[E, U]$, where $E \in \mathcal{K}$ and U is an open neighborhood of E in X. Let $|E| = n$ and for each $m \in N$, let $\mathcal{K}_m = \{G \in \mathcal{K} : |G| = m\}$, where $\mathcal{K}_m = \emptyset$ for $m \in N$ and $m < n$. Since $\mathcal{K}$ is uncountable, there is an $m \in N$ such that $\mathcal{K}_m$ is uncountable. Let m be the least such a natural number. Then $n < m$. Since $E \in \mathcal{K}$, we have $\{G \in \mathcal{K} : \mathcal{K}_m \cap [G, U]$ is uncountable$\} \neq \emptyset$. Let s be the largest of $\{t \in N : \mathcal{K}_m \cap [G, U]$ is uncountable for some $G \in \mathcal{K}_t$ and $t < m\}$ and let $G = \{x_i : i \leqq s\} \in \mathcal{K}_s$ such that $\mathcal{K}_m \cap [G, U]$ is uncountable. Let $\mathcal{J} = \mathcal{K} \cap \mathcal{F}_m[X] \cap [G, U] - \cup\{[H, U] : H \in \mathcal{K}_t$ and $s < t < m\}$. Then $\mathcal{J}$ is an uncountable compact subset of $\mathcal{F}[X]$. If $\mathcal{J}' = \mathcal{J} - \{G\}$, then it is clear that $\mathcal{J}$ is the one-point compactification of the uncountable discrete space $\mathcal{J}'$. Thus $\mathcal{F}[X]$ contains a copy of $A(\omega_1)$.

Assume that X is a Hausdorff space and let us show the implication (b)→(c). Suppose that $\mathcal{F}[X]$ contains a copy of $A(\omega_1)$. We denote it by $\mathcal{J} = \mathcal{J}' \cup \{G\}$, where $\mathcal{J}'$ is a discrete space of cardinality ω_1. Let $G = \{x_i : i \leqq n\}$ and let $U(x_i)$ be an open neighborhood of x_i for $i \leqq n$ such that $\{U(x_i) : i \leqq n\}$ is pairwise disjoint in X. Since $\mathcal{J} - [G, \cup\{U(x_i) : i \leqq n\}]$ is finite, without loss of generality, we may assume that $\mathcal{J} \subset [G, \cup\{U(x_i) : i \leqq n\}]$. For $i \leqq n$, let $A_i = \{y \in U(x_i) : y \in H$ for some $H \in \mathcal{J}\}$. Then $x_i \in A_i$ for $i \leqq n$. Since $|\mathcal{J}| = \omega_1$, we have $|A_i| = \omega_1$ for some i $(i \leqq n)$. Since $\mathcal{J}$ is the one-point compactification of $\mathcal{J}'$, A_i and $A_i - \{x\}$, where $x \in A_i - \{x_i\}$, are compact. Since X is a Hausdorff space, $A_i - \{x\}$ is closed in A_i for each $x \in A_i - \{x_i\}$. Thus $A_i - \{x_i\}$ is discrete and hence, A_i is a copy of $A(\omega_1)$. The proof is completed.

We cannot omit the condition "X is Hausdorff" in Theorem 3.4.

EXAMPLE 3.5. There is a compact space X for which $\mathcal{F}[X]$ contains an uncountable compact subset, but X does not contain a copy of $A(\omega_1)$.

Let X be a set of cardinality ω_1. We topologize it as follows: closed subsets of X are $\emptyset$, X and finite subsets of X. It is clear that X does not contain a copy of $A(\omega_1)$. If $\mathcal{A} = \{\{x, y\} : y \in X - \{x\}\} \cup \{x\}$ for some $x \in X$, then $\mathcal{A}$ is an uncountable compact subset of $\mathcal{F}[X]$.

If X is a set linearly ordered by $<$, then X with the usual order topology

$\lambda(<)$ induced by $<$ is said to be a *linearly ordered topological space* (=LOTS). Intervals are denoted in the usual way. For example, we denote $\{x\in X: a\leqq x\leqq b\}$ by $[a, b]$ for $a, b\in X$ satisfying $a\leqq b$. A subset C of a LOTS X is said to be *order-convex* if whenever $a, b\in C$ satisfying $a\leqq b$, then $[a, b]\subset C$. If Y is a set linearly ordered by $<$ and τ is a topology on Y such that (1) $\lambda(<)\subset\tau$ and (2) τ has a base consisting of order-convex sets, then $X=(Y, \tau)$ is said to be a *generalized ordered space* (=*GO* space) [24] and we often say that the GO space X is constructed on the LOTS Y. Every GO space is known to be a hereditarily collectionwise normal space.

A Hausdorff space X is said to be *of pointwise countable type* [2] if for each $x\in X$, there is a compact subset K of X containing x such that K has a countable character in X. p-spaces and first countable spaces are of pointwise countable type.

For GO spaces, we obtain

THEOREM 3.6. *If X is a GO space, then the following conditions are equivalent.*

(a) *$\mathcal{F}[X]$ is a Moore space,*

(b) *$\mathcal{F}[X]$ is of pointwise countable type,*

(c) *X is first countable.*

PROOF. We shall prove the implication (b)→(c). By E.K. van Douwen [14], it suffices to prove that $\mathcal{F}[X]$ is first countable. It is well known that $A(\omega_1)$ is not a GO space. Since every subspace of a GO space is also a GO space, by using Theorem 3.4, every compact subset of $\mathcal{F}[X]$ is countable (hence metrizable). Thus for each $F\in\mathcal{F}[X]$, there is a compact metric space $\mathcal{K}(F)$ containing F such that $\mathcal{K}(F)$ has a countable character in $\mathcal{F}[X]$. Hence $\mathcal{F}[X]$ is first countable. The proof is completed.

By *a space of ordinals,* we mean a subspace of some ordinal.

THEOREM 3.7. *If X is a space of ordinals, then the following conditions are equivalent.*

(a) *$\mathcal{F}[X]$ is metrizable,*

(b) *$\mathcal{F}[X]$ is of pointwise countable type.*

PROOF. (a)→(b). Obvious.

(b)→(a). In [31] or [33], it was pointed out that if X is a space of ordinals, then $\mathcal{F}[X]$ is paracompact. Thus this follows from Theorem 3.6 and [15, 5.4.1].

The proof is completed.

§ 4. Partially separated spaces.

DEFINITION 4.1. A space X is said to be *weakly separated* [35] if there is a reflexive and antisymmetric relation $\leqq$ defined on X such that for each $x\in X$, $\{y\in X: y\leqq x\}$ is an open set of X. If, in addition, the relation is transitive (i.e. the relation is a partial order of X), then X is said to be *partially separated* [4], [7].

As was seen in [35], a space X is weakly separated if and only if for each $x\in X$, there is an open neighborhood $U(x)$ of x in X such that if $y\in U(x)$ and $x\in U(y)$, then $x=y$, or equivalently, X has an antisymmetric neighbornet in the sense of H. J. K. Junnila [21]. Similarly partially separated spaces are characterized as follows.

LEMMA 4.2. *A space X is partially separated if and only if there is a weak separation $\mathfrak{U}=\{U(x): x\in X\}$ of X such that if $y\in U(x)$, then $U(y)\subset U(x)$, or equivalently, X has an antisymmetric and transitive neighbornet.*

A space X is said to be *scattered* if X has no dense-in-itself subsets. It is well known that X is scattered if and only if for some ordinal α, $X^{(\alpha)}=\emptyset$, where $X^{(\alpha)}$ is the α-th derivative of X. A space X is said to be *locally countable* if every point of X has a neighborhood with cardinality at most ω_0. σ-discrete spaces, scattered spaces and locally countable spaces are known to be partially separated spaces by M. G. Bell [4]. Furthermore,

LEMMA 4.3. *For a space X, $\mathcal{F}[X]$ is partially separated.*

PROOF. For each $F\in\mathcal{F}[X]$, let $\mathcal{U}(F)=[F, X]$ and let $\mathfrak{U}=\{\mathcal{U}(F): F\in\mathcal{F}[X]\}$. It is to prove easy that $\mathfrak{U}$ is a partial separation of $\mathcal{F}[X]$. The proof is completed.

As mentioned in the introduction, weakly (partially) separated spaces play the fundamental role for the paracompactness of Pixley-Roy hyperspaces.

LEMMA 4.4 (Bell [4], Bennett, Fleissner and Lutzer [7]). *If X is partially separated, then $\mathcal{F}[X]$ is paracompact.*

LEMMA 4.5 (Przymusiński [31], Tanaka [33]). *For a space X, the following conditions are equivalent.*

(a) $\mathcal{F}_2[X]$ *is paracompact,*

(b) X *is weakly separated.*

Let us define $\mathcal{F}^1[X]=\mathcal{F}[X]$ and $\mathcal{F}^{n+1}[X]=\mathcal{F}[\mathcal{F}^n[X]]$ for each $n\in N$. The following result is an immediate consequence of Lemmas 4.3 and 4.4. However, it has already obtained by T. Przymusiński in [31].

LEMMA 4.6. *Let* $n\geqq 2$. *For a space* X, $\mathcal{F}^n[X]$ *is paracompact.*

THEOREM 4.7. *For a space* X, *there are a paracompact Hausdorff space* Y *and an open finite-to-one mapping* Φ *from* Y *onto* $\mathcal{F}[X]$.

PROOF. By Lemma 4.6, $\mathcal{F}^2[X]$ is a paracompact Hausdorff space. As the space Y, we take $\mathcal{F}^2[X]$ and define $\Phi: \mathcal{F}^2[X]\to\mathcal{F}[X]$ by $\Phi(\{F_1, \cdots, F_n\})=F_1\cup\cdots\cup F_n$, where $F_i\in\mathcal{F}[X]$ for $i\leqq n$. It is clear that the mapping Φ is finite-to-one. For $\{F_1, \cdots, F_n\}\in\mathcal{F}^2[X]$, let $[\{F_1, \cdots, F_n\}, [F_1, U_1]\cup\cdots\cup[F_n, U_n]]$ be a basic open neighborhood of $\{F_1, \cdots, F_n\}$. Then

$$\Phi([\{F_1, \cdots, F_n\}, [F_1, U_1]\cup\cdots\cup[F_n, U_n]])$$
$$=[F_1\cup\cdots\cup F_n, U_1\cup\cdots\cup U_n].$$

Thus Φ is an open mapping. It is similar to the above argument that Φ is a continuous mapping. The proof is completed.

COROLLARY 4.8. *For a space* X, *the following conditions are equivalent.*

(a) $\mathcal{F}[X]$ *is a Moore space,*

(b) $\mathcal{F}[X]$ *is the open finite-to-one image of a metric space,*

(c) $\mathcal{F}[X]$ *is contained in MOBI in the sense of A. V. Arhangel'skii* [3].

(d) X *is first countable.*

RROOF. We shall prove the implications (a)→(b) and (c)→(d).

(a)→(b). If $\mathcal{F}[X]$ is a Moore space, then $\mathcal{F}^2[X]$ is a paracompact Moore space and hence, metrizable. Thus this implication follows from the proof of Theorem 4.7.

(c)→(d). If $\mathcal{F}[X]$ is contained in the class MOBI, then $\mathcal{F}[X]$ is first countable. Hence X is first countable. The proof is completed.

REMARK 4.9. In [12], J. Chaber characterized open finite-to-one images of metric spaces as follows: A space X is the open finite-to-one image of a metric space if and only if X is a metacompact developable space having a countable cover by closed metrizable subspaces.

Next we prove that partial separatedness is preserved by perfect mappings.

THEOREM 4.10. *Let $f: X \to Y$ be a perfect mapping from X onto Y. Then*
(a) *if X is partially separated, then so is Y, and*
(b) *if Y is partially separated, X is a Hausdorff space and $\mathfrak{F}[X]$ is paracompact, then X is partially separated.*

PROOF. (a) Let $\mathfrak{U}=\{U(x): x\in X\}$ be a partial separation of X. For each $F\in\mathfrak{F}[X]$, let $U(F)=\cup\{U(x): x\in F\}$. Then $\mathfrak{U}'=\{U(F): F\in\mathfrak{F}[X]\}$ satisfies the condition (b) of Lemma 2.6 (i) (for details, see M. G. Bell [4] and H. R. Bennett, W. G. Fleissner and D. J. Lutzer [7]). For each $y\in Y$, let F_y be a finite subset of $f^{-1}(y)$ such that $f^{-1}(y)\subset U(F_y)$. Since $\mathfrak{U}$ is a partial separation of X, for each $y\in Y$, $\cup\{U(x): x\in f^{-1}(y)\}=U(F_y)$. For each $y\in Y$, let $V(y)=Y-f(X-U(F_y))$ and let $\mathfrak{B}=\{V(y): y\in Y\}$. We shall prove that $\mathfrak{B}$ is a partial separation of Y. To see this, assume that $y\in V(z)$ and $z\in V(y)$ for some $y, z\in Y$. Then $f^{-1}(y)\subset U(F_z)$ and $f^{-1}(z)\subset U(F_y)$. Since $\mathfrak{U}$ is a partial separation of X, $U(F_y)=U(F_z)$. Thus $F_y\cap F_z\neq\emptyset$ and hence, $y=z$. Thus $\mathfrak{B}$ is a weak separation of Y. Since $\mathfrak{U}$ is a partial separation of X, $\mathfrak{B}$ is transitive. Thus $\mathfrak{B}$ is a partial separation of Y.

(b) Let $\mathfrak{U}=\{U(y): y\in Y\}$ be a partial separation of Y. By T. Przymusiński [31], every fiber is a scattered subset of X. For $x\in f^{-1}(y)$ and $y\in Y$, if $x\in (f^{-1}(y))^{(\alpha)}-(f^{-1}(y))^{(\alpha+1)}$ for some α, let $V(x)=f^{-1}(V(y))-((f^{-1}(y)^{(\alpha)}-\{x\})$. Let $\mathfrak{B}=\{V(x): x\in X\}$. Assume that $x\in V(z)$ and $z\in V(x)$ for some $x, z\in X$. Then $f(x)\in U(f(z))$ and $f(z)\in U(f(x))$. It follows that $f(x)=f(z)$. By the definition of $\mathfrak{B}$, $x=z$. Hence $\mathfrak{B}$ is a weak separation of X. Since $\mathfrak{U}$ is a partial separation of Y, it is easy to prove that $\mathfrak{B}$ is transitive. Thus $\mathfrak{B}$ is a partial separation of X. The proof is completed.

EXAMPLE 4.11. We cannot omit the condition "$\mathfrak{F}[X]$ is paracompact" in Theorem 4.10 (b).

Let X be a partially separated space and let $f: X\times I\to X$ be a perfect mapping from $X\times I$ onto X, where I is the closed unit interval. By M. E. Rudin's theorem in [32], $\mathfrak{F}[X\times I]$ is not normal. Thus by Lemma 4.4, $X\times I$ is not partially separated.

In [33], the author has shown that for any GO space X, $\mathfrak{F}[X]$ is paracompact if and only if X is weakly separated (i.e., $\mathcal{F}_2[X]$ is paracompact by Lemma 4.5) as the affirmative answer to H.R. Bennett's problem in [5]. Furthermore, we obtain the following theorem.

THEOREM 4.12. *Let X be a GO space. Then X is partially separated if and only if X is weakly separated.*

PROOF. It suffices to prove the "if" part. Let X be a GO space constructed on a LOTS $(Y, <)$. Let $\mathfrak{U}=\{U(x): x\in X\}$ be a weak separation of X. Without loss of generality, we may assume that each $U(x)$ is order-convex. For each $x\in X$, define a subset $V(x)$ of X as follows: $y\in V(x)$ if and only if there is a finite sequence $\{x_1, \cdots, x_n\}$ of points of X such that $x_1\in U(x)$, $x_{i+1}\in U(x_i)$ for $i<n$ and $y\in U(x_n)$.

We shall prove that the collection $\mathfrak{V}=\{V(x): x\in X\}$ is a partial separation of X. It suffices to prove that $\mathfrak{V}$ satisfies the following conditions:

(1) $V(x)$ is an open neighborhood of x for each $x\in X$,

(2) if $x\in V(y)$ and $y\in V(z)$, then $x\in V(z)$, and

(3) if $x\in V(y)$ and $y\in V(x)$, then $x=y$.

For each $x\in X$, we have $U(x)\subset V(x)$. Since $\mathfrak{U}$ is a weak separation of X, $\mathfrak{V}$ satisfies (1). From the definition of $\mathfrak{V}$, it is clear that $\mathfrak{V}$ satisfies (2). It follows from the following claim that $\mathfrak{V}$ satisfies (3).

CLAIM. Let $\{x_1, \cdots, x_n\}$ be a finite sequence of points of X such that $x_{i+1}\in U(x_i)$ for each $i\leqq n$, where $x_{n+1}=x_1$. Then $x_1=\cdots=x_n$.

PROOF OF CLAIM. Let us call such a sequence $\{x_1, \cdots, x_n\}$ a *cycle with length n*. We shall prove by induction on the length. If $n=1$, the claim is obvious. Let $n>1$ and we have already proven the claim with length $<n$. Let $\{x_1, \cdots, x_n\}$ be a cycle with length n. Then there is an $i\leqq n$ such that $x_{i-1}\leqq x_i$ and $x_{i+1}\leqq x_i$, or $x_{i-1}\geqq x_i$ and $x_{i+1}\geqq x_i$, where $x_0=x_n$ and $x_{n+1}=x_1$. We consider the first case only. If $x_{i+1}\leqq x_{i-1}$ $(\leqq x_i)$, then $x_{i-1}\in[x_{i+1}, x_i]\subset U(x_i)$, because $U(x_i)$ is order-convex. Since $\mathfrak{U}$ is a weak separation of X, $x_{i-1}=x_i$ and hence, $x_{i+1}\in U(x_{i-1})$. If $x_{i-1}\leqq x_{i+1}$ $(\leqq x_i)$, then $x_{i+1}\in[x_{i-1}, x_i]\subset U(x_{i-1})$. In either case, we have $x_{i+1}\in U(x_{i-1})$. Hence $\{x_1, \cdots, x_{i-1}, x_{i+1}, \cdots, x_n\}$ is a cycle with length $n-1$. By the induction hypothesis, $x_1=\cdots=x_{i-1}=x_{i+1}=\cdots=x_n$. Moreover $x_i=x_{i+1}$, because $x_i\in U(x_{i-1})=U(x_{i+1})$ and $x_{i+1}\in U(x_i)$. The proof is completed.

THNOREM 4.13. *Let X be a GO space. Then the following conditions are equivalent.*

(a) $\mathcal{F}[X]$ *is paracompact,*

(b) $\mathcal{F}_2[X]$ *is paracompact,*

(c) X *is partially separated.*

PROOF. This follows from Lemmas 4.4, 4.5 and Theorem 4.12 immediately.

COROLLARY 4.14. *Let X be a GO space. Then the following conditions are equivalent.*

(a) $\mathcal{F}[X]$ *is metrizable,*

(b) $\mathcal{F}_2[X]$ *is metrizable,*

(c) X *is a first countable partially separated space.*

PROOF. By D. J. Lutzer [25], $\mathcal{F}_2[X]$ is first countable if and only if X is first countable. Thus this follows from Theorem 4.13.

We furthermore characterize GO spaces constructed on separable LOTS's whose Pixley-Roy hyperspaces are metrizable.

THEOREM 4.15. *Let $X=(Y, \tau)$ be a GO space constructed on a separable LOTS $(Y, <)$. Then the following conditions are equivalent.*

(a) $\mathcal{F}[X]$ *is metrizable,*

(b) $\mathcal{F}_2[X]$ *is metrizable,*

(c) *if we define*

$$I=\{x\in Y : \{x\}\in\tau\},$$

$$L=\{x\in Y-I :]\leftarrow, x]\in\tau\},$$

$$R=\{x\in Y-I : [x, \rightarrow[\in\tau\},$$

$$E=Y-(I\cup L\cup R),$$

then

(i) E *is countable,*

(ii) R *(resp. L) can be written as $R=\cup\{R_n : n\in N\}$ (resp. $L=\cup\{L_n : n\in N\}$) such that $cl_\tau R_n\cap L=\emptyset$ (resp. $cl_\tau L_n\cap R=\emptyset$) for each $n\in N$,*

(iii) *if $x\in E\cap cl_\tau R_n$, then for some $y<x$, $]y, x[\cap R_n=\emptyset$, and*

(iv) *if $x\in E\cap cl_\tau L_n$, then for some $z>x$, $]x, z[\cap L_n=\emptyset$.*

REMARK 4.16. In [6], H. R. Bennett, W. G. Fleissner and D. J. Lutzer obtained another characterization.

PROOF OF THEOREM 4.15. (a)$\rightarrow$(b). Obvious.

(b)$\rightarrow$(c). This implication was in fact proved by H. R. Bennet, W. G. Fleissner and D. J. Lutzer in [6].

(c)→(a). By Theorem 4.12, it suffices to prove that X is weakly separated. (Notice that X is first countable.) Since $X=E\cup(\cup\{cl_\tau R_n : n\in N\})\cup(\cup\{cl_\tau L_n : n\in N\})\cup I$, by M.G. Bell [4], it suffices to prove that $cl_\tau R_n$ and $cl_\tau L_n$ are weakly separated for $n\in N$. Fix n and we shall prove that $cl_\tau R_n$ is weakly separated. Using (ii), $cl_\tau R_n\cap L=\emptyset$. Clearly $cl_\tau R_n\cap I=\emptyset$. For each $x\in cl_\tau R_n$, define an open neighborhood $U(x)$ of x in X as follows:

(1) if $x\in R\cap cl_\tau R_n$, let $U(x)=[x, \rightarrow[$.

(2) if $x\in E\cap cl_\tau R_n$, then, using (iii), there is a $y<x$ such that $]y, x[\cap R_n=\emptyset$. Let $U(x)=]y, \rightarrow[$.

Let $\mathfrak{U}=\{U(x): x\in cl_\tau R_n\}$. Then $\mathfrak{U}$ is a weak separation of $cl_\tau R_n$. To see this, assume that $y\in U(x)$ and $x\in U(y)$ for some $x, y\in cl_\tau R_n$. We devide two cases.

Case 1. $x\in R\cap cl_\tau R_n$. Since $U(x)=[x, \rightarrow[$, we have $x\leqq y$. If $y\in E\cap cl_\tau R_n$, then there is a $z<y$ such that $]z, y[\cap R_n=\emptyset$ and $U(y)=]z, \rightarrow[$. Then $]z, y[\cap cl_\tau R_n=\emptyset$. Thus it follows that $x\leqq z$ and hence, $x\notin U(y)$, which is a contradiction. Thus $y\in R\cap cl_\tau R_n$. Hence $y\leqq x$. We have $x=y$.

Case 2. $x\in E\cap cl_\tau R_n$. Then there is a $z<x$ such that $]z, x[\cap R_n=\emptyset$ and $U(x)=]z, \rightarrow[$. Hence $]z, x[\cap cl_\tau R_n=\emptyset$. Since $y\in cl_\tau R_n$, we have $x\leqq y$. If $y\in R\cap cl_\tau R_n$, then $x\neq y$ and hence, $x<y$, which contradicts the fact that $x\in U(y)$. Thus $y\in E\cap cl_\tau R_n$. Hence we obtain $y\leqq x$ similarly. We have $x=y$. The proof is completed.

Remark 4.17. The collection $\mathfrak{U}=\{U(x): x\in cl_\tau R_n\}$ constructed in the proof of Theorem 4.15 is in fact a partial separation of $cl_\tau R_n$.

As other classes of spaces for which weak separatedness implies partial separatedness,

Theorem 4.18. *Let X be a semi-stratifiable space. Then the following conditions are equivalent.*

(a) *$\mathfrak{F}[X]$ is paracompact,*

(b) *$\mathfrak{F}_2[X]$ is paracompact,*

(c) *X is σ-discrete,*

(d) *X is partially separated.*

Proof. This follows from H. J. K. Junnila [21].

The following results are essentially proved by T. Przymusiński (see [31, § 4]) and thus, the proofs are omitted.

THEOREM 4.19. *Let X be a locally Čech complete Tychonoff space. Then the following conditions are equivalent.*

(a) $\mathcal{F}[X]$ *is paracompact,*

(b) $\mathcal{F}[X]$ *is normal,*

(c) $\mathcal{F}_2[X]$ *is paracompact,*

(d) $\mathcal{F}_2[X]$ *is normal,*

(e) X *is scattered,*

(f) X *is partially separated.*

THEOREM 4.20. *Let X be a compact Hausdorff space. Then the following conditions are equivalent.*

(a) $\mathcal{F}[X]$ *is metrizable,*

(b) $\mathcal{F}_2[X]$ *is metrizable,*

(c) X *is countable,*

(d) X *is a first countable partially separated space.*

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