TSUKUBA J. MATH.
Vol. 7 No. 2 (1983). 215–226

UNIFORM VECTOR BUNDLES OF RANK $(n+1)$ ON P_n

By

Edoardo BALLICO

Introduction.

Here vector bundle (or sometimes bundle) means algebraic vector bundle on an algebraic variety. Every variety is defined over an algebraically closed field K with $ch(K)=0$. We write $\boldsymbol{P}_n:=\boldsymbol{P}_n(K)$. A vector bundle E on $\boldsymbol{P}_n$ is uniform if there exists a sequence of integers $(k; r_1, \cdots, r_k; a_1, \cdots, a_k)$ (called the splitting type of E) with $a_1>\cdots>a_k$ and such that for every line L of $\boldsymbol{P}_n$: $E_L\cong\bigoplus_{i=1}^{K} r_i\mathcal{O}_L(a_i)$. If the rank r of E is low with respect to the dimension n of $\boldsymbol{P}_n$, there are only a few uniform vector bundles of rank r. See [1], [2], [5] for the following

THEOREM. *For $r\leqq n$, $n\geqq 2$, $r=3$ and $n=2$, the uniform vector bundles of rank r on $\boldsymbol{P}_n$ are (up to isomorphism) direct sum of line bundles, $\Omega^1_{\boldsymbol{P}_n}(a)$, $T\boldsymbol{P}_n(b)$, $S^2T\boldsymbol{P}_n(c)$, with a, b, c integers.*

In particular every such bundle is homogeneous, i.e. for every automorphism g of $\boldsymbol{P}_n$, $g^*(E)\cong E$. But for $r\geqq 2n$ there exists uniform vector bundles of rank r on $\boldsymbol{P}_n$ which are not homogeneous. Thus it remains open the range $n+1\leqq r<2n$. Ph. Ellia in [3] proved that a uniform rank-$(n+1)$ vector bundle on $\boldsymbol{P}_n$ is decomposable if $n=3, 4, 5$ or $n=p-1$ where p is a prime number. His methods give also many other partial results on rank-$(n+1)$ vector bundles on $\boldsymbol{P}_n$, giving evidence to the following

TEEOREM 1. *Every uniform vector bundle of rank $n+1$ on $\boldsymbol{P}_n$ is isomorphic either to a direct sum of line bundles or to the direct sum of a line bundle and of $\Omega^1_{\boldsymbol{P}_n}(b)$ or $T\boldsymbol{P}_n(a)$.*

In this paper we prove theorem 1, using the methods of [3]. To pass from [3] to theorem 1 no geometry is involved; the only problems are about roots of unity, roots of polynomials or decomposition of polynomials. Thus the proofs are tricky.

We want to thank U. Zannier for useful conversations.

Received September 8, 1982.

§ 0. Notations. For more details, see [2], [3].

Every vector bundle E on $\boldsymbol{P}_1$ is a direct sum of line bundles and thus it has a natural filtration. If $E\cong\bigoplus_{i=1}^{K} r_i \mathcal{O}_{\boldsymbol{P}_1}(a_i)$ with $a_1>\cdots>a_k$, the j-th term of the filtration HN^jE is the unique subbundle of E isomorphic to $\bigoplus_{i=1}^{j} r_j \mathcal{O}_{\boldsymbol{P}_1}(a_i)$. This is the Harder-Narasimhan filtration. Now we define the relative Harder-Narasimhan filtration. Let $G(1, n)$ be the grassmannian of lines in $\boldsymbol{P}_n$ and $F_n:=\{(x, 1)\in \boldsymbol{P}_n\times G(1, n): x\in 1\}$ the incidence variety. We have the projections $p: F_n\to \boldsymbol{P}_n$, $q: F_n\to G(1, n)$.

PROPOSITION [2] *Let E be a uniform vector bundle on $\boldsymbol{P}_n$ of splitting type $(k; r_1, \cdots, r_k; a_1, \cdots, a_k)$. There are bundles E_i of rank r_i on $G(1, n)$ such that p^*E has a filtration by subbundles whose graded bundle is $\bigoplus_{i=1}^{k}[q^*E_i\otimes p^*\mathcal{O}_{\boldsymbol{P}_n}(a_i)]$.* This is the HN (or Harder-Narasimhan) filtration:

$$HN^jp^*E:=\mathrm{Im}[q^*q_*p^*E(-a_j)\otimes p^*\mathcal{O}(a_j)\to p^*E].$$

We write $G:=G(1, n)$ and $F:=F_n$ if there is no possibility of misunderstanding. Let H be the tautological subbundle on $\boldsymbol{P}_n$ i.e. let H be $\mathcal{O}_{\boldsymbol{P}_n}(-1)$. Let Q be the tautological quotient bundle on $\boldsymbol{P}_n$ i.e. let $Q=T\boldsymbol{P}_n(-1)$. Let N be the tautological quotient bundle of rank $(n-1)$ on G. F is naturally identified to $\boldsymbol{P}(Q)$ and this identification determines on F a relative tautological subline bundle H_Q. We consider the Chern classes (in $H^*(F, \boldsymbol{Z})$ over $\boldsymbol{C}$ or, if you prefer, in general in the Chow ring) $U:=c_1(p^*H)$, $V:=c_1(H_Q)$.

Consider the polynomial

$$R(X, Y)=X^n+\cdots+X^lY^{n-l}+\cdots Y^n$$

In [2] it is proved the following result (Leray-Hirsch's theorem):

a) The natural morphism t of $\boldsymbol{Z}[U, V]$ into $H^*(F, \boldsymbol{Z})$ induces an isomorphism of $H^*(F, \boldsymbol{Z})$ with $\boldsymbol{Z}[U, V]/(R(U, V), U^{n+1})$.

b) The subalgebra $p^*H^*(\boldsymbol{P}_n, \boldsymbol{Z})$ is the image by t of the algebra of polynomials in the variable U.

c) The subalgebra $q^*H^*(G, \boldsymbol{Z})$ is the image by t of the algebra of symmetric polynomials in U, V.

d) The Picard group of F is the free abelian group generated by p^*H and H_Q. Every vector bundle E of rank r on a projective variety has the Chern polynomial

$$C_E(T):=T^r-c_1(E)T^{r-1}+\cdots+(-1)^rc_r(E).$$

The Chern polynomial has the following properties:

i) if L is a line bundle, then $C_{E\otimes L}(T)=C_E(T-c_1(L))$;

ii) if E has a filtration with graduation $\oplus_i E_i$, then $C_E=\Pi_i C_{E_i}$.

Now let E be a uniform vector bundle of rank r on $\boldsymbol{P}_n$ of splitting type $(k; r_1, \cdots, r_k; a_1, \cdots, a_k)$. Consider $P(T, U)=T^r+c_1UT^{r-1}+\cdots+c_rU^r$ the Chern polynomial of p^*E, where c_i are the Chern classes of E (recall the definition of U). Then ii) applied to the HN-filtration of E gives the following relation in $\boldsymbol{Z}[T, U, V]$:

$$P(T, U)+Q(T, U, V)U^{n+1}+M(T, U, V)R(U, V)=\Pi_{i=1}^k S_i(T+a_iU, U, V)$$

where $Q(T, U, V)$ is a homogeneous polynomial of degree $r-n-1$, $M(T, U, V)$ is a homogeneous polynomial of degree $r-n$ and $S_i(T, U, V)$ is the Chern polynomial of q^*E_i (it is homogeneous of degree r_i and symmetric in U and V). In particular let E be a uniform vector bundle on $\boldsymbol{P}_n$ of rank $n+1$ and splitting type $\{k; r_1, \cdots, r_k; a_1, \cdots, a_k)$. We have the following fundamental relation

$$(\mathcal{E})\qquad P(T, U)+xU^{n+1}+(aT+bU+cV)R(U, V)=\Pi_{i=1}^k S_i(T+a_iU, U, V)$$

with $P(T, U)=T^{n+1}+c_1UT^n+\cdots+c_nU^nT$ the Chern polynomial of p^*E, x, a, b and c integers. $(\mathcal{E}_j)$ is the relation obtained by $(\mathcal{E})$ replacing T by $T-a_jU$:

$$(\mathcal{E}_j)\qquad P_j(T, U)+x_jU+(aT+b_jU+cV)R(U, V)=\Pi_{i=1}^k S_i(T+(a_i-a_j)U, U, V)$$

with $P_j(T, U)$ Chern polynomial of $p^*(E(-a_j))$ and $b_j=-aa_j+b$. In this paper x_j, a, b_j, c will have always the meaning given by $(\mathcal{E}_j)$. From the symmetry of $S_j(T, U, V)$ it follows [3 lemma III. 1.2] that either $x_j=0$ or $x_j=c-b_j$.

§1. **We fix a uniform vector bundle of rank** $n+1$ and splitting type $(k; r_1, \cdots, r_k; a_1, \cdots, a_k)$. For simplicity we consider always the geometrical situation of $(\mathcal{E}_j)$, avoiding the case in which $(\mathcal{E}_j)$ does not come from the HN-filtration of such a bundle. If $k=1$ or $k=2$, $r=n$ or 1, then theorem 1 is satisfied [3, IV. 2]. Thus we may assume $k\neq 1$, if $k=2$, $r\neq n$ or 1, $n\geqq 7$ [3, Chapter 6] and that the a_i's are consecutive (otherwise E splits by [2]). With these assumptions the proof of theorem 1 is purely algebraic: it follows from the relations $(\mathcal{E}_j)$.

Ellia's machinery permits to handle easily the case $c=0$ [3, Chapter III] and, with much more efforts, the case "$x_j=c-b_j$ for every j". The main technical point of this paper is the following lemma, proved in the second paragraph:

LEMMA 1. *If* $x_j=0$, *then* $c=b_j=0$.

For the proof of lemma 1 we will show that if $c\neq 0$ or $b_j\neq 0$, then $P_j(T, 1)$

has $(n+2)$ roots, impossible. But for some detail we use the techniques of the first paragraph. The reader can verify that this is not a circular proof. We say that t is a primitive solution of $(\mathcal{E}_i)$ if in $(\mathcal{E}_i)$:

1) $x_i=c-b_i$;
2) t is a root of $S_i(0, 1, V)$, $t\neq 1$, and t is a simple root of the polynomial

$$D_i(V):=cV^{n+1}+(c+b_i)(V^n+\cdots+V)+c.$$

Ellia assume $x_i\neq 0$ instead of condition 1). By [3, lemma III. 1.2], $x_i\neq 0$ implies $x_i=c-b_i$. The condition 1) is sufficient for us, even if $b_i=c$.

LEMMA 2. *Let $t_1, \cdots, t_s$ be primitive solutions of $(\mathcal{E}_i)$. If for every $1\leqq h\leqq 1$ there exists $s(h)$ such that $t_{s(h)}^{n+1-h}\neq 1$, then $c_{n+1-h}=0$ for $1\leqq h\leqq 1$.*

The proof is exactly the same of [3, lemme V. 1.1].

Recall that Ellia [3, lemme III. 1.3] proved that the polynomial $D_i(V)$ defined in (1) has, for $c\neq 0$, at most 3 real roots and that every multiple root of D_i is a real root.

Copying [3, Remarque V. 3.3] we have the following

REMARK 1. Consider $S(v)=Mv^2+Dv+M$, $A(v)=Mv^3+Zv^2+Zv+M=(1+v)(Mv^2+(Z-M)v+M)$. Then $S(v)$ has a double roots if and only if $S(1)=0$ or $S(-1)=0$. Thus if $x_i=c-b_i$, $r_i=2$ or $r_i=3$ and there is no primitive solution of $(\mathcal{E}_i)$, then either $c=-(nb_i)/(n+2)$ or, if n is odd, $c=b_i$ or, if n is even, $c=(nb_i)/(n+2)$.

LEMMA 3. *A primitive root of unity of order r, $2<r\leqq n$, is a root of the polynomial $A(x)=cx^{n+2}+bx^{n+1}-bx-c$, $c\neq 0$, if and only if $b=0$, $\pm c$ or, for $n\equiv 1 \bmod 6$, $b=-2c$, for $n\equiv 3 \bmod 6$, $2b=-c$.*

PROOF. If $r>12$ or $r=5, 7, 9, 11$ this is in [3, V. 4.4 and V. 4.6]. The remaining cases can be checked directly. Q.E.D.

By lemma 2 and lemma 3 if there exists an index i with $x_i=c-b_i$, $c\neq 0$, $-\frac{1}{2}b_i$, $-2b_i$, except in a few cases in $(\mathcal{E}_i)$ we have $c_1=\cdots=c_n=0$. We want to show that there exists always an index i such that in $(\mathcal{E}_i)$ $c_1=\cdots=c_n=0$. By [3, Chapter III] this is the case if $c=0$. Thus by lemma 1 we may assume for this problem $x_i\neq 0$ for every i and $c\neq 0$.

LEMMA 4. *Assume $r\geqq 3$, $x_i=c-b_i$, $c\neq 0$, $b_i=-2c$ if $n\equiv 1 \bmod 6$, $2b_i=-c$ if $n\equiv 3 \bmod 6$. Then in $(\mathcal{E}_i)$ we have $c_1=\cdots=c_n=0$.*

PROOF. Under both assumptions the polynomial $A(x)=cx^{n+2}+b_ix^{n+1}-b_ix-c=(x-1)D_i(x)$ has no multiple root since it is easy to check that it has no real multiple root and by [3, lemme III. 1.3] any multiple root of $A(x)$ is real. The only cyclotomic polynomial which divide $D_i(x)$ is x^2-x+1. Thus if $r\geqq 3$, we may apply lemma 2. Q.E.D.

To use the general machinary of [3], we have to control the case of primitive solutions of $(\mathcal{E}_i)$ which are roots of unity.

LEMMA 5. *Assume $a=0$ and either $b_i=0$ or $b_i=-c$. Then we have $c=0$ or $k=1$.*

PROOF. Assume $c\neq 0$. We have $b_i=b_j$ for every i, j. We put $b:=b_i$. Suppose $b=-c$. Then the left-hand side of $(\mathcal{E}_i)$ is $P_i(T, U)+c(V^{n+1}+U^{n+1})$. If in $(\mathcal{E}_i)$ we put $U=1$, $V=z$ with $z^{n+1}=-1$, $S_j(j-i, 1, z)=0$, we obtain that $(j-i)(-1+z)$ is a root of $P_i(T):=P_i(T, 1)$ (see the proof of lemma 8 in the next paragraph). In the same way, taking $V=1$, $U=z$ as above, we obtain the roots $-(j-i)(-1+z^{-1})$. We obtain $2n-2r_i+2$ distinct roots of $P_i(T)$ since $1(-1+z)=k(-1+w)$ with 1, k non-zero integers and $z^{n+1}=w^{n+1}=-1$ implies $z=w$ by lemma 7 in the next paragraph. Thus we have $k\leqq 2$. Assume $k=2$, thus $r_1=r_2$. We have shown that $P_2(T)=P_1(T-1)$ has $2n-2r_1+2$ roots of type $\pm(-1+z)$ with $z^{n+1}=-1$ and $P_1(T)$ has $2n-2r+2$ roots of type $\pm(-1+w)+1$ with $w^{n+1}=-1$. An equality $\pm(-1+z)=\pm(-1+w)+1$ for such w, z implies $z=w$, $-w$, w^{-1}, $-w^{-1}$ and z of order 3 or 6. This is impossible since n is odd $(r_1=r_2)$ and $z^6=1$ implies $z^{n+1}=1\neq -1$. For $b=0$ a different proof is given in [3, V. 5.1.1]. Q.E.D.

LEMMA 6. *Assume $a=0$, $c\neq 0$, $k\neq 1$, $x_j=c-b_j$. Then there exists an index i such that in $(\mathcal{E}_i)$ we have $c_t=0$ for $1\leqq t\leqq n$.*

PROOF. By lemma 5 we may assume that $b:=b_j\neq -c$. The proof of [3, V. 3.6.] shows that there exists an index i and a primitive solution u for $(\mathcal{E}_i)$ with

$$\pi/(n+1)<\arg(u)<5\pi/(n+1) \tag{2}$$

In particular $r_i\geqq 2$. If we cannot apply lemma 2, i.e. if u is a root of unity, then either $b=c$ (case solved by lemma 1 or lemma 10) or $2\leqq r_i\leqq 3$, $n\equiv 1 \bmod 6$ or $n\equiv 3 \bmod 6$, $u^6=1$. But for $n\geqq 14$ this contradicts (2). If there are at least 4 odd r's, then $c=0$ (same proof as [3, V. 3.1]. Thus there exists an index j with $r_j=2$. Since $S_j(k-j, 1, x)$ is a different factor of $D_j(x)$ (unless the linear term of T in $S_j(T, U, V)$ vanishes) then we obtain easily $r_1=2$ or $r_2=2$ and then, for the same reason, $r_s=2$ for every index s. We have only to control the cases $n=7$, 9 or 13. Consider $S_1(T, U, V)=T^2+dT(U+V)+A(U^2-UV+V^2)$. Since $c\neq 0$, $A\neq 0$. By the restriction to a fiber of the Harder-Narasimhan filtration we obtain $A\geqq 0$ since a subbundle of a trivial bundle has non-negative even Chern classes. We consider the decomposition of $D_1(V)$ by the factors $S_1(s-1, 1, V)$. From the terms of degree $n+1$, n and 0 we obtain $12/A+6d/A=9/2$ for $n=7$, $42/A+21d/A=15/2$ for $n=13$ and $30/A+15d/A=7$ for $n=9$. This is impossible. Q.E.D.

PROPOSITION 1. *Assume* $c\neq 0$, $k>1$. *Then there exists an index* i *such that in* $(\mathcal{E}_i)$ *we have* $c_s=0$ *for* $1\leqq s\leqq n$.

PROOF. We may assume $a\neq 0$, $x_j\neq 0$ for every index j. Assume $b_j=0$. Then $(cV+b_j)R(1, V)+c-b_j=c(V^{n+1}+\cdots+V+1)$. We have $c_s=0$ for $1\leqq s\leqq n$ in $(\mathcal{E}_j)$ by lemma 2 if the order of the roots of $S_j(0, 1, V)$ have $n+2$ as minimum common multiple. This happens if $r_j\geqq(n+2)/2$, for examples by the degrees of cyclotomic polynomials [4, pag. 206]. Assume $b_h=-c$. Then $(cV+b_h)R(1, V)+c-b_h=c(V^{n+1}+1)$. As above we have $c_1=\cdots=c_n=0$ in $(\mathcal{E}_h)$ if $r_h\geqq(n+1)/3$. In fact $v^k=1$, $k\leqq n+1$, $v^{n+1}=-1$ implies k even, say $k=2s$, $v^s=-1$, $s\leqq(n+1)/3$. Suppose the thesis does not hold. There can be other factors S_i, but, if we have j, h with $b_j=0$, $b_h=-c$, at most one factor S_i with $r_i=2$. This factor can exist only if $n\equiv 1$, 3 mod 6. In fact the case $b_i=cn/(n+2)$ cannot occur if $b_j=0$, $b_h=0$, since $b_i=b_j+(j-i)a$. Furthermore if there exists j with $b_j=0$, there exists at most a factor S_i with degree $r_i=1$ and it exists only for n even. Thus we have $(n+2)/2+(n+1)/3+2+1\geqq n+1$ i.e. $n\leqq 20$. The factor with $r_i=2$ could exist only for $n=3$, 7, 13, 15 or 19; if it does not exist, we have the better inequality $n\leqq 8$. Thus we may assume $n=7$, 8, 9, 13, 15 or 19. For $n=19$, $V^{20}+1=\phi_{h0}(V)\cdot\phi_8(V)$, where ϕ_d is the cyclotomic polynomial of order d; since $\deg\phi_{40}=16>(n+1)/2$, $b_h=-c$ cannot happen. For $n=13$, $V^{14}+1=\phi_4(V)\cdot\phi_{28}(V)$ and $\deg\phi_{28}=12>(n+1)/2$. For $n=9$, 15 $n+2$ is prime, $V^{n+1}+V^n+\cdots+V+1$ is irreducible, thus $b_j=0$ implies $k=1$. The remaining possibility (when either $b_j\neq 0$ for every j or $b_h\neq -c$ always or $n=7$, 8) can be checked directly. We have to use remark 1 to analyze the existence of primitive solution for $(\mathcal{E}_i)$ if $r_i=2$, 3 and use [3, V. 3.1] and its extension to the case n odd. Q.E.D.

If $c=0$, then ([3, Chapter III]) there exists an index j such that in $(\mathcal{E}_j)$ we have $c_1=\cdots=c_n=0$; furthermore if $c=0$, $S_j(0, 1, V)$ is divided by V. We use always the above notations, i.e. we assume $c_s=0$ in $(\mathcal{E}_j)$ by prop. 1. At this point, modulo the proof of lemma 1 given in the next paragraph, to prove theorem 1 it is sufficient to copy, with mild simplifications, the proofs in [3, V. 6]. We put $b:=b_j$, $u_i:=a_i-a_j$, $1\leqq i\leqq k$. We have $u_j=0$ and the u_i are consecutive by assumptions. Thus $u_i=j-i$. In $(\mathcal{E}_{i+j})$ the left-hand side is

$$T^{n+1}+\cdots+(-1)^n(n+1)i^nTU^n+U^{n+1}((-1)^{n+1}i^{n+1}+c-b)$$
$$+(aT+(-ai+b)U+cV)R(U, V).$$

Since either 1) $x_{i+j}=0$ or 2) $x_{i+j}=c-b_{i+j}$ for every i by [3, lemme III. 1.2], we have respectively either 1) $i^{n+1}=(-1)^n(c-b)$ or 2) $i^n=-(-1)^n a$. The condition $c_1=0$ in $(\mathcal{E}_j)$ implies

$$\sum_{i=1}^{k} r_i(i-j)=0 \tag{3}$$

and thus $k\neq 2$ and $j\neq 1$, k. 1) and 2) implies $k\leqq 4$. If n is odd, $x_{j-1}=x_{j+1}=0$. Thus the left-hand side of $(\mathcal{E}_j)$ is T^{n+1} for n odd by lemma 1 and, for n odd, the vector bundle E splits and the theorem is proved.

Thus we may assume n even. We have $a=-1$. Suppose $k=4$. Taking eventually the dual vector bundle, we may assume $j=2$. Then $x_4=0$ and by lemma 1 $b_4=c=0$. The condition $b_4=0$ is equivalent to $b=-2$. From 1) we have $2^{n+1}=-b$, contradiction.

Thus we may assume n even, $k=3$, $a=-1$, $j=2$. (3) implies $r_1=r_3$. It cannot happen $x_1=0$ or $x_3=0$. For example $x_1=0$ implies $b_1=b-1=0$, $c=0$. The left-hand side of $(\mathcal{E}_2)$ is $(b=1,\ a=-1,\ c=0)$

$$T^{n+1}-TR(U, V)+UR(U, V)-U^{n+1}=(T-U)(R(T, U)-R(U, V))$$
$$=(T-U)(T-V)\sum_{n-1}(T, U, V)$$

where we write

$$\sum_{n-1}(T, U, V)=\sum_{r+s+t=n-1}T^rU^sV^t.$$

$\sum_{n-1}(T, U, V)$ is irreducible, thus $x_1\neq 0$, because this contradicts the hypotheses that, for $c=0$, V devides $S_2(0, 1, V)$. Now assume $r_1\geqq 4$. As in [3, V. 6.3.1] we obtain $c=0$ and, taking $T=0$ in $(\mathcal{E}_2)$ the left-hand side is $b_1V(V^{n-1}+\cdots+1)$. If $b_1\neq 0$, as in [3, pag. 48-49], we obtain a contradiction. If $b_1=0$, i.e. $b=-1$, we are in the case $a=-1$, $b=-1$, $c=0$, just solved. Thus we may assume $r_1=r_3\leqq 3$. First assume r_1 odd. Since n is even, by [3, lemme V. 3.1] we have $c=0$. The relation $(\mathcal{E}_r)$ gives, for $T=0$, the identity

$$bV(V^{n-1}+\cdots+1)=S_1(-1, 1, V)S_2(0, 1, V)S_3(1, 1, V)$$

and, since n is even, every S has a real root, which is absured unless $b=0$. Assume $b=0$. The left-hand side of $(\mathcal{E}_2)$ is $T(T^n-R(U, V))$ and $T^n-R(U, V)$ is irreducible by the Eisenstein's criterion, contradiction. The case $r_1=r_3=2$ is verbatim [3, V. 6.4.2 case (2)]. The proof of theorem 1 is finished, modulo the proof of lemma 1.

§2. **In this paragraph we prove lemma 1.** Thus we assume $x_j=0$ and write b, r instead of b_j, r_j; $P(T):=P_j(T, 1)$ where $P_j(T, U)$ is defined by $(\mathcal{E}_j)$. We will prove, under the assumption $c\neq 0$ or $b\neq 0$, that $P(T)$ has $n+2$ roots, a contradiction.

We use freely particular cases of the following lemma.

LEMMA 7. *Let d, s be non zero integers, v, w, z roots of unity with $v\neq 1$. Assume*

$$d(-1+z)w=s(-1+v) \tag{4}$$

Then $zw^2=v$. Furthermore z and v are conjugate unless

1) $s=2d,\ w^3=-1,\ z=-1,\ v=w^{-1}$;
2) $s=-2d,\ z=-1,\ w^3=-1,\ v=-w^{-1}$;
3) $2s=d,\ w^3=1,\ v=-1,\ z=w$;
4) $2s=-d,\ w^3=1,\ v=-1,\ z=-w$.

Furthermore if $w^2=1$, then the $z=v$, $s=dw$.

PROOF. We have $\arg(-1+z)^2+\arg(w)^2\equiv\arg(-1+v)^2 \bmod 2\pi$. Since $-1+e^{ix}=-2ie^{ix/2}\sin(x/2)$, we have $\arg(z)-\pi/2+2\arg(w)\equiv\arg(v)-\pi/2 \bmod 2\pi$ i.e. $v=zw^2$. If $w^2=1$, then we have finished. Thus we may assume w not rational. From $zw^2=v$ and (4) it follows $z, v\in \boldsymbol{Q}(w)$ and the minimal polynomials of w over $\boldsymbol{Q}(z)$ and $\boldsymbol{Q}(w)$ have degree at most 2. Thus either $w^3\in\boldsymbol{Q}(z)$ or $w^2\in\boldsymbol{Q}(z)$. But $w^2\in\boldsymbol{Q}(z)$ implies $w\in\boldsymbol{Q}(z)$ by (4). Assume $w^3\in\boldsymbol{Q}(z)$, $w\notin\boldsymbol{Q}(z)$; we have $\mathrm{ord}(w)=3\mathrm{ord}(z)$ or $\mathrm{ord}(w)=6\mathrm{ord}(z)$. From $d(-1+z)w^2=s(-w+zw^3)$, we obtain $-dw^2+dzw^2+sw\in\boldsymbol{Q}(z)$; $szw^2-dzw+dw\in\boldsymbol{Q}(z)$, i.e. $-dw^2+d^2w/s-d^2w/(sz)$ is in $\boldsymbol{Q}(z)$, implies $dzw^2+sw-d^2w/s+d^2w/(sz)\in\boldsymbol{Q}(z)$ i.e.

$$w(-2d^2/s+d^2/k^2+s+d^2z/s)\in\boldsymbol{Q}(z).$$

Thus, since by assumption $w\notin\boldsymbol{Q}(z)$, $d^2z^2+z(-2d^2+s^2)+d^2=0$. This implies either $z=-1$, $4d^2=s^2$, or $-2d^2+s^2=\pm d^2$. In the last case $d^2=s^2$ (since $3d^2=s^2$ is impossible) and taking absolute values in (4) we obtain $z=v$ or $z=v^{-1}$ i.e. z and v are conjugate. If $z=-1$, $s=2d$, we have case 1), otherwise case 2). By symmetry if $w\notin\boldsymbol{Q}(v)$, either z and v are conjugate or we are in cases 3) or 4). Thus we may assume $\boldsymbol{Q}(z)=\boldsymbol{Q}(w)=\boldsymbol{Q}(v)$. Hence either z is conjugate to v or z is conjugate to $-v$. Assume for example $\mathrm{ord}(z)<\mathrm{ord}(v)$. Then $\mathrm{ord}(v)=2\mathrm{ord}(z)$

and either $\operatorname{ord}(w)=\operatorname{ord}(z)$ or $\operatorname{ord}(w)=\operatorname{ord}(v)$. In both cases $w^{2\operatorname{ord}(z)}=1$ and $v=zw^2$ gives the contradiction. Q.E.D.

LEMMA 8. *$x_j=0$ implies either $c=b_j=0$ or $2r_j\geqq n$.*

PROOF. Assume $c\neq 0$ or $b:=b_j\neq 0$; recall $r=r_j$. Then from $(\mathcal{E}_j)$ we obtain, taking $T=0$, the fundamental relation

$$(cV+bU)R(U, V)=\prod_{i=1}^{k}S_i((i-j)U, U, V) \tag{5}$$

Fix $i\neq j$. Let A_i be the set of root of unity w satisfying $(S_i(i-j, 1, w))/(cw+b)=0$. For some F_i, F_i', F_i'', $S_i(T+(i-j)U, U, V)=TF_i+S((i-j)U, U, V)$ implies $S_i(T, U, V)=(T-(i-j)U)F_i'+S_i((i-j)U, U, V)=(T-(i-j)V)F_i''+S_i((i-j)V, V, U)$ since $S_i(T, U, V)$ is symmetric in U, V. Thus we have

$$S_i(T+(i-j)U, U, V)=(T+(i-j)U-(i-j)V)G_i+S_i((i-j)V, V, U) \tag{6}$$

for some G_i. If in (6) we take $U=1$, $V=t\in A_i$, we obtain $P((i-j)(-1+t))=0$ because $S_i((i-j)U, U, V)$ is a product of symmetric divisor of $R(U, V)$ and eventually a constant multiple of $(cV+b)$.

If in (6) we take $U=t$, $V=1$, we obtain that $(i-j)(-1+t)t^{-1}=-(i-j)(-1+t^{-1})$ is a root of $P(T)$. Since t and t^{-1} are conjugate, they are both roots of $S_i(i-j, 1, V)$. Thus $P(T)$ has at least $2n-2r$ distinct roots (by lemma 7) of a very particular form. Thus $2n-2r\leqq n$. Q.E.D.

REMARK 2. The proof of lemma 8 shows that if $x_j=0$, c and b_j not both 0, $P(T)$ has at least $2n-2r$ non-zero distinct roots of a very particular type.

LEMMA 9. *$x_j=0$ implies $b_j=0$ or $r_j\geqq n$.*

PROOF. Take $S_j(T, U, V)=\sum_{h\geq 0}T^hB_h(U, V)$. Let w be a root of $B_0(1, V)=0$. From $(\mathcal{E}_j)$, deriving with respect to T at the point $T=0$, $U=1$, $V=w$, we obtain

$$c_n=(\prod_{i\neq j}S_i(i-j, 1, w)(B_1(1, w):=(cw+b)B(w)B_1(1, w)$$

In the same way for $T=0$, $U=w$, $V=1$, we obtain

$$c_nw^n=(c+bw)B(w)B_1(1, w)$$

From this relation it follows either $c=0$ or $bw=bw^n$ for any w with $B_0(1, w)=0$. Assume $b\neq 0$. Then since $2r\geqq n$, we obtain $c_n=0$. Thus $B_1(1, V)=0$ since it has degree $r-1$ and r distinct roots. Let t be the largest integer n such that $c_t\neq 0$. If $t=0$, $P(T)=T^{n+1}$ and the proof of lemma 8 shows that $r\geqq n$ (in fact in this case we have $k=1$ and E is a direct sum of line bundles). Now assume

$t>0$. We have $c_s=0$, $B_s(U, V)=0$ for $s>t$ exactly as above. Deriving $(\mathcal{E}_j)$ with respect to T at $T=0$, $U=1$, $V=w$ and at $T=0$, $U=w$, $V=1$, we obtain $(c+bw)=w^t(cw+b)$ i.e. $(cx^{t+1}+bx^t-bx-c)$ has cyclotomic polynomials as divisor. Assume $c\neq 0$. Then by lemma 3 this implies $b=0$ or $b=\pm c$. Suppose $b=\pm c$; we have $w^t=1$ for every root w of $B(1, V)=0$. Since $2r\geqq n$, we have $t=r=[(n+1)/2]$ and $P(T)$ has 0 as a root of multiplicity at least $n/2+1$. Thus remark 2 gives the contradiction. If $c=0$, the proof is even simpler. Q. E. D.

LEMMA 10. *$x_i=x_j=0$ for $i\neq j$ implies $c=b_j=a=0$.*

PROOF. We may assume $b_i=b_j=0$ and thus $a=0$. Assume $c\neq 0$. Then $2r_j\geqq n$, $2r_i\geqq n$ implies $r_i+r_j\geqq n$, thus $k\leqq 3$ and if $k=3$, $r_h=1$. It is easy to prove, as in the proof below of lemma 1, that $P(T)$ has more than $n+1$ roots, contradiction. We use the relation $P(T)=P_i(T+j-i)$ and the fact, easily checked directly, that an equation $1\pm(-1+t)=\pm(-1+w)$ with t, w roots of unity has only a few solutions. Q. E. D.

Now we are ready for the proof of lemma 1. We may assume $n\geqq r$, $c\neq 0$, $b_j=0$ and in $(\mathcal{E}_j)$ $c_n\neq 0$ (see proof of lemma 9). We may assume $a\neq 0$ by the proof of lemma 10. Taking $U=1$, $V=w$ with $R(1, w)=0$, $S_j(0, 1, w)\neq 0$, we obtain r non-zero roots of $P(T)$ from the roots of $S_j(T, 1, w)$. Taking $U=1$, $V=w$ with $S_j(0, 1, w)=0$, from the equation $S_j(T, 1, w)=0$ we obtain $r-1$ non-zero roots of $P(T)$ because $c_n\neq 0$. Since $r(n-r)+r(r-1)=(n-1)r>n(r-1)$, there exists $h\neq 0$ with $S_j(h, 1, w)=0$ for at least r different w's with $R(1, w)=0$. Thus $P(T)$ has $r+1$ roots of the type h, $hw_1, \cdots, hw_r$: since $S_j(h, 1, w)=S_j(h, w, 1)$, if $S_j(h, 1, w)=R(1, w)=0$, hw^{-1} is a root of $P(T)$. Since $2n-2r+r+1>n$ for $n\geqq r$, we may assume that $P(T)$ has a set $A=\{d, dw_1, \cdots, dw_r\}$, w_i distinct $(n+1)$-th roots of unity $(w\neq 1)$, of $r+1$ roots, where $d=s(-1+v)$ or $d=-s(-1+v)$ for some v with $S_{s+j}(s, 1, v)=0$.

We distinguish 3 cases (the assertions follows from lemma 7):

1) if $v^6\neq 1$, from the roots of B of $P(T)$ given by lemma 8 at most $\pm s(-1+v)$, $\pm s(-1+v^{-1})$ are in A;
2) if $v^6=1$ but $v\neq -1$, then $B\cap A$ contains at most $\pm 2s$, $\pm s(-1+v)$, $\pm s(-1+v^{-1})$;
3) if $v=-1$, then $B\cap A$ contains at most $2s$, $\pm s_h(-1+v_h)$, $\pm s_h(-1+v_h^{-1})$, $h=1, 2$, where $v_1^3=1$, $v_2^3=-1$ and the s_h's are given by lemma 7.

In case 1) we have $2n-2r-4+r+1\leqq n$ i.e. $n\leqq r+3$. In case 2) we have $n\leqq r+5$ while in case 3) we have $n\leqq r+9$. Furthermore in case 2) if $n\geqq r+4$ we have $k\geqq 3$ and n odd; in case 3) n is odd and if $n\geqq r+2$ we have $k\geqq 3$, since for

$k=2$ only $2s$ can be in $A\cap B$ by lemma 7.

First we assume $k\geqq 3$. If for some index i, $r_i=1$, $S_i(T+i-j, 1, V)$ is of the form $T+H(1+V)$ or $T+dV$. In both cases it is easy to show that $P(T)$ has at least $\{-H(1+w)\}$ or $\{-dw\}$, with $R(1, w)=0$, as roots. By remark 2 this is impossible. Thus we assume $r_i\geqq 2$ for every i. By the first paragraph we may assume $b_j=0$, $c=-b_i$, $c=-(nb_h)/(n+2)$, $c=b_s$ (n odd) or $c=(b_s n)/(n+2)$ (n even) or $b_l=-2c$ ($n\equiv 1 \bmod 6$), $2b_l=-c$ ($n\equiv 3 \bmod 6$). If $n\equiv 1, 3 \bmod 6$, then 3 does not divide $n+1$ and case 2) do not occur; furthermore in case 3) we have necessarily $n\leqq r+1$ and thus $k=2$; in case 1) we have at most $k=3$, $r_h=r_l=2$, $r_j=n-3$: this case can be handle taking $U=1$, V roots of unity in the polynomials $S_i(T, U, V)$, $i\neq j$ (we know their constant part since only -1 and i give in this case cyclotomic polynomials of degree at most 2). But such a cumbersome calculation can be avoid with the following remark; if in the case above there is a cyclotomic polynomial of degree 2, then 4 devides $n+1$; if $c=b_s$ see below; if $c=-b_i$, $V^{n+1}+1$ has no factor of degree 2 and we win; otherwise there is a primitive solution of $(\mathcal{E}_i)$ for some index i since $c=-(nb_l)/(n+2)$ is impossible if $2b_l=-c$ or $2c=-b_l$, because $b_i=b_j+(i-j)a$; we use the last part of the first paragraph to conclude, in particular (3) gives the contradiction since there is an index i such that in $(\mathcal{E}_i)$, $c_1=\cdots=c_n=0$. If $c=b_k$ (n odd) we have $x_k=0$ and $c=0$ by lemma 10. Again if $c\neq b_k$ and n is odd, the contradiction comes from the last part of the first paragraph, where, for n odd, it is not necessary to use lemma 1, lemma 10 is sufficient. If n is even, we have necessarily $n=r+3$ by the discussion of 2) and 3). Since -1 is not a root of $R(1, V)$ for n even, in $(\mathcal{E}_j)$ there cannot be two factors of degree 2.

Thus we may assume $k=2$, $x_1=0$, $x_2\neq 0$.

Assume $n=r+3$. Then $S_2(1, 1, V)$ has a factor $(1+v)$ and a factor $(1+dV+V^2)$ with $d=0$ or $d=1$ or $d=-1$; the order of the root of unity is respectively 4, 6, 3. The factor $(1+V)$ implies that n is odd. $P(T)$ has as roots 0, the elements of A, ± 2. Thus $P(\pm 2t)\neq 0$ if $R(1, t)=0$, $t\neq -1$, and thus ± 2 is never a root of $S_r(T, 1, t)$ for such a t and it is at most a simple root of $S_r(T, 1, -1)$. We have $r(r-1)+r-2+r(n-r-1)=(n-2)r+r-2$. Thus there exists $z\in \boldsymbol{C}$, $z\neq 0$, ± 2, such that for at least $n-1$ roots of $R(1, V)$, $S_2(z, 1, t)=0$. As at the begining of the proof of the lemma, the elements of $A':=\{e, ew_1, \cdots, ew_{n-1}, \pm 2\}$ with w_i roots of unity, are roots of $P(T)$. This is easily seen impossible. Now assume $n=r+2$. $S_2(1, 1, V)$ has $(1+dV+V^2)$ as a factor, $d=0$, 1 or -1. Suppose n odd. Since $x^{n+1}+1$ has no factor of degree 1 or 3, we have necessarily $c=-(b_2 n)/(n+2)$ by the first paragraph, since for $r_2=3$, if $n\equiv 1, 3 \bmod 6$ and b_2 has an exceptional value, then $(\mathcal{E}_2)$ has a primitive solution. From $(\mathcal{E}_2)$ it follows that $D_2(x)=cx^{n+1}+(c+b_2)(x^n+\cdots+x)+c$ has at least 4 real roots (r is odd and $S_2(0, 1, V)$ has 3 real roots). This implies $c=0$ by [3, lemme III. 1.4]. Suppose n even and thus $d=1$. Then in A there is at most one of $(-1+v)$, $-(-1+v)$ and one of $-(-1+v^{-1})$ and $(-1+v^{-1})$ with $v^3=1$, since $R(1, -1)\neq 0$. Thus $n\leqq r+1$, absurd.

Assume $n=r+1$. Then $S(1, 1, V)=dV(1+V)$ and thus n is odd. For every $(n+1)$-root of unity w, we have $P(2w)=0$ since $2w$ is either ± 2 or conjugate to an element in A. $P(T)$ would have $n+2$ roots, absurd. Thus theorem 1 is proved.

References

[1] Elencwajg, G., Les fibrés uniformes de rang 3 sur $\boldsymbol{P}^2(\boldsymbol{C})$ sont homogènes, Math. Ann. **231** (1978), 217-227.

[2] Elencwajg, G.—Hirschowitz, A.—Schneider, M., Les fibrés uniformes de rang au plus n sur $\boldsymbol{P}^n(\boldsymbol{C})$ sont ceux qu'on croit, Proceedings of the Nice Conferences 1979 on Vector Bundles and Differential Equations. Birkhauser, Boston (1980).

[3] Ellia, Ph., Sur les fibres uniformes de rang $(n+1)$ sur $\boldsymbol{P}^n$, Memoires Soc. Math. France n. 7, 1982.

[4] Lang, S., Algebra, Addison-Wesley, Reading, Massachusetts (1971).

[5] Sato, E., Uniform vector bundles on a projective space, J. Math. Soc. Japan **28** (1976), 123-132.

Scuola Normale Superiore
56100 Pisa
ITALY