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LOCALIZATIONS WITH RESPECT TO NON-HEREDITARY TORSION THEORIES

By

Tamotsu IKEYAMA

As generalizations of quotient rings localizations of rings and modules with respect to hereditary torsion theories have been introduced and studied by many authors. Such localizations can be defined categorically and hence extended to ones with respect to arbitrary torsion theories (Cf. [4, Proposition 2.5, p. 37]). Few considerations, however, are made on the localizations of modules with respect to non-hereditary torsion theories. The main purpose of the present paper is to show the existence of the localization of a module with respect to a torsion theory, but not to any hereditary torsion theory by characterizing the localizations of modules with respect to some torsion theory.

Throughout this paper, unless otherwise stated, we consider in the category mod-R of unital right R-modules over a ring R with unit. $E(M)$ denotes the injective hull of a module M. For the definition and basic properties of torsion theories for an abelian category see [2].

Following [6], the localization of a module M (with respect to a torsion theory $(\mathcal{T}, \mathcal{F})$) is a homomorphism $f: M\rightarrow L$ with the properties that Ker (f) and Coker (f) are torsion, that L is torsionfree and that L is $\mathcal{T}$-injective; i.e., $\mathrm{Hom}_R(-, L)$ is exact on all exact sequences of the form $0\rightarrow X\rightarrow Y\rightarrow T\rightarrow 0$ with a torsion module T. Then it is known by [6] that such a localization of a module is unique up to isomorphism if it exists and that, for a torsion theory, every module has the localization if and only if the torsion theory is hereditary where a hereditary torsion theory means a torsion theory whose torsion class is closed under taking submodules. Hence the localization of a module with respect to a torsion theory does not exist in general. We show, however, that there exists the localization of a module with respect to a torsion theory but not to any hereditary torsion theory. To give such an example we need the following arguments. For $\mathcal{T}$-injective objects for a torsion theory $(\mathcal{T}, \mathcal{F})$ for an abelian category the following is known.

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LEMMA 1 ([1, p. 19]). *Let L be a subobject of an injective object E and $(\mathcal{T}, \mathcal{F})$ a torsion theory for an abelian category. Then L is $\mathcal{T}$-injective if and only if it is isomorphic to a direct summand of a subobject K of E such that E/K is torsionfree.*

PROOF. For the convenience of readers we restate the proof given in [1]. If L has the above indicated form, then it is obvious by diagram chase that K is $\mathcal{T}$-injective. Hence L is also $\mathcal{T}$-injective. Conversely, suppose that L is $\mathcal{T}$-injective. Let X denote the torsion subobject of E/L. Then we obtain the following canonical exact sequence

$$0 \longrightarrow L \longrightarrow K \longrightarrow X \longrightarrow 0$$

with some subobject K of E. Since the foregoing sequence splits, we have that L has the desired form.

Now, we consider the localization of a module M with respect to a torsion theory. It a homomorphism $f: M \to L$ is the localization of M, then so is the inclusion $\mathrm{Im}(f) \to L$ of $\mathrm{Im}(f)$. For the localization of a module given by the inclusion we have the following.

THEOREM 2. *Let $i: M \to L$ be the inclusion of modules. Then it is the localization of M with respect to some torsion theory if and only if there exist a module N and an injective module E with a monomorphism $f: L \oplus N \to E$ such that*

$$\mathrm{Hom}_R(\mathrm{Coker}(i) \oplus N,\ \mathrm{Coker}(f) \oplus L) = 0\,.$$

PROOF. Assume that i is the localization of M with respect to some torsion theory. Let E be a injective module containing L. Taking the same submodule K of E as the preceding lemma, $K = L \oplus N$ with some submodule N of E. Put f to be the inclusion $K \to E$. Then $\mathrm{Coker}(f)$ is torsionfree by the definition of K. Hence so is $\mathrm{Coker}(f) \oplus L$. On the other hand N is torsion, for so is $N \cong K/L$ which is torsion by the definition of K. Thus $\mathrm{Coker}(i) \oplus N$ is also torsion. Conversely, suppose that the modules M and L satisfy the above indicated conditions. Let $(\mathcal{T}_1, \mathcal{F}_1)$ be the torsion theory cogenerated by $\mathrm{Coker}(f) \oplus L$, that is

$$\mathcal{T}_1 = \{X \in \text{mod-}R \mid \mathrm{Hom}_R(X, \mathrm{Coker}(f) \oplus L) = 0\}\,.$$

Then it is easy to see that i is the localization of M with respect to $(\mathcal{T}_1, \mathcal{F}_1)$ by the preceding lemma.

REMARK. Let the inclusion $i: M \to L$ of modules satisfies the above indicated conditions and $(\mathcal{T}_0, \mathcal{F}_0)$ the torsion theory generated by $\mathrm{Coker}(i) \oplus N$, that is

$$\mathcal{T}_0=\{X\in \text{mod-}R \mid \mathrm{Hom}_R(\mathrm{Coker}\,(i)\oplus N,\ X)=0\}.$$

Then $\mathcal{T}_0 \leqq \mathcal{T}_1$ where $\mathcal{T}_1$ is the same as above. Put $(\mathcal{T}, \mathcal{F})$ to be a torsion theory such that $\mathcal{T}_0 \leqq \mathcal{T} \leqq \mathcal{T}_1$. Then i is the localization of M with respect to $(\mathcal{T}, \mathcal{F})$ as above.

It is easily seen that the preceding theorem with its proof is available for any abelian category with enough injectives. Hence its dual is also true for colocalizations in any abelian category with enough projectives. For the definition of colocalizations of modules and for the dual of Lemma 1 see [6] and [1, 2.9 Theorem, p. 17] respectively.

THEOREM 2*. *Let* $j: C \to M$ *be an epimorphism of modules. Then it is the colocalization of* M *with respect to some torsion theory if and only if there exist a module* N *and a projective module* P *with an epimorphism* $f: P \to C \oplus N$ *such that*

$$\mathrm{Hom}_R(\mathrm{Ker}\,(f)\oplus C,\ \mathrm{Ker}\,(j)\oplus N)=0\,.$$

COROLLARY 3. *Let* $i: M \to L$ *be the inclusion of modules. Then the following conditions are equivalent:*

(a) *i is the localization of M with respect to some torsion theory.*
(b) $\mathrm{Hom}_R(L/M,\ (E(L)/L)\oplus L)=0$.
(c) (1) $\mathrm{Hom}_R(L/M,\ L)=0$,
  (2) $\mathrm{Hom}_R(L/M,\ E(L)/L)=0$.
(d) (1) $\mathrm{Hom}_R(L/M,\ E(L))=0$,
  (2) $\mathrm{Hom}_R(L/M,\ E(L)/L)=0$.

Since the torsionfree class of a hereditary torsion theory is closed under taking injective hulls and the torsion theory cogenerated by an injective module is hereditary, we have the following.

COROLLARY 4 ([4, Corollary 1, p. 37] with [5, Footnote 1]). *Let* $i: M \to L$ *be the inclusion of modules. Then it is the localization of* M *with respect to some hereditary torsion theory if and only if the following conditions are satisfied:*

(1) $\mathrm{Hom}_R(L/M,\ E(L))=0$,
(2) $\mathrm{Hom}_R(L/M,\ E(E(L)/L))=0$.

For the independence of the conditions (1) and (2) in (c) and (d) in Corollary 3 as well as (1) and (2) in the preceding corollary see [3].

We give an example of the localization of a module with respect to a torsion theory, but not to any hereditary torsion theory.

EXAMPLE 5. Let K be a field,

$$R=\left\{\begin{pmatrix} a & K & K & K \\ 0 & K & K & K \\ 0 & 0 & a & K \\ 0 & 0 & 0 & K \end{pmatrix} \middle| a\in K\right\}$$

the subring of 4×4 upper triangular matrices over K and $E=(K\ K\ K\ K)$, $L=(0\ K\ K\ K)$, $X=(0\ 0\ K\ K)$ and $M=(0\ 0\ 0\ K)$ right R-modules by matrices operations. We denote the matrix units of R by e_{ij}. We first note that E is injective, since it is isomorphic to the K-dual of the projective left R-module ${}_R Re_{44}$. Thus it is obvious that $E(L)=E$ as right R-modules. We show that the inclusion $M\to L$ is the localization of M with respect to a torsion theory for mod-R, but not to any hereditary torsion theory for mod-R. According to the preceding corollaries, it is enough to show that the following three conditions are satisfied:

(i) $\mathrm{Hom}_R(L/M, E(L))=0$,

(ii) $\mathrm{Hom}_R(L/M, E(L)/L)=0$,

(iii) $\mathrm{Hom}_R(X/M, E(L)/L)\neq 0$.

We first show that the condition (i) is satisfied. Let $f: L/M\to E(L)$ and $f((\overline{0\ 1\ 0\ 0}))=(a\ b\ c\ d)$. Then

$$f((\overline{0\ 1\ 0\ 0}))=f((\overline{0\ 1\ 0\ 0})e_{22})=(a\ b\ c\ d)e_{22}=(0\ b\ 0\ 0)$$

implies $a=c=d=0$. Moreover, we have

$$0=f((\overline{0\ 1\ 0\ 0})e_{24})=(0\ b\ 0\ 0)e_{24}=(0\ 0\ 0\ b)\,,$$

for $(0\ 1\ 0\ 0)e_{24}\in M$. Thus $f=0$, since $L=(0\ 1\ 0\ 0)R$. Next, we show that the condition (ii) is satisfied. Let $f: L/M\to E(L)/L$ and $f((\overline{0\ 1\ 0\ 0}))=(\overline{a\ 0\ 0\ 0})$, Then

$$f((\overline{0\ 1\ 0\ 0}))=f((\overline{0\ 1\ 0\ 0})e_{22})=(\overline{a\ 0\ 0\ 0})e_{22}=0\,.$$

Thus $L=(0\ 1\ 0\ 0)R$ implies $f=0$. Put

$$f: X/M \longrightarrow E(L)/L((\overline{0\ 0\ a\ 0})\longmapsto(\overline{a\ 0\ 0\ 0}))\,.$$

Then f is a non-zero R-homomorphism by routine calculations. Thus the modules M and L satisfy the condition (iii).

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Institute of Mathematics
University of Tsukuba
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