TSUKUBA J. MATH.
Vol. 7 No. 1 (1983). 105-135

# PEAKLESS AND MONOTONE FUNCTIONS ON $G$-SPACES

By

H. BUSEMANN and B. B. PHADKE

## Introduction.

In recent years an extensive and significant theory of convex sets and functions on complete Riemann manifolds has been created. A good and up-to-date survey is found in Walter [14].

The present paper provides the foundations of an analogous theory for functions on non-Riemannian spaces, namely $G$-spaces[1], which include the smooth complete Finsler spaces. The principal difference (unrelated to the possible absence of smoothness) is that in many cases peaklessness (see below) which is weaker than convexity, proves the adequate concept for obtaining results corresponding to those in Riemann spaces. Our methods have the somewhat suprising effect that applied to the Riemannian case from which they originated they often yield stronger results than the original ones because a peakless function on a Riemann space need not be convex.

We gratefully emphasize that we sent the original version of this paper as a preprint to N. Innami. He not only discovered some inaccuracies, but strengthened our Theorems (22) and (25) materially and permitted us to include the results in the present paper.

We are going to introduce various types of peakless functions, one of which we call "nearly peakless". The concept coincides with the continuous "geodesically quasiconvex" functions ("geodesically" is often omitted) in the Riemannian case.

Although we dislike changing accepted terminology, the change is practically forced on us here both by semantic and mathematical reasons. Requiring that a function is nearly peakless, peakless or convex are increasingly stringent conditions, moreover peaklessness may be considered as a limiting case of convexity, see Section 1. Thus quasiconvex, if used at all, ought to be reserved for peakless functions. Also, a completely arbitrary monotone function is "quasiconvex" which

Received March 24, 1982.

1) The term $G$-space was introduced by the first author in 1944, but disregarded by others who (much later) used it with a different meaning often encountered in the literature.

is surely not related to the intuitive idea of convexity.

As this remark shows "quasiconvex" functions need not be continuous. Nevertheless we will only consider continuous functions, $f$, because the brevity gained by knowing that the sets like $[f<\alpha]$ or $[f\leqq\alpha]$ are open resp. closed without mentioning continuity every time seemed to outweigh the usually not important generality gained by omitting continuity. Furthermore, only continuous peakless functions can replace convex functions for many significant results in Riemannian spaces. At this early stage of a theory of functions on $G$-spaces we feel that questions of continuity are secondary to the principal problems; but where we give a proof valid without continuity we will mark the result by the subscript $i$, for example $(5)_i$.

Because the emphasis here is less on the individual theorems than on the theory as a whole we refrain from mentioning such theorems here, although we believe that the reader will find many of independent interest.

Monotone functions are special peakless functions, but they lead to entirely different problems, so the paper naturally divides into two parts.

An introduction does not usually end with an example. But in our case there is an outstanding one which may facilitate the understanding for a reader accustomed to Riemannian geometry.

Hilbert noticed, see [5], Section 18 ([5] will be quoted as $G$), that if in the definition of the Klein model of hyperbolic geometry the bounding ellipsoid is replaced by a closed strictly convex hypersurface $C$ and distance is defined as in the hyperbolic case, the open affine segments with endpoints on $C$ are the geodesics. Of course, the metric is smooth when $C$ is.

If $z(t)$ is a geodesic in terms of arclength and $p$ a point not on it then the distance $pz(t)$ is a strictly peakless function of $t$ which in contrast to the hyperbolic case need not be convex, see $G$ (18.12) p. 109 and [8]. Moreover, there are two ways of generalizing negative curvature to $G$-spaces ($G$ Sections 36 and 41) which coincide in the Riemannian case. Unless hyperbolic, a Hilbert geometry has negative curvature only in the weaker sense which is expressible in terms of peakless functions, whereas the stronger one requires convexity. We will come back to these facts later in greater detail.

## Part I. Peakless functions

### 1. The basic concepts.

Let $N$ be a connected set with more than one point of the real $t$-axis. A continuous real valued function $f(t)$ defined on $N$ is nearly peakless if $t_1<t_2<t_3$,

$t_i \in N$ implies

$$(*) \quad f(t_2) \leqq \max [f(t_1),\ f(t_3)] .$$

The function is peakless if it is nearly peakless and equality holds in (*) only if $f(t_1)=f(t_2)=f(t_3)$. This implies in contrast to near peaklessness that $f(t)=c$ on a proper interval only if $c=\min f(t)$.

The function $f(t)$ is strictly peakless when it is peakless and not constant on any proper interval. This means that $f(t)$ is either strictly monotone or takes its minimum at exactly one point $t_0$ and is strictly monotone for $t \geqq t_0$ and $t \leqq t_0$.

As pointed out in $G$ p. 109 peaklessness may be regarded as a degenerate form of convexity in this sense: If $f(t)$ is positive and convex then so is $f^{\alpha}(t)$, for $\alpha > 1$, so that requiring $f^{\alpha}(t)$ to be convex becomes weaker with increasing $\alpha$. For $\alpha \to \infty$ this leads to peaklessness. Because the proof in $G$ is not satisfactory we give a better one: The convexity of $f^{\alpha}(t)$ means that for $t_1 < t_3$ and $t_2 = (1-\theta)t_1 + \theta t_3$ $(0 < \theta < 1)$

$$f(t_2) \leqq [(1-\theta) f^{\alpha}(t_1) + \theta f^{\alpha}(t_3)]^{1/\alpha} \leqq \max [f(t_1),\ f(t_3)] ,$$

with equality in either case only if $f(t_1)=f(t_2)=f(t_3)$. On the other hand $\lim\limits_{\alpha \to \infty} [(1-\theta) f^{\alpha}(t_1) + \theta f^{\alpha}(t_3)]^{1/\alpha} = \max [f(t_1),\ f(t_3)]$.

Let $R$ be a $G$-space. For the basic definitions and facts we refer to Sections 6, 7, 8 in $G$. Here we merely repeat some notation. The space $R$ is metric with distance $xy$. A segment $T(x,\ y)$, $(x \neq y)$ is an isometric map of an interval $\beta \leqq t \leqq \beta + xy = \gamma$ of the real $t$-axis into $R$ i.e. $z(\beta)=x$, $z(\gamma)=y$ and $z(t_1)z(t_2) = |t_1 - t_2|$. A geodesic is a locally isometric map $z(t)$ of the entire $t$-axis into $R$, and $z(\eta t + \beta)$, $\eta = \pm 1$, $\beta$ real represents the same geodesic. The point set $\tilde{z}$ carrying $z(t)$ determines $z(t)$ up to the above transformation, see $G$ p. 39.

The restriction of a geodesic $z(t)$ to a closed interval is a geodesic curve (often called geodesic segment in the literature). This occurs so frequently that, where there is no chance of ambiguity, we will put $z(t)^{1,2} = z(t) | [t_1,\ t_2]$ and $z(t_i) = z_i$.

A globally isometric map of the $t$-axis into $R$ is a straight line. If all geodesics are straight lines then $R$ is called straight.

$(xyz)$ means that $x \neq y$, $y \neq z$ and $xy + yz = xz$. The open sphere $\{x \mid px < \rho\}$ is denoted by $S(p,\ \rho)$. Similarly we put $\{x \mid px = \rho\} = K(p,\ \rho)$ and $\{x \mid px \leqq \rho\} = B(p,\ \rho)$.

For functions $f(x)$ defined on $R$ we use, following Hausdorff, $[f < \alpha] = \{x \mid f(x) < \alpha\}$ when $\alpha > \inf f$ and define $[f = \alpha]$ and $[f \leqq \alpha]$ analogously when $\alpha \in \operatorname{range} f$.

For any set $M$ we denote its boundary by $\dot{M}$ and its closure by $M^c$ in particular $[f<\alpha]^{\cdot}$ and $[f<\alpha]^c$ are the boundary and closure of $[f<\alpha]$.

$M\neq\emptyset$ is convex if with any points $b\neq c$ it contains all $T(b, c)$. This deviates from the definition in $G$ p. 117 and also some others, but coincides with that of Soetens [13]. An open hemisphere shows that the closure of a convex set need not be convex. By general agreement $M$ is called totally convex if it contains a geodesic curve $z(t)^{1,2}$ whenever $z(t_i)\in M$.

We consider continuous functions $f(x)$ defined on $R$ with different degrees of peaklessness:

$f(x)$ is weakly peakless if $(bqc)$ implies

$$(**) \quad f(q)\leqq \max[f(b), f(c)],$$

or equivalently, if $foz(t)$ is nearly peakless on any segment $z(t)$, $t_1\leqq t\leqq t_2$.

$f(x)$ is nearly peakless, peakless, or strictly peakless if for any geodesic $\tilde{z}$ and one (and hence all) of its representations $foz(t)$ has the corresponding property.

In a straight space every weakly peakless function is nearly so. If we had omitted continuity, then for any strictly monotone $\varphi(t)$ on the $t$-axis the function $f(x)=f(x_1, x_2)=\varphi(x_1)$ would be a peakless function on the cartesian $(x_1, x_2)$-plane.

We also use the generally adopted term convex function when $foz(t)$ is convex for every geodesic $z(t)$.

In the Riemann and Finsler spaces convex functions are, according to Bangert [*'Analytische Eigenschaften konvexer Funktionen auf Riemannschen Mannigfaltigkeiten'*, J. für die Reine u. Angew. Math. 307/308 (1979) p. 307-319] always continuous. His proof cannot be applied to $G$-spaces, in particular because small spheres need not be convex (see *'Non-convex Spheres in G-spaces'* by H. Busemann and B.B. Phadke, J. Ind. Math. Soc. 44 (1979) p. 39-50) and normal co-ordinates do, in general, not exist. We were not able to prove continuity of convex functions in $G$-spaces. But a rather simple modification of classical arguments shows that a function which satisfies locally the condition $2f(y)\leqq f(x)+f(z)$ when $xy=yz=xz/2$ is either continuous and convex or not bounded in any nonempty open set.

## 2. Consequences of the definitions.

As far as we know weakly peakless functions have not been studied (did not even get a name) in the literature, but they will prove important here. We begin with a nearly obvious remark.

$(1)_i$ *$f(x)$ is weakly peakless (always on the $G$-space $R$) iff each $[f\leqq\alpha]$ is* *convex.*

For if $f$ is weakly peakless then any point $q\neq b$, $c$ on a segment $T(b, c)$ satisfies $(bqc)$. Therefore $b, c\in[f\leqq\alpha]$ and $f(q)\leqq\max[f(b), f(c)]\leqq\alpha$ and $T(b, c)\subset[f\leqq\alpha]$.

Conversely let the $[f\leqq\alpha]$ be convex, then $q$ with $(bqc)$ lies on some (actually exactly one) $T(b, c)$. Put $\alpha=\max[f(b), f(c)]$. Then $T(b, c)\subset[f\leqq\alpha]$ implies $q\in[f\leqq\alpha]$ i.e. $f(q)\leqq\alpha$.

$(2)_i$ *$f(x)$ is nearly peakless iff all $[f\leqq\alpha]$ are totally convex.*

The proof is practically the same as for (1), besides (2) is well known[2].

For the sake of completeness we prove but will not use

(3) *If $f(x)$ is weakly peakless then each $[f<\alpha]$ is a regular open set, i.e. it is the interior of its closure.*

If $[f<\alpha]$ is not regular then there would be a $S(p, \sigma)\subset[f<\alpha]^{\cdot}$ with $p\notin[f<\alpha]$; $\sigma<\rho(p)$. There are points $q\in[f<\alpha]\frown S(p, \sigma)$ and since $[f<\alpha]$ is open there is $S(q, \tau)\subset[f<\alpha]\frown S(p, \sigma)$. For $x\in S(q, \tau)$ let $D_x$ be the diameter of $\bar{S}(p, \sigma)$ through $x$ and $E_x$ that radius on $D_x$ which does not contain $x$. Then $E_x\frown[f<\alpha]=\emptyset$ because $f$ is weakly peakless. But $\bigcup_{x\in S(q,\tau)} E_x$ contains an open set; which would mean that $[f<\alpha]$ is not dense in $[f<\alpha]^c$. This proves (3).

(4) *Let $f(x)$ be nearly peakless. It is peakless iff $[f\leqq\alpha]=[f<\alpha]^c$. The condition is equivalent to $[f<\alpha]^{\cdot}=[f=\alpha]$.*

The equivalence is obvious because $[f<\alpha]^c=[f<\alpha]\smile[f<\alpha]^{\cdot}$, $[f\leqq\alpha]=[f<\alpha]\smile[f=\alpha]$ and the summands in either union are disjoint.

Let $f$ be peakless. $[f\leqq\alpha]\supset[f<\alpha]^c$ is obvious. Assume $u\in[f\leqq\alpha]/[f<\alpha]^c$ exists and let $f(w)<\alpha$. Coming from $w$ on a segment $T(u, w)$ there is a first point $v$ with $f(v)=\alpha$ and $v\neq u$ because $v\in[f<\alpha]^c$. But $(uvw)$ and $f(w)<f(u)=f(v)$ contradicts peaklessness.

Conversely let $f$ be nearly peakless and that $[f\leqq\alpha]=[f<\alpha]^c$ for each $\alpha>\inf f$. Assume for $t_1<t_2<t_3$, $f(z(t_2))=\alpha=\max(f(z(t_1)), f(z(t_3))=f(z(t_1))>f(z(t_3))=\beta$. Then since $f(x)$ is nearly peakless, $f(z(t))=\alpha$ for $t_1\leqq t\leqq t_2$. Thus there is a largest $t_0$ between $t_1$ and $t_3$ such that $f(z(t))=\alpha$ for $t_1<t\leqq t_0<t_3$. Then we

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2) "Wellknown" without comments means "known in Riemann spaces with a simple proof valid under our hypothesis".

have points $a_i=z(t_i)$, $i=1, 2, 3$, such that $(a_1a_2a_3)$, $f(a_1)=f(a_2)=\alpha$ and $f(a_3)=\beta<\alpha$. Put $Q=[f<\alpha]^c$. Then $a_1\in Q$ and $a_3\in$the interior of $Q$. By the result in [13], p. 204, $a_2\in$the interior of $Q$ i.e. $f(a_2)<\alpha$, a contradiction. Hence $f(x)$ is peakless.

$(5)_i$ *If* $f(x)$ *is nearly peakless on* $R$ *then* $f(x)$ *is constant on each closed geodesic* $z(t)$.

This is wellknown and an immediate consequence of the periodicity of $z(t)$.

$(6)_i$ COROLLARY. *If a strictly peakless function* $f$ *exists on* $R$ *then* $R$ *has no closed geodesics, hence is simply connected if compact and* $f$ *is not continuous.*

The second part holds because in a compact space any nontrivial free homotopy class contains a closed geodesic, see $G(32.2)$. It follows from (11) that $f$ cannot be continuous.

For nearly peakless functions we have

$(7)_i$ *If for all* $p\neq q$ *a nearly peakless function* $f$ *exists on* $R$ *with* $f(p)>f(q)$ *then all geodesics are simple and none is closed. Hence* $R$ *is simply connected.*

For assume $z(t_1)=z(t_2)$ with $t_1<t_2$ and choose $t$ with $t_1<t<t_2$ and $z(t)\neq z(t_i)$, $i=1, 2$. By hypothesis there is a nearly peakless function $f$ with $foz(t)>foz(t_i)$ which contradicts the near peaklessness of $f$.

If $R$ were not simply connected then every point would be the vertex of a geodesic monogon (possibly a closed geodesic) not homotopic to 0, see $G(28.2)$.

Thus the only 2-dimensional spaces satisfying $(7)_i$ topologically are the plane and the sphere. Whether the latter actually occurs is an open question, see end of this section.

$(8)_i$ *If* $\dim R=2$ *and* $f(x)$ *is weakly peakless, moreover constant on two non-collinear segments* $T_1$, $T_2$ *with the same midoint* $p$, *then* $f(x)$ *is constant on a (non-degenerate) geodesic triangle with one vertex at* $p$.

Let $T_i=T(q_i, q_i')$, where we assume $q_ip<\rho(p)/2$, (for the definition of $\rho(p)$ see $G$ p. 33). Let $f(x)=\alpha$ on $T_1\cup T_2$. It follows from weak peaklessness that $f(x)\leqq\alpha$ in the triangle $pq_1q_2$. If $f(x)=\alpha$ in $pq_1q_2$ then there is nothing to prove. Assume that $f(x)<\alpha$ at some point $u$ of $pq_1q_2$. Then $u\notin T_1\cup T_2$. Because $q_ip<\rho(p)/2$ the segments $T(u, w)$; $w\in T(p, q_2')$ are unique and intersect $T(p, q_1)$ for sufficiently small $sp$ and $w\in T(p, s)$ in points $v$ because $\dim R=2$. Since

$f(v)=f(w)=\alpha$ and $(uvw)$ we have $f(x)=\alpha$ on $T(v, w)$. If $T(u, q_1)\cap T(p, q_1)=r$ then $f(x)=\alpha$ in the triangle $prs$. See Figure 1.

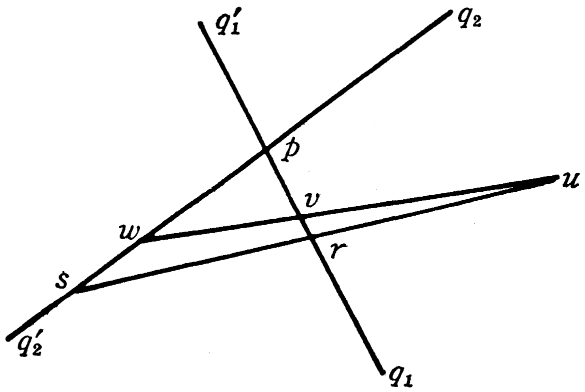


Figure 1

A function $f$ is called locally nonconstant if no nonempty open set exists on which $f$ is constant. These functions play an important rôle in the work of Innami discussed below. A corollary of (5) and (8) is

$(9)_i$ *If* $\dim R=2$ *and* $R$ *admits a locally nonconstant nearly peakless function then no two closed geodesics intersect.*

$(10)_i$ *A peakless function does not attain a maximum unless it is constant. Therefore no strictly peakless function on any G-space attains a maximum.*

For if $f(q)=\alpha$ is its maximum then for $(bqc)$ with $bq, qc<\rho(q)$ we have $\alpha=f(q)\leqq\max(f(b), f(c))\leqq\alpha$ so that the equalities hold, and, by hypothesis $f(q)=f(b)=f(c)=\alpha$. This means that the nonempty set $[f=\alpha]$ contains $S(q, \rho(q))$ whenever it contains $q$. Therefore $[f=\alpha]$ is open. If $x_n\to x$, put $\rho(x)=3\varepsilon>0$ and choose $n$ so large that $x_n x<\varepsilon$. Then $\rho(x_n)>2\varepsilon>x_n x$, see $G$ p. 33. Thus $x\in S(x_n, \rho(x_n))$ and by the above argument, $f(x)=\alpha$ when $f(x_n)=\alpha$. Hence $[f=\alpha]$ is also a closed set. As $R$ is connected this means that $R=[f=\alpha]$ i.e. $f(x)$ is constant.

(11) COROLLARY. *On a compact space every peakless function is constant.*

The assertion is false for weakly peakless functions as the following example shows: On the ordinary sphere $x_1^2+x_2^2+x_3^2=1$ in the cartesian $(x_1, x_2, x_3)$-space define $f(x_1, x_2, x_3)=0$ if $x_3\geqq0$ and $f(x_1, x_2, x_3)=x_3$ if $x_3\leqq0$. This example also shows explicitly, because of (5), that a weakly peakless function is, in general, not nearly peakless, which is implicitly contained in (1) and (2).

The following weaker form of (11) holds for weakly peakless functions:

(12) *On a compact space a weakly peakless function cannot be locally nonconstant.*

This is an immediate consequence of

(13) *If $f(x)$ is weakly peakless and attains its maximum $\alpha$ then $[f=\alpha]$ is the closure of its nonempty interior which is regular.*

For if $f(q)=\alpha$ and if $(bqc)$ with $b$ and $c$ as diametrically opposite points in $S(q, \rho)$, $\rho<\rho(p)$ then either $f(b)=\alpha$ or $f(c)=\alpha$ as $f(x)$ is weakly peakless. Similarly if for $x\in T(b, q)$, $f(x)<\alpha$ then for all $y\in T(q, c)$, $f(y)=\alpha$. Thus in the sphere $S(q, \rho)$ on any diameter $T(b, c)$ at least one radial segment $T(b, q)$ or $T(q, c)$ is such that $f(x)=\alpha$ on the whole of that radial segment. Hence $S(q, \rho)$ contains an $S(u, \sigma)\subset[f=\alpha]$. Thus the interior points of $[f=\alpha]$ are dense in $[f=\alpha]$ which is therefore the closure of its nonempty interior. As $[f=\alpha]$ is closed this means that the set of interior points of $[f=\alpha]$ is a regular open set.

For compact $G$-surfaces $R$ Innami proved that a locally nonconstant, not necessarily continuous, nearly peakless function does not exist. From this and (9) he draws (without continuity) the surprising conclusion that $R$ is homeomorphic to the plane, cylinder or Moebius Strip, so that this weak hypothesis has the same topological implications as for Riemannian surfaces of curvature 0. Innami shows with more complicated arguments involving peakless functions and a condition on their growth, that the level sets of the function in question have at most two components. We owe this information to a preprint which Innami sent us and some correspondence arising from it. Innami's paper entitled "A classification of Busemann $G$-surfaces which possess convex functions" will appear in Acta Math. 148 (1982).

The statement (10) is false for nearly peakless functions. Define $f(t)=0$ for $t\leqq 0$, $f(t)=t$ for $0\leqq t\leqq 1$, $f(t)=1$ for $t\geqq 1$, and in a Cartesian $x=(x_1, x_2, x_3)$ space consider the cylinder $C$: $x_3=f(x_1)$ with generators parallel to the $x_2$-axis. $C$ is with its intrinsic metric isometric to the euclidean plane. The function $f(x)=x_3$ on $C$ is nearly peakless but attains its maximum.

This example exhibits a general feature of nearly peakless functions which attain a maximum $\alpha_m$. If $z(t)^{1,2}$ is a geodesic curve with $f(z(t_1))<\alpha_m$ and $f(z(t_2))=\alpha_m$ then $z(t)\in[f=\alpha_m]$ for $t\geqq t_2$. If not then $t_3>t_2$ with $foz(t_3)\neq\alpha_m$, hence

$<\alpha_m$, would exist and $f(x)$ would not be nearly peakless.

We do not know whether (11) holds for nearly peakless functions. The answer is probably affirmative because of the preceding remark and because it holds in Riemann spaces by a theorem of Bangert [2], quoted as [2.7] in Walter [14].

## 3. Surfaces without nonconstant nearly peakless functions.

Some very nice examples for the nonexistence of nonconstant nearly peakless functions can be obtained for surfaces of revolution in $E^3$.

But we begin with a general observation.

(14) *If $f(x)$ is nearly peakless on $R$ and the geodesic $z(t)$ is dense in a set $M$ which is the closure of a nonempty open set then $f(x)$ is constant on $M$.*

If $f o z(t)$ is constant on $z(t)$ then trivially $f(x)$ is constant on $M$ so we show that for an $f$ which is not constant on $M$ no geodesic $z(t)$ can be dense in $M$. Because $g(t)=f o z(t)$ is nearly peakless the limits $\Lambda^+=\lim_{t\to\infty} g(t)$ and $\Lambda^-=\lim_{t\to-\infty} g(t)$ exist if we admit $\pm\infty$. Since $g(t)$ is not constant in $M$ there is an interior point $p=z(t_0)$ of $M$ for which $f(p)\neq\Lambda^\pm$ and hence

$$\min\{|f(p)-\Lambda^+|,\ |f(p)-\Lambda^-|\}\geqq 2\eta>0\,.$$

Therefore $t_1<t_0<t_2$ exist with $|g(t)-f(p)|\geqq\eta$ for $t\leqq t_1$ and $t\geqq t_2$. Choose $\rho>0$ such that $|f(p)-f(x)|<\eta$ for $px<\rho$. Then $z(t)\frown S(p,\rho)=\emptyset$ for $t\leqq t_1$ and $t\geqq t_2$ and $z(t)^{1,2}$ is not dense in $M\frown S(p,\rho)$.

Many compact $G$-surfaces are known to carry dense, even transitive geodesics, $G$ Theorem (34.14), so that any nearly peakless function is constant on them. In view of our remark at the end of last section this is probably true for all compact $G$-spaces.

However there are many noncompact surfaces with transitive geodesics. For a particularly attractive example, because of its connection with continued fractions, see E. Artin [1]. Here and in other examples the curvature has a constant sign. We prove that a nearly peakless function must be constant on many surfaces of revolution in $E^3$ where the curvature does change sign. The discussion is based on the following facts found in Darboux [9], Section 582.

Let $S$ be a surface of revolution about the $x_3$-axis in Cartesian $x=(x_1, x_2, x_3)$-space, whose meridian in the $(x_1, x_3)$-plane is $x_1=g(x_3)>0$, $-\infty<x_3<\infty$, where $g(x_3)$ is smooth (class $C^3$ suffices). A zone on $S$ is the closed part of $S$ bounded by two distinct parallels $x_3=c_i$.

A special zone $Z$, $x_3 \in [\sigma_1, \sigma_2]$ is defined by the properties that a $\tau_1$ with $\sigma_1 < \tau_1 < \sigma_2$ exists with $g'(x_3) > 0$ in $[\sigma_1, \tau_1)$, $g'(x_3) < 0$ in $(\tau_1, \sigma_2]$ and $g(\sigma_1) = g(\sigma_2)$.

The parallel circle $C: x_3 = \tau_1$ is a geodesic because $g'(\tau_1) = 0$, but no other parallel circle in $Z$ is since $g'(x_3) \neq 0$.

The principal result on special zones in Darboux is that a geodesic tangent to one $x_3 = \sigma_i$ stays in $Z$ and meanders from $x_3 = \sigma_1$ to $x_3 = \sigma_2$ and back. Each of these geodesics is either closed or dense in $Z$. Because *all geodesics in $Z$ are obtained from one by rotation about the $x_3$-axis, $C$ intersects all of them.*

If the geodesics are closed then a nearly peakless function $f(x)$ defined on $S$ is constant on each of the closed geodesics and also on $C$ by (5). Hence $f(x)$ is constant on $Z$. In the case of a dense geodesic $f(x)$ is constant by (14). Thus

(15) *A nearly peakless function $f(x)$ defined on a surface of revolution in $E^3$ is constant on any special zone.*

Among various applications we discuss two:

(I) *The case whose prototype is* $g(x_3) = \exp(-x_3^2)$.

The corresponding general case is: $g(x_3)$ has exactly one maximum, $c$, say at $x_3 = 0$ and $g'(x_3) < 0$ for $x_3 > 0$, $g'(x_3) > 0$ for $x_3 < 0$ and $\lim_{x_3 \to \infty} g(x_3) = \lim_{x_3 \to -\infty} g(x_3) = b \geqq 0$.

For any value $a$ with $b < a < c$ there is exactly one special zone $Z_a$ bounded by the two parallel circles with radius $a$, so that a nearly peakless $f(x)$ on the surface $S$ is constant in each $Z_a$ and hence on $S = \bigcup_{a > b} Z_a$.

(II) *The case whose prototype is* $g(x_3) = \sin x_3 + 2$.

The general case is this: $g(x_3)$ attains its minimum $m$ at a discrete set $x_3 = \sigma_\nu$, $\nu \in I$ with $\sigma_\nu < \sigma_{\nu+1}$, $\sigma_{|\nu|} \to \infty$ for $|\nu| \to \infty$, so that $g(\sigma_\nu) = m$. In each $[\sigma_\nu, \sigma_{\nu+1}]$, $g(x_3)$ attains its maximum $m_\nu$ at exactly one $\tau_\nu$, $\sigma_\nu < \tau_\nu < \sigma_{\nu+1}$ (note that the maxima are arbitrary, they need not be equal) and $g'(x_3) \neq 0$ if $x_3 \neq \sigma_\nu, \tau_\nu$. Then $[\sigma_\nu, \sigma_{\nu+1}]$ does not define a special zone because $g'(\sigma_\nu) = 0$. Therefore we choose $\xi_\nu^n$ and $\eta_\nu^n$ with $\sigma_\nu < \xi_\nu^n < \tau_\nu < \eta_\nu^n < \sigma_{\nu+1}$ such that $g(\xi_\nu^n) = g(\eta_\nu^n)$ and $\xi_\nu^n - \sigma_\nu < n^{-1}$, $\sigma_{\nu+1} - \eta_\nu^n < n^{-1}$.

Then $[\xi_\nu^n, \eta_\nu^n]$ defines a special zone so that $f(x)$ is constant on it and by continuity on $[\sigma_\nu, \sigma_{\nu+1}]$, hence on $S$ since $\bigcup [\sigma_\nu, \sigma_{\nu+1}]$ is the $x_3$-axis.

Thus

(16) *A nearly peakless $f(x)$ defined on a surface in $E^3$ of type (I) or type*

*(II) is constant.*

We note also:

(17) *There are surfaces of revolution in $E^3$ homeomorphic to $\boldsymbol{R}^2$ on which no peakless function exists.*

For Walter [14], p. 6 constructs a rather simple surface $S$ of revolution homeomorphic to $E^2$ on which there exists a maximal compact locally convex connected proper subset $M$ of $S$. If $f(x)$ were a peakless function on $S$ then $S=\bigcup[f\leqq\alpha]$. For $\beta=\max_{x\in M} f(x)$ we have $[f\leqq\alpha]\supset M$ when $\alpha\geqq\beta$. Since $M$ is maximal $[f\leqq\alpha]=M$ for $\alpha\geqq\beta$ or $\beta=\max f(x)$. But a peakless function does not have a maximum.

## 4. The Functions $px$ and $Tx$

We now consider special functions, the simplest of which are the functions $px$, where $p$ is fixed for each function.

(18) THEOREM. *If the functions $px$ are weakly peakless then the space has finite dimension, is straight with strictly convex spheres, or equivalently, the $px$ are strictly peakless.*

This theorem weakens materially the hypothesis of the important Theorem (20.9) in $G$ which contains that in a straight space convexity of the spheres is equivalent to strict peaklessness of the $px$. The improvement was made possible by two deeper theorems not known at the time $G$ was written.

To prove (18) we show first that segments are unique. Assume two segments $T(a, a')$ exist. Then $\rho(a)<aa'$. On the two segments take the points $b$, $c$ with $ab=ac=\rho(a)/2$. If $(bqc)$ then $qa<\rho(a)$ hence $p$ with $(qap)$ exists. See Figure 2. Then

$$pb<ab+ap\,,\quad pc<ac+ap\,,\quad qp=qa+ap\,.$$

$$pq\leqq\max(pb,\ pc),\ \text{so}\ qa=pq-pa\leqq\max(pb-ba,\ pc-ca)<ab=ac\,.$$

Also $qa'\leqq\max(a'b,\ a'c)=a'b=a'c$; hence $aa'\leqq aq+qa'<ab+a'b=aa'$; which is false.

The weak peaklessness of the $px$ now means that the $S(p, \rho)$ are convex. According to Berestovskiĭ [3] this makes the space finite dimensional and this together with the uniqueness of segments yields that the space is straight; see [6], p. 19, (12.12, c).

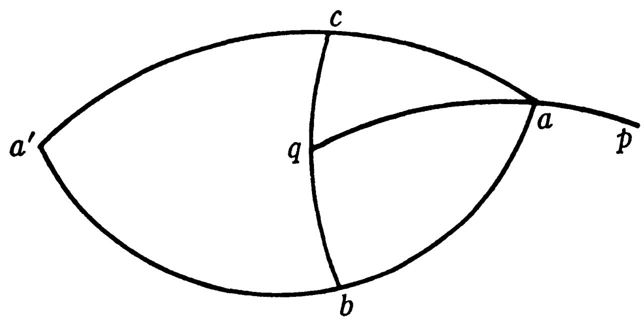


Figure 2

This shows that the weak peaklessness of $pz(t)$ for all points $p$ and all lines $z(t)$ of a straight space implies their strict peaklessness.

If the line is kept fixed and $p$ varies it is shown in [8] that the convexity of the $px(t)$ for $p \notin L$ implies their strict convexity. This has an analogue for peakless functions:

(19) *If for a fixed line $L=z(t)$ in a straight space the functions $pz(t)$ are peakless then they are strictly peakless.*

The case $p \in L$ is no exception here because for $p=z(t_0)$ the function $pz(t) = |t-t_0|$ is strictly peakless (but not strictly convex).

Let $p \notin L$ and assume that $pz(t)$ is not strictly peakless. Then $pz(t)$ is constant in an interval or $t_1<t_2<t_3$ exist such that the $pz(t_i)$ are equal. If $(qpz(t_2))$ then we find as above that $qz(t_2)>qz(t_1)$, $qz(t_2)>qz(t_3)$, thus $qz(t)$ is not even nearly peakless.

The functions $pz(t)$ provide a good example where convexity in Riemann spaces must be replaced by peaklessness in general spaces. There are Hilbert geometries, which in the Riemannian case are hyperbolic, where the $pz(t)$ are peakless, but not convex, in contrast to the hyperbolic case, see $G$ p. 109 and [8].

The next simplest case of distance functions are the $Tx$, where $T$ is a segment.

Assume that a positive continuous function $\lambda(p)$ exists on $R$ such that $Tx$ is a weakly peakless function whenever $T \subset S(p, \lambda(p))$ for suitable $p$, i.e. $Tq \leqq \max(Tb, Tc)$ when $(bqc)$.

Let $T$ be a segment with $p$ as one end point and $u \in T$ with up $<\lambda(p)$. Then for $(bqc)$,

$$T(u, p)q \leqq \max [T(u, p)b, T(u, p)c] .$$

For $u \to p$ we obtain $pq \leqq \max(pb, pc)$ so that the hypothesis of (14) is satisfied and the space is straight with strictly convex spheres. For the following see $G$

sections 36 and 41.

The capsule $C_T$ in a $G$-space $R$ is the set $\{x \mid Tx \leqq \alpha\}$. By definition $R$ has (strictly) convex capsules if every point has a neighborhood $U_p$ such that the capsules in $U_p$ are (strictly) convex. The capsules are globally (strictly) convex if every capsule is (strictly) convex. In the present context the global convexity would have been perferable as a definition of convex capsules, but the restriction to $U_p$ has an important differential geometric reason. A Riemann space has nonpositive curvature ($K \leqq 0$) iff the capsules are convex. $K<0$ implies their strict convexity, but the converse need not hold everywhere.

The (strict) convexity of the capsules is equivalent to the existence of a positive continuous function $\eta(p)$ defined on $R$ such that for any two segments $T(a, b)$ and $T$ in $S(p, \eta(p))$ the function $Tx$ is weakly peakless for $x \in T(a, b)$, (strictly peakless unless $T \cap T(a, b)$ is a proper segment).

In a straight space the (strict) convexity of the capsules implies their global (strict) convexity or, equivalently, that the functions $Tx$ are (for $T \not\subset \mathfrak{z}$ strictly) peakless. Collecting our results we have:

(20) THEOREM. *If a positive continuous function $\lambda(p)$ exists on $R$ such that $Tx$ is weakly peakless for any $T$ contained in some $S(p, \lambda(p))$ then* $\dim R < \infty$, *the space is straight and the capsules are globally convex. The function $Tx$ is peakless. When $T$ and $\mathfrak{z}$ have at most one point in common $Tz(t)$ is strictly peakless when the capsules are strictly convex.*

In $G$-spaces there is a stronger condition than the convexity of the capsules which is called nonpositive curvature in $G$ and also coincides in the Riemannian case with nonpositive curvature.

The stronger condition implies that the functions $Tx$ are convex, $G$(36.12).

In Hilbert Geometry the $Tx$ are in general only peakless, $G$ pp. 110-111 and [8], in fact the hyperbolic geometry is the only Hilbert Geometry which has nonpositive curvature in the stronger sense, see Kelly and Straus [11].

## 5. The distance loci $C^{\alpha}(Q)$.

In this section we study the distance loci $C^{\alpha}(Q)$ which are defined for a closed set $Q$ and $\alpha>0$ by $C^{\alpha}(Q)=\{x \mid xQ \leqq \alpha\}$.

(21) *If, in a simply connected space the capsules are convex then $C^{\alpha}(Q)$ is convex for any $\alpha>0$ and any closed convex set $Q$; the functions $Qx$ are peakless. If $Q$ is strictly convex then so is $C^{\alpha}(Q)$ and $Qz(t)$ is strictly peakless when $\mathfrak{z}$*

*intersects Q in at most one point.*

Clearly the relevant case to which the others can be reduced is $T(a, b) \subset C^{\alpha}(Q)$ when $a, b \in C^{\alpha}(Q)/Q$. Because the spheres are convex the feet $c, d$ of $a, b$ on $Q$ are unique and $T = T(c, d) \subset Q$. Since $ac \leqq \alpha$, $bd \leqq \alpha$ the segment $T(a, b) \subset C^{\alpha}(T) \subset C^{\alpha}(Q)$. For $(apb)$ let $q$ be the foot of $p$ on $T$. If $Q$ is strictly convex then $q \neq c, d$ is impossible because $T(p, q)$ intersects $\dot{Q}$ in a point $r$ with $(prq)$ and $pQ \leqq pr < pq$ so that $q$ is not a foot of $p$ on $Q$. Thus $q = c$ or $q = d$, i.e. $c = d$. But $B(c, \alpha)$ is strictly convex, hence $p \in S(c, \alpha) \subset C^{\alpha}(Q)$. Hence $C^{\alpha}(Q)$ is strictly convex. Thus the function $Qx$ is nearly peakless for convex $Q$, see $G$ section 36.

Let $t_1 < t_2 < t_3$ and $\alpha = \max(Qz(t_1), Qz(t_3)) = Qz(t_1)$. If $Qz(t_2) = Qz(t_1)$ then the convexity of $C^{\alpha}(Q)$ implies $Qz(t) \geqq \alpha$ for $t \geqq t_2$ by $G$ (20.1) (applied to $z(t)$ with $t > t_2$ and $z(t_1)$). Therefore $Qz(t_3) = \alpha$ and $Qz$ is peakless.

If $Q$ and hence the $C^{\alpha}(Q)$ are strictly convex and $\tilde{z}$ intersects $Q$ in one point then $Qz(t)$ is strictly peakless because it is peakless and attains its minimum 0 at one point only. If $\tilde{z} \cap Q = \emptyset$, but $Qz(t)$ attains its minimum $\alpha$ at $t_0$ say then $\tilde{z}$ touches $C^{\alpha}(Q)$ at $z(t_0)$ only because $C^{\alpha}(Q)$ is strictly convex. Finally, when $Qz(t)$ does not attain a minimum then as a peakless function it is strictly peakless.

Let $R$ be straight and $M$ $(\neq R)$ a closed subset of $R$. A ray $\tilde{r}: r(t)$, $t \geqq 0$, with origin $r(0) \in \dot{M}$ is called perpendicular to $\dot{M}$ at $r(0)$ if for every $t$ the point $r(0)$ is a, and hence by $G$(20.6) the only, foot of $r(t)$ on $\dot{M}$.

(22) THEOREM. *Let $Q$ $(\neq R)$ be a closed convex set in a straight space with convex spheres. Then at each point $p \in \dot{Q}$ a ray perpendicular to $\dot{Q}$ exists. $R$ is the union of $Q$ and these rays and every point not in $Q$ lies on exactly one of these rays.*

This version is one of two essential improvements by Innami mentioned in the introduction. We had only discussed the case where $R$ has convex capsules.

PROOF. Let $y \in R - Q$ and $f$ the (by $G$(24.12)) unique foot of $y$ on $Q$. For $(yxf)$ the foot of $x$ is $f$. We must show that this also holds for $(fyx)$. Denote the line through $y$ and $f$ by $L$ and let $V$ be the (closed) set formed by all transversals to $L$ at $f$. By $G$(20.11) the set $V$ decomposes $R$ into two arcwise connected sets $U$ and $W$. Let $y \in U$. We show that $Q \cap U = \emptyset$. Assume $a \in Q \cap U$ exists. Since the line through $a$ and $f$ is not a supporting line of $B(y, yf)$ the segment $T(a, f) \subset Q$ must enter $S(y, yf)$. Hence for some $b$ with $(abf)$ in $Q$ $yb < yf$ which is a contradiction to $f$ being the foot of $y$ on $Q$.

Thus $Q \subset V \cup W$. For $Q \subset L$ the assertion that $f$ is the foot of $x$ is obvious.

When $w \in Q - L \cap Q$, then $T(w, x)$ intersects $V$ in some point $w' \neq f$ and $xw \geqq xw' \geqq xf$. Thus $f$ is the foot of $x$ and the ray $\tilde{r}$ from $f$ through $y$ is perpendicular to $\dot{Q}$ at $f$.

If $\tilde{r}_v$ is perpendicular to $\dot{Q}$ at $z'_v$ and $z'_v \to z$, then any converging subsequence of $\{\tilde{r}_v\}$ tends to a ray $\tilde{r}$ perpendicular to $\dot{Q}$ at $z$. Let $z \in \dot{Q}$ be given and $z_v \notin Q$ with $z_v \to z$. If $z'_v$ is the foot of $z_v$ on $\dot{Q}$, then $z'_v z \leqq z z_v \to 0$. Since rays perpendicular to $\dot{Q}$ at $z'_v$ exist, there is a ray perpendicular to $\dot{Q}$ at $z$.

Finally, since every point $x \notin Q$ has a foot on $\dot{Q}$, $R - Q$ is covered by rays perpendicular to $\dot{Q}$ and every point of $R - Q$ lies on exactly one perpendicular since it has only one foot on $Q$. This completes the proof of (22).

In the Riemannian case this result is found in parts (1), (2) and (3) of (3.4) in [4]. The proof in [4] which is much shorter uses a typically Riemannian fact (in addition to nonpositive curvature which is stronger than convex capsules in Finsler spaces) namely that perpendicularity of line elements corresponds to their forming an angle of measure $\pi/2$. This implies, of course, that perpendicularity is symmetric which is not the case in Finsler spaces of dimension $>2$ unless they are Riemannian and only for a few very special cases when $n=2$ see $G$ pp. 84, 103, 104.

(23) THEOREM. *If $M$ is a closed set, $\dot{C}^{\alpha}(M) = \{Mx = \alpha\}$, $\alpha > 0$, and $C^{\beta}(M)$ is convex for $\beta \leqq \alpha$, moreover $z(t)$ is a geodesic with $Mz(t_0) = \max_t Mz(t) = \alpha$ then $Mz(t) \equiv \alpha$*

First, $t_1$, $t_2$ with $t_1 < t_0 < t_2$, $|t_i - t_0| < \rho(z(t_0))/2 = \sigma$ and $Mz(t_i) < \alpha$ cannot exist. For if $\beta = Mz(t_2) \geqq Mz(t_1)$ then $T(z_1, z_2)$ has $z_i \in C^{\beta}(M)$, $(z_i = z(t_i))$ but does not lie in $C^{\beta}(M)$.

If follows that $Mz_1 = \beta < \alpha$ implies $z(t) = \alpha$ for $t_0 \leqq t \leqq t_0 + \sigma$. But $Mz_1 = \beta < \alpha$ for $t_0 - t_1 \leqq \sigma$ is also impossible because $S(z_1, \alpha - \beta) \subset C^{\alpha}(M)$ implies that $z_1$ is an interior point of $C^{\alpha}(M)$. By hypothesis $z(t_0) \in \dot{C}^{\alpha}(M)$ whereas by Soetens [13] $z(t_0)$ would have to be an interior point of $C^{\alpha}(M)$. Thus $Mz(t) = \alpha$ in $[t_0 - \sigma, t_0 + \sigma]$.

We can apply the same argument to $z(t_0 \pm \sigma)$ replacing $\sigma$ by $\sigma' = \rho(z(t_0 \pm \sigma))/2$. Repetition of this argument yields the assertion.

The following observation is simple and important and has almost certainly been made by others for Riemann spaces but we did not see it anywhere in print.

(24) *If $\bar{R}$ is a covering space of $R$ related to $R$ by a locally isometric map $\Omega$ of $\bar{R}$ onto $R$ and $M$ is a totally convex set in $R$, then $\Omega^{-1}M = \bar{M}$ is totally convex in $\bar{R}$.*

NOTE. The total convexity of $M$ is essential. For convex $M$ the set $\Omega^{-1}M$ will in general not be connected and Soetens [13] has pointed out various unexpected phenomena which can occur in $\Omega^{-1}M$.

The proof of (24) is trivial: If $\bar{z}(t)^{1,2}$ is a geodesic curve in $\bar{R}$ with $\bar{z}_i \in \overline{M}$, then $z(t)=\Omega\bar{z}(t)$ is a geodesic curve in $R$ from $z_1=\Omega\bar{z}_1$ to $z_2=\Omega\bar{z}_2$. By hypothesis $z(t)^{1,2}\subset M$ where $\bar{z}(t)^{1,2}\subset\overline{M}$.

Whereas the previous theorems, especially (22), contain the first 3 parts of [4], (3.2) the fourth (and last) part (as proved) depends on non-trivial analysis and the Riemannian metric. In fact, simple (smooth) examples show that part 4 does not have a complete analogue under our weak conditions. But the most important implication of part 4 is a simple consequence of (22) and (24).

(25) THEOREM. *Let $M$ be a closed totally convex set in a space $R$ with convex capsules. Let $p_1$, $p_2$ ($p_1=p_2$ admitted) be points of $\dot{M}$ ($M\neq R$) and $z_i(t)$ distinct halfgeodesics, $i=1, 2$, $t\geqq 0$, $z_i(0)=p_i$ such that for some $\beta>0$ the point $z_i(0)$ is the foot of $z_i(t)$ on $\dot{M}$ for $t\leqq\beta$. Then $z_1(t_1)\neq z_2(t_2)$ for any $t_i>0$. $R$ is not compact.*

*In particular, $M$ is a simple Tchebyshev set.*

A closed set in a $G$-space is a Tchebyshev set when each point has a unique foot on it and is simple when in addition the segment from a point $p\notin M$ on $M$ is unique. It is clear that the last statement in (25) follows from the case $p_1=p_2$.

Originally we had proved only $z_1(t)\neq z_2(t)$ for $t>0$. The stronger statement is the second improvement due to Innami who bases his argument on the following interesting

LEMMA. *Let $M$ be closed set in the $G$-space $R$ with $\bar{R}$ as universal covering space and $\Omega$ as locally isometric map of $\bar{R}$ on $R$. Define $f(x)=xM$ on $R$ and $\bar{f}(\bar{x})=\bar{x}\overline{M}$ where $\overline{M}=\Omega^{-1}M$ on $\bar{R}$. Then $\bar{f}=f\circ\Omega$.*

For, let $\bar{x}\in\bar{R}$ and $z\in R$ be the foot of $\Omega\bar{x}=x$ on $M$, moreover $T(\bar{x}, \bar{z})$ the segment over $T(x, z)$. We can assume $x\notin M$ then $\bar{x}\notin\overline{M}$. We have $xz=xM$, $\bar{x}\bar{z}=xz$, $\bar{x}\overline{M}=xM$. If the latter were false the $\bar{u}\in\overline{M}$ with $\bar{x}\bar{u}<xM=xz$ would exist, hence $xu<xz$, $\bar{x}\bar{u}<xz$ and $xu\leqq\bar{x}\bar{u}<xz$, so that $z$ would not be the foot of $x$. Thus $\bar{f}(\bar{x})=f\circ\Omega\bar{x}=zx=\bar{z}\bar{x}\geqq\bar{f}(\bar{x})$, which proves the Lemma.

It has various applications. The one needed for (25) is where $M$ is a closed totally convex set in a space $R$ ($\neq M$) with convex capsules. Since $xM$ is peakless on $R$, the space is by (11) non-compact. Under the hypothesis of (25) it follows from the lemma and $z(t)M=t$ for $t\leqq\beta$ that $z(t)M=t$ for all $t$ for if $\bar{z}(t)$

lies over $z(t)$ then (22) yields $\bar{z}(t)\bar{M}=t$ for all $t$ when $\bar{M}=\Omega^{-1}M$.

We mention that a space with convex capsules is straight when some point $q$ exists for which $\{q\}$ is totally convex. This follows from the simple observation that any $G$-space in which such a $q$ exists is simply connected, because, if not, then a non-trivial homotopy class based on $q$ contains a geodesic monogon (which may be a closed geodesic), so that $\{q\}$ is not totally convex. Therefore the space is straight when its universal covering space is straight.

(26) *If $R$ has convex capsules and $\tilde{z}$ is a closed geodesic not contained in (and hence disjoint from) the closed totally convex set $M\neq\phi$ then $\tilde{z}$ lies entirely in some $[Mx=\alpha]$.*

*Moreover $\tilde{z}$ is freely homotopic to a closed geodesic in $M$ and the homotopy can be accomplished by gliding $\tilde{z}$ along a cylinder consisting of segments $T$ of length $\alpha$ connecting points of $\tilde{z}$ to their feet on $M$, and for each $0<\beta<\alpha$ the points on these $T$ with distance $\beta$ from $M$ form a closed geodesic.*

*If the capsules are strictly convex then all closed geodesics lie in $M$.*

The first part follows at once from (23). The sets $[Mx\leqq\alpha]$ are totally convex and contain a closed geodesic with one of its points. (4) yields $\dot{C}^{\alpha}(M)=\{Mx=\alpha\}$. Therefore, if $\max_t Mz(t)=\alpha$ then $Mz(t)\equiv\alpha$ by (23).

To prove the assertion on homotopy we pass to the universal covering space $\bar{R}$ of $R$ which is straight (see $G$(38.2)). This theorem assumes domain invariance which holds because the space has a finite dimension by Berestovskiĭ [3] and hence domain invariance by [6], p. 16. The function $\bar{M}\bar{x}$ is peakless.

Let $z(t_0)$ be a simple point of $\tilde{z}$ (see $G$(9.4)) and $f_{t_0}$ be a foot of $z(t_0)$ on $M$, moveover $T$ a segment from $z(t_0)$ to $f_{t_0}$. If $\bar{f}_{t_0}$ lies over $f_{t_0}$ then a unique segment $\bar{T}$ over $T$ beginning at $\bar{f}_{t_0}$ and ending at a point $\bar{z}_0$ over $z(t_0)$ exists and $\bar{T}$ also has length $\alpha$. Because $z(t_0)$ is a simple point of $\tilde{z}$, there is exactly one geodesic $\bar{z}(t)$ over $z(t)$ with $\bar{z}(t_0)=\bar{z}_0$, see $G$ p. 169. $\bar{z}(t)$ is a straight line, let $\bar{f}_t$ be the foot of $\bar{z}(t)$ on $\bar{M}$. Since $z(t)$ is periodic and $\bar{M}\bar{z}(t)$ is peakless we have $\bar{z}(t)\bar{f}_t\equiv\alpha$, which could also have been seen from the first part of the proof.

If $t_1<t_2<t_3$ and $\bar{z}_i$ has the foot $\bar{f}_{t_i}=f_i$ on $M$, $0<\beta<\alpha$ and $(\bar{z}_i\bar{u}_i\bar{f}_i)$ with $\bar{u}_i\bar{f}_i=\beta$ then we want to prove that $(\bar{u}_1\bar{u}_2\bar{u}_3)$. If $\bar{v}$ is the foot of $\bar{u}_2$ on $T(\bar{u}_1, \bar{u}_3)$ and the geodesic $\bar{y}(t)$ carries $T(\bar{u}_1, \bar{u}_3)$ then the peaklessness of $\bar{y}(t)M$ and $\bar{y}(t)\bar{v}$ yield

$$\alpha=\bar{z}_i\bar{f}_i,\ \bar{z}_1\bar{u}_1=\bar{z}_3\bar{u}_3\geqq\bar{z}_2\bar{v}\,,\quad \beta=\bar{u}_1\bar{f}_1=\bar{u}_3\bar{f}_3\geqq\bar{v}\bar{f}_2 \quad \text{and}$$

$$\alpha=\bar{z}_2\bar{f}_2\leqq\bar{z}_2\bar{v}+\bar{v}\bar{f}_2\leqq\bar{z}_1\bar{u}_1+\bar{u}_1\bar{f}_1=\alpha\,.$$

This implies $vf_2=\beta$. Therefore the points $\bar{u}(t)$ with $\bar{u}(t)\bar{f}_t=\beta$ and $(\bar{z}(t)\bar{u}(t)\bar{f}_t)$

form a straight line.

Translating this back into $R$ we find the assertion on homotopy.

The constancy of $\bar{z}(t)\bar{f}_t$ is incompatible with strictly convex capsules which yields the last remark in (26).

This has the following corollary noted in [4] (for convex $f$).

(27) *If $R$ has strictly convex capsules and admits a peakless function $f(x)$, then no closed geodesics exist when $f$ does not have a minimum. If $f$ has a minimum $\beta$ then all closed geodesics lie in $[f=\beta]$.*

By the last remark in (26) each closed geodesic must be in $[f\leqq\alpha]$. If $f$ has no minimum then $\bigcap_{\alpha\in \text{range of } f}[f\leqq\alpha]=\emptyset$. If $\min f=\beta$ then $[f=\beta]=[f\leqq\beta]$ is totally convex and contains all closed geodesics.

We stop here following [4] because the preceding results, in particular (27), elucidate clearly our contention that convexity is often too strong a condition and that peaklessness suffices.

Without major difficulties many more results of [4] can be established for $G$-spaces with nonpositive curvature; see $G$ Section 36, but this does hot belong here because convexity enters instead of peaklessness. Most of the results are nontrivial and the question, whether or to which extent convex capsules would suffice, is often difficult to decide.

But some of the results were largely known to E. Cartan, (for example [14], (7.1) which is ascribed to [4]) and are merely disguised by recent terminology.

$f(x)$ defined on $R$ is an exhaustion function or exhaustive when all $[f\leqq\alpha]\neq\emptyset$ are compact. [14], (7.1) states that on any Hadamard manifold all $px$ are exhaustive and convex, that the $h(\tilde{r}, x)$ (see next section) are convex but neither the $h(\tilde{r}, x)$ nor the $-h(\tilde{r}, x)$ are exhaustive.

Now the $px$ are exhaustive in any $G$-space (and according to Hopf-Rinow in any complete Finsler space, see [6], Section 1). In any straight space (hence any simply connected complete Finsler space without conjugate points) the $h(\tilde{r}, x)$ and $-h(\tilde{r}, x)$ are nonexhaustive, see $G$(23.1, 2). That (on an Hadamard manifold) the $px$ are convex was proved by Cartan and without smoothness in $G$(36.7). This leads readily to the convexity of the $h(\tilde{r}, x)$, see (29) below.

## 6. The Functions $h(\tilde{r}, x)$.

The function $h(\tilde{r}, x)$ where $\tilde{r}$ is a ray (i.e. $r(t)$ is defined for $t\geqq k$ and $r(k)r(t)=t-k$) by the always existing (see $G$ p. 131)

$$h(\tilde{r}, x)=\lim_{t\to\infty}(xr(t)-t)$$

$|h(\tilde{r}, x)-h(\tilde{r}, y)|\leqq xy$ so that $h(\tilde{r}, x)$ is continuous in $x$.

If $\tilde{r}'$ is a subray, $t\geqq t_0$, of $\tilde{r}$ then

$$h(\tilde{r}', x)=\lim(xr(t_0)-t)=\lim(xr(t)-t-t_0)=h(\tilde{r}, x)+t_0\,.$$

These functions play a considerable rôle in $G$-spaces and lately also in Riemann spaces, where they are known as Busemann functions. The definition of $h(\tilde{r}, x)$ presupposes that a ray exists, which is the case whenever the space is not bounded. In general there are few essentially different functions $h$. Taking the ordinary cylinder as an example the rays are parallel to one of the orientations of the axis (or a generator) and for parallel rays the functions differ only by a constant, so that there are, up to constants, only two distinct functions $h$. There is only one if we take a semicylinder and cap it by a hemisphere.

To assure the existence of sufficiently many $h(\tilde{r}, x)$ we assume that the space is straight. In comparison to monotone $h(\tilde{r}, x)$, see Part II 4, the requirement that the $h(\tilde{r}, x)$ are weakly peakless is relatively weak.

(28) *In a straight space the $h(\tilde{r}, x)$ are weakly peakless iff the spheres are convex or the $px$ are strictly peakless. Also $h(\tilde{r}, x)$ are peakless if they are weakly peakless.*

For a proof denote the balls $S(p, \rho)\cup K(p, \rho)$ by $B(p, \rho)$. Similarly the limit sphere $K_\infty(\tilde{r}, p)$ with central ray $\tilde{r}$ through $p$ is given by $h(\tilde{r}, x)=\alpha=h(\tilde{r}, p)$. Its interior is by definition $h(\tilde{r}, x)<\alpha$ (See $G$ Section 22). The limit sphere and its interior form the limit ball $B_\infty(\tilde{r}, p)$, i.e. the set $[h(\tilde{r})\leqq\alpha]$ in our previous notation.

If the $S(p, \rho)$, hence the $K(p, \rho)$ and $B(p, \rho)$ are convex then the $K_\infty(p, x)$ or $h(\tilde{r}, x)=\alpha$ are convex as limits of spheres and so are the limit balls $[h(\tilde{r})\leqq\alpha]$; hence $h(\tilde{r}, x)$ is a weakly peakless function by (1).

Conversely let $h(\tilde{r}, x)$ be a weakly peakless function, then the $B_\infty(\tilde{r}, p)$ are convex. Consider $K(p, \rho)$ and denote by $\tilde{r}_x$ the ray from $x\in K(p, \rho)$ through $p$. From $G$ pp. 132, 133 we know that $B_\infty(\tilde{r}, x)$ contains $B(p, \rho)$ in its interior except for $x$. Therefore

$$B(p, \rho)=\bigcap_{x\in K(p,\rho)} B_\infty(\tilde{r}_x, x)\,,$$

and $B(p, \rho)$ is convex as intersection of convex sets.

We note that the interior of $B_\infty(\tilde{r}, p)$ is $[h(\tilde{r})<h(\tilde{r}, p)]$. Hence from (4) we see that $h(\tilde{r}, x)$ are peakless if they are weakly peakless. This completes

the proof of (28).

If $px$ is convex then

$$2(r(t)b-t)\leqq r(t)a-t+r(t)c-t \quad \text{hence}$$

$$2h(\tilde{r}, b)\leqq h(\tilde{r}, a)+h(\tilde{r}, c) \quad \text{where } ab=bc=ac/2.$$

Because $h(\tilde{r}, z(t))$ is continuous this implies that it is convex. Thus:

(29) *If $px$ are convex then $h(\tilde{r}, x)$ are convex.*

As an application (well known in Riemann spaces) we have:

(30) *If $R$ is simply connected and has nonpositive curvature in the sense of $G$ Section 36 then the functions $h(\tilde{r}, x)$ are convex.*

We also have:

(31) *If in a straight space $R$ the $h(\tilde{r}, x)$ are convex then the $px$ are convex.*

For given $(bqc)$ with $q$ as midpoint of $b$ and $c$ and given $p$ let $r(t)$ be the ray emanating from $q$ and containing $p$. We note, see [8, p. 26], that

$$[2r(t)q-r(t)b--r(t)c]=[2(r(t)q-t)-(r(t)b-t)-(r(t)c-t)]$$

increases with $t$. Hence $2pq-pb-pc\leqq 2h(\tilde{r}, q)-h(\tilde{r}, b)-h(\tilde{r}, c)\leqq 0$. Therefore $px$ is convex.

In [8] we proved that $px$ is convex if $px^{\alpha}$ is convex for some $\alpha>1$.

## 7. A special peakless function.

The principal result of this section rests on two lemmas which we discuss first.

(32) *If in a straight space for a fixed line $L$ the function $xL$ is weakly peakless, then a line parallel to $L$ is equidistant from $L$.*

We remind the reader of the concept "parallel". If $z(t)$ represents $L$ and $q\notin L$ denote by $A_t$ the line through $q$ and $z(t)$. For $t\to\infty$ $A_t$ tends to a line $A^+$ and for $t\to-\infty$ to a line $A^-$, the asymptotes to $L$ through $q$, $G$ p. 138. If $A^+$ and $A^-$ fall on the same line $P$, then $P$ is the parallel to $L$ through $q$ (and any other point of $P$, l.c.).

Let $a_t(\tau)$ represent $A_t$ such that $a_t(0)=q$. Since $xL$ is weakly, hence nearly peakless $a_t(\tau)L$ is nonincreasing for $\tau\geqq 0$. When $t\to\pm\infty$ then $a_t(\tau)$ tends for $t\to\pm\infty$ to the two opposite orientations of $P$. Therefore, if $u(t)$ represents $P$, then $u(t)L$ is both nondecreasing and nonincreasing, hence constant.

The Parallel Axiom ($G$ p. 141) requires that $A^+=A^-$ for any $q$ and $L$ and in addition that $L$ be parallel to $P$ (which is not always the case, l.c.). The above arguments then yield readily:

(33) *If $xL$ is weakly peakless for all lines $L$ in a straight space and the Parallel Axiom holds, then parallels $P_1$, $P_2$ are equidistant from each other. If $v_i \in P_i$ is a foot of $v_j \in P_j$ on $P_i$ then $v_j$ is a foot of $v_i$ on $P_j$.*

We note that since $h(\tilde{r}, r(t))=-t$ no $h(\tilde{r}, x)$ is constant and that we observed before that the conditions $(C)$ and $(B)$ of our previous paper [8] are satisfied if the functions $h(\tilde{r}, x)$ are respectively peakless and convex. We call a function $f(x)$ defined on $R$ linear if $f(x)$ is not constant on $R$ and $f \circ z(t)$ is linear for each geodesic $z(t)$. The condition $(D)$ of [8] is equivalent to the linearity of $h(\tilde{r}, x)$. In [8] we have (for straight spaces) a further condition $(A)$ related to peaklessness which lies between $(D)$ and $(B)$ i.e. $D \rightarrow A \rightarrow B \rightarrow C$.

This condition $(A)$ requires that the functions $xL$ have a very special form: If $L$ is a line and if $z(t)$ is a representation of another line such that $z(o) \in L$ then $z(t)L=\lambda_1(\tilde{z})t$ for $t \geqq 0$ and $z(t)L=\lambda_2(\tilde{z})|t|$ for $t \leqq 0$.

$z(t)L$ is then a very special strictly peakless (and also convex) function for $\tilde{z} \neq L$. We proved in [8] that this condition characterizes the Minkowskian geometries.

We want to show here that the special form can be replaced by the condition that $R$ satisfies the parallel axiom and that $z(t)L$ is convex.

(34) THEOREM. *If in a straight space satisfying the parallel axiom, the functions $xL$ are convex for any line $L$ then the space is Minkowskian. The converse is also true.*

In a Minkowskian space parallel lines make equal intercepts on a pair of parallel lines so that the converse is easy. To prove the main statement let the parallel axiom hold and assume that the functions $xL$ are convex in a straight space $R$.

Let $L$ and $M$ be lines meeting at $z$. Then by convexity of $xL$, the functions $xL$ are increasing on each ray of $M$ starting at $z$. This implies, since the parallel axiom holds, that parallel lines are equidistant from each other, see (33).

Now consider points $a'$, $a$ on $M$ with $(a'za)$ and $a'z=za$. Let $L'$ be the parallel to $L$ through $a'$. Let $f$ be a foot of $a$ on $L$ and $h$ be a foot of $f$ on $L'$. Let $m$ be a foot of $z$ on $L'$ and $f'$ be a foot of $a'$ on $L$. See figure 3.

By convexity of $xL'$, $2zm \leqq aL' \leqq af+fh$. Since $L$ and $L'$ are equidistant, $zm=fh$. Thus $zm \leqq af$. Also $a'f'=zm$. Hence $a'f' \leqq af$. Similarly we can show that $af \leqq a'f'$. Therefore $a'f'=af$.

Now $ah \geqq aL' \geqq 2zm=zm+zm=af+fh \geqq ah$, thus $(afh)$.

Similarly if we draw the parallel $L''$ to $L$ through $a$ and $u$ is a foot of $b$ on $L''$ ($b$ is a point on $M$ such that $az=ab$), $g$ is a foot of $u$ on $L$ and $k$ is a foot of $g$ on $L'$ then we have $(bug)$ and $(ugk)$ and $bu=ug=gk$.

It follows, in particular therefore, that $af=ug=1/2.bg$; so the condition $(A)$ is satisfied. From [8], Theorem 2 the space $R$ is now Minkowskian. This completes the proof of (34).

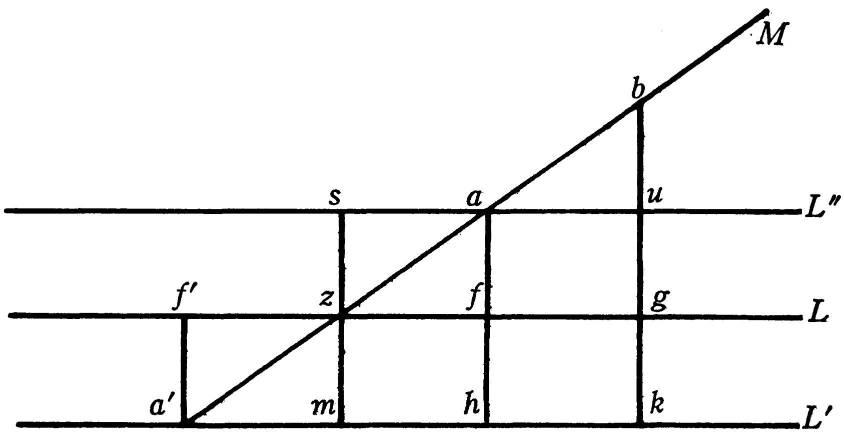


Figure 3

## Part II. Monotone functions

### 1. Basic properties

The continuous function $f(x)$ defined on $R$ is weakly (strictly) monotone if $foz(t)$ is (strictly) monotone on each geodesic $z(t)$. We will see presently that the strictly monotone functions are uninteresting.

A weakly monotone function is nearly peakless. From our previous results (4) and (2) we conclude that such a function is constant on each closed geodesic and that the sets $[f \leqq \alpha]$ are totally convex. But we can say more:

(35) *$f(x)$ is weakly monotone iff the $[f \leqq \alpha]$ and $[f=\alpha]$ or the $[f \geqq \alpha]$ and the $[f=\alpha]$ are totally convex.*

For if $f$ is weakly monotone, then $t_1<t_2$ and $f(z(t_i))=\alpha$; $i=1, 2$; implies $foz(t)=\alpha$ in $[t_1, t_2]$. The other parts of (35) follow from the observation that $-f$ is monotone with $f$.

Conversely assume $[f=\alpha]$ and $[f\leqq\alpha]$ to be totally convex. Then $f(x)$ is nearly peakless by (2). If $f$ were not weakly monotone then a geodesic $z(t)$ would exist on which $foz(t)$ is not monotone. Since it is nearly peakless this means that it must attain its minimum at some $t_1$ or in a finite interval $[t_1, t_2]$. Then $t'<t_1<t''$ resp. $t'<t_1<t_2<t''$ exist with $foz(t')>f(t_1)$, $foz(t'')>f(t_1)$ and $foz(t')=foz(t'')=\alpha$. But then $[f=\alpha]$ would not be totally convex.

The function constructed to show that (11) is false for nearly peakless functions (where several steps could have been constructed instead of one) illustrates (35) well.

For a strictly monotone function no $[f=\alpha]$ can contain a proper segment, so that each $[f=\alpha]$ consists of a point. Since a strictly monotone function is strictly peakless it does not attain a maximum by (10) and hecause $-f$ is also strictly monotone it does not attain a minimum. Therefore each $[f=\alpha]$ separates the space, hence dim $R=1$. $R$ is a straight line or a great circle, but the latter is impossible, so that a strictly monotone function is nothing but an ordinary strictly monotone function on the $t$-axis.

The most interesting class of monotone functions with which we will deal exclusively and for which we reserve the term monotone is this:

$f(x)$ is monotone on $R$ if on each geodesic $z(t)$ either $foz(t)=$constant or $foz(t)$ is strictly monotone.

To avoid trivialities we also require that $f(x)$ is not constant on all of $R$.

Because a monotone function is peakless we conclude from (10):

(36) *On a compact space no monotone function exists.*

A flat subset of a $G$-space $R$ is a set which with the metric of $R$ is a $G$-space. Here we encounter flats satisfying a much stronger condition.

A flat subset $S$ of $R$ is totally flat or a total flat if any geodesic curve $z(t)|[t_1, t_2]$, $t_1\neq t_2$, for which $z(t_1)$ and $z(t_2)$ lie in $S$ lies entirely in $S$.

This implies among other properties that $S$ contains a closed geodesic with one of its points and a geodesic with one of its multiple points. But notice that in straight spaces flats are always total.

The ordinary cylinder $\xi_1^2+\xi_2^2=r^2$ in a Cartesian $(\xi_1, \xi_2, \xi_3)$-space provides a good example. Its flat subspaces are the generators and circles but the generators are not totally flat, whereas the circles are. They are the level sets $f(\xi)=\alpha$ of the linear function $f(\xi)=\xi_3$. (A linear function is monotone.)

(37) *A continuous function on a $G$-space is monotone iff the sets $[f=\alpha]\neq\emptyset$ are totally flat.*

For if $f$ is monotone and $foz(t_1)=foz(t_2)=\alpha$ with $t_1\neq t_2$, then $foz(t)$ is not strictly monotone, hence constant so that $\tilde{z}\subset[f=\alpha]$.

Assume the $[f=\alpha]$ are totally flat. Any geodesic $z(t)$ for which $foz(t)$ is not constant intersects an $[f=\alpha]$ at most once and $foz(t)$ is therefore strictly monotone.

$G$-spaces and hence flats in $G$-spaces have the same dimension at all points. We notice next

(38) *If $f(x)$ is a monotone function on $R$ and* $\dim R=n$ *then each* $[f=\alpha]\neq\emptyset$ *has dimension* $n-1$.

Put $m=\dim[f=\alpha]$. Because $f$ is not constant there is a point $r\notin[f=\alpha]$. For any point $p\in[f=\alpha]$ the segment $T(r, p)$ and the geodesic containing it intersect $[f=\alpha]$ only in $p$. Choose $q$ with $(qup)$ and $qp\leqq\rho(p)/4$. Then $M=[f=\alpha]\cap\bar{S}(p, \rho(p)/4)$ is compact and so is $N=\bigcup_{x\in M}T(q, x)$. Moreover the $T(q, x)$ have only $q$ in common. A theorem of Hurewicz [10] yields $\dim N=m+1$, so that $\dim[f=\alpha]\leqq n-1$.

Because $qp\leqq\rho(p)/4$ we have $qx\leqq\rho(p)/2$ for $x\in M$ and $\rho(q)\geqq\rho(p)-pq\geqq 3\rho(p)/4$. Therefore if $u$ is the midpoint of $q$ and $p$ and $U$ is a sufficienty small neighborhood of $u$, every segment of length $\rho(q)$ from $q$ meeting $\bar{U}$ intersects $M$ in some point $y$. The union $N'$ of these $T(q, y)$ has dimension $n$ since $N'\supset U$. On the other hand $N\supset N'$ so $\dim[f=\alpha]=n-1$.

The method of this proof gives more generally

(39) *If the total flat $S_1$ is a proper subset of the flat $S_2$ wish* $\dim S_2<\infty$ *then* $\dim S_1<\dim S_2$.

We give some examples. An $n$-dimensional Desarguesian space is a $G$-space defined on a nonempty open set of the real projective space $\boldsymbol{P}^n$ whose geodesics fall on projective lines. The space is either all of $\boldsymbol{P}^n$ or a convex subset of the affine space $A^n$, see [6], p. 37. Since $\boldsymbol{P}^n$ is compact, it does not carry a monotone function, but every $R\subset A^n$ does. In fact if $f(x)=\sum_{i=1}^{n}k_i x_i+\lambda_i$ is any affine function in $A^n$ (the $x_i$ are affine cöordinates) which is not constant, then the restriction of $f(x)$ to $R$ is monotone, but in general not linear. If $n$ points $a_1, a_2, \cdots, a_n$ are given then an $f$ and an $\alpha$ exist such that $a_i\in[f=\alpha]$, $i=1, 2, \cdots, n$.

For any strictly monotone function $h(u)$, $u\in(-\infty, \infty)$, of one variable $h\circ L(x)$ will be monotone when $L(x)$ is linear, so that the range of a monotone function can be any open interval.

Individual linear functions for which an arbitrary $G$-space $R'$ appears as a $[f=\alpha]$ are easily constructed. As $R$ we take the product of $R'$ with the real $t$-axis so that the points of $R$ have the form $(p, t)$, $p\in R'$. The distance is defined by

$$(p_1, t_1)(p_2, t_2)=[p_1p_2^\alpha+|t_1-t_2|^\alpha]^{1/\alpha}, \qquad \alpha>1.$$

The discussion in $G$, pp. 42, 43 shows that $f(p, t)=t$ is linear and each $[f=\alpha]$ is congruent to $R'$.

## 2. Desarguesian spaces.

Assume that $f$ is monotone on $R$, $\alpha<\beta$ and $[f=\alpha]\neq\emptyset$, $[f=\beta]\neq\emptyset$, $\alpha<\gamma<\beta$. Let $z(t)|[t_1, t_2]$ be a geodesic curve connecting a point of $[f=\alpha]$ to a point of $[f=\beta]$. It intersects $[f=\gamma]$ in exactly one point $z(\bar{t})$ with $t_1<\bar{t}<t_2$ and

$$\frac{foz(t_2)-foz(\bar{t})}{foz(t_2)-foz(t_1)}=\frac{\beta-\gamma}{\beta-\alpha}.$$

If $f$ is linear and $foz(t)$ is not constant, say $foz(t)=kt+\lambda$, $k\neq0$, then no $[f=\alpha]=\emptyset$, Moreover if $t_2>t_1$, then

$$foz(t_2)-foz(t_1)=k(t_2-t_1)=k\lambda_{t_1}^{t_2}z(t)$$

where $\lambda_{t_1}^{t_2}$ is the length of $z(t)^{1,2}$. If $z(t)$ represents a segment in $[t_1, t_2]$ then $\lambda_{t_1}^{t_2}z(t)=t_2-t_1$ and we obtain with the above notation

$$\frac{t_2-\bar{t}}{t_2-t_1}=\frac{\beta-\gamma}{\beta-\alpha}.$$

In particular

(40) *If $f$ is linear on $R$, $\alpha\neq\beta$, then the points which divide geodesic arcs $z(t)$, $(0\leqq t\leqq l)$, $z(0)=x\in[f=\alpha]$, $z(l)=y\in[f=\beta]$ in the ratio $\rho:1$ (i.e. $xw:xy=\rho$) all lie on $[f=\gamma]$ with $\gamma=(1-\rho)\alpha+\rho\beta$.*

We observe next:

(41) *If* dim $R=2$ *and for any $p\neq q$ a monotone $f$ with $f(p)=f(q)$ exists then $R$ is straight.*

By (38) an $[f=\alpha]$ has dimension 1 and is therefore a straight line or a great circle. There are no other geodesics since each $T(p, q)$ lies on some $[f=\alpha]$ by hypothesis. We conclude from $G$ (31.3) that $R$ is straight or compact. The latter is excluded by (36).

(42) *If* dim $R=2$ *and for any $p\neq q$ a linear $f$ with $f(p)=f(q)$ exists then $R$*

*is Minkowskian.*

It follows from (41) and its proof that $R$ is straight and any line is an $[f=\alpha]$ and from (40) applied to a fixed $p\in[f=\alpha]$ and $[f=\beta]=\mathfrak{z}$ that $[f=(\alpha+\beta)/2]$ bisects every segment $T(p, z(t))$. It now follows from the two papers [7] and [12] that $R$ is Minkowskian.

We now prove a strengthened analogue of (41) for $n>2$.

(43) THEOREM. *If* $\dim R=n>2$ *and for any n points* $a_1, a_2, \cdots, a_n$ *a monotone* $f$ *and* $\alpha$ *exist with* $a_i\in[f=\alpha]$, $i=1, 2, \cdots, n$ *then* $R$ *is straight and Desarguesian.*

*Moreover for any plane (=2-flat)* $P$ *in* $R$ *and given* $p\neq q$ *in* $P$ *there is a monotone function* $\bar{f}$ *on* $P$ *with* $\bar{f}(p)=\bar{f}(q)$.

NOTE. We saw earlier that Desarguesian spaces satisfy the hypothesis of (43).

PROOF. Let points $a_1, a_2, \cdots, a_n$ be given. Let $f_1$ be a monotone function and $\alpha_1$ a number such that $a_i\in[f_1=\alpha_1]=F_1$, $i=1, 2, \cdots, n$. Then as $f_1$ is not constant on $R$, $F_1\neq R$. Hence there exists $b_1\notin F_1$. Let $f_1^1$ be a monotone function and $\alpha_1^1$ a number such that $[f_1^1=\alpha_1^1]=F_1^1$ contains $a_1, a_2, \cdots, a_{n-1}, b_1$. Since $b_1\notin F_1$ and $f_1^1$ is not constant on $R$ the restriction of $f_1^1$ to $F_1$ is not constant on $F_1$ also. That $b_1\notin F_1$ further implies that $F_1^1\supsetneqq F_1\cap F_1^1$ so that by (39) $\dim F_1\cap F_1^1 <\dim F_1^1=n-1$ and $a_1, a_2, \cdots, a_{n-1}$ lie in the total flat $H_1=F_1\cap F_1^1$.

Let $h_1$ denote the restriction of $f_1^1$ to $F_1$ and put $\beta_1=\alpha_1^1$. We have thus shown that for any $a_1, a_2, \cdots, a_{n-1}$ on the total flat $F_1$ there exists a (nonconstant) monotone function $h_1$ on $F_1$ such that $a_1, a_2, \cdots, a_{n-1}$ lie in the total fiat $H_1=[h_1=\beta_1]$. In other words, the hypothesis of (43) is satisfied if we replace $R$ by $F_1$ i.e. this hypothesis is hereditary on any total flat of co-dimension 1.

Applying the above argument therefore to $F_1$ and $H_1$ (instead of $R$ and $F_1$) we see that the points $a_1, a_2, \cdots, a_{n-2}$ lie in a total flat $G_1$ of $H_1$ with $\dim G_1 \leqq n-3$.

Clearly this procedure continues. For two distinct points it yields a total 1-flat $L$ (since $<1$ is impossible), hence a great circle or a straight line. Any geodesic which has two common points with $L$ coincides with $L$ since $L$ is total. Thus the geodesic through two points is unique and $R$ is straight or of the elliptic type, see $G$ section 31. The latter is excluded by (36).

Next consider three noncollinear points $a_1, a_2, a_3$. They lie in a 2-flat $P$, therefore the space is Desarguesian, see $G$ (14.1).

Our procedure above of constructing monotone functions on flats of successively lower dimensions shows that for $p\neq q$ in the 2-flat $P$ there is a monotone function $\bar{f}$ defined on $P$ with $\bar{f}(p)=\bar{f}(q)$.

A corollary of Beltrami's Theorem and (43) is

(44) *A Riemann space of dimension $n>2$ in which for $n$ given points a monotone function $f$ and an $\alpha$ exists such that $[f=\alpha]$ contains these points is euclidean or hyperbolic.*

Another consequence of (42) and (43) is

(45) *If* $\dim R=n>1$ *and for any $n$ points a linear function $f$ and an $\alpha$ exists such that $[f=\alpha]$ contains these points, then $R$ is Minkowskian.*

For, the assertion is (42) for $n=2$. If $n>2$ then $R$ is Desarguesian by (43) and any three noncollinear points lie in a Minkowski plane. This makes the metric of $R$ Minkowskian. For the parallel axiom holds, hence $R$ is all of $A^n$ and the affine midpoint of two points is also the metric midpoint. This characterizes Minkowskian geometry, see $G$ p. 94.

However (45) can be considerably improved, see (49).

## 3. A dual approach.

We now discuss a method which may be considered as the dual to that of Section 2.

Instead of requiring that $n$ given points lie in an $[f=\alpha]$, we postulate that a point is the intersection of $[f_i=\beta_i]$. We prove that $n$ of these suffice if $\dim R=n$. It turns out that under certain conditions on the ranges of $f$ and $foz(t)$, parts of (45) hold when a single point is the intersection of $[f_i=\alpha_i]$ and that the space is Minkowskian when linear $f_i$ exist such that one point $q=\bigcap_i [f_i=\alpha_i]$.

We say that a set $M$ of monotone functions $f$ isolates the point $q$ if

$$(\ast\ast\ast) \qquad \bigcap_{f\in M}[f=f(q)]=q\,.$$

(46) *If* $\dim R=n>1$ *and* $(\ast\ast\ast)$ *holds then a minimal set of monotone functions isolating $q$ consists of exactly $n$ functions.*

The considerations of the preceding section imply that any level sets $F_1, F_2, \cdots$ through $q$ satisfy

$$\dim\bigcap_{i=1}^{j}F_i\geqq n-j\,, \qquad j\leqq n\,,$$

hence a minimal set isolating $q$ consists of at least $n$ functions. We will construct one with exactly $n$ functions.

Take any $f_1$ and put $F_1=[f_1=f_1(q)]$. Then $\dim F_1=n-1$. Because $q$ can be isolated there is an $f_2$ such that $F_2=[f_2=f_2(q)]\not\supset F_1$ and $\dim F_1\cap F_2=n-2$ because the restriction of $f_2$ to $F_1$ is monotone on $F_1$. For $n=2$ we are finished. If $n>2$ we proceed in the same way: Because of (**) there is an $f_3$ such that $F_3=[f_3=f_3(q)]$ does not contain $F_1\cap F_2$, then $\dim(F_1\cap F_2\cap F_3)=n-3$, etc.

The next statement although simple is the clue to our present method.

(47) LEMMA. *If $f_1$ and $f_2$ are monotone and the range of every nonconstant $f_1\circ z(t)$ contains the range of $f_1$, the level sets $[f_1=\alpha_1]$ and $[f_2=\alpha_2]$ intersect but are distinct, then for any point $q$, putting $f_i(q)=\beta_i$ the sets $[f_i=\beta_i]$ are distinct.*

Since $[f_1=\alpha_1]\neq[f_2=\alpha_2]$ there is a segment $T$ in $[f_2=\alpha_2]$ which intersects $[f_1=\alpha_1]$ in a point. If $z(t)$ is the geodesic containing $T$, then $z(t)$ lies in $[f_2=\alpha_2]$ and $f_1\circ z(t)$ is not constant and range $f_1\circ z(t)\supset$range $f_1$ implies that $f_1\circ z(t)$ takes the value $\beta_1$, so that $[f_1=\beta_1]=[f_2=\beta_2]$ would imply $[f_2=\alpha_2]\cap[f_2=\beta_2]\neq\emptyset$ which is impossible.

The general form of (47) is

(48) THEOREM. *If $\dim R=n$, the monotone functions $f_1, f_2, \cdots, f_n$ isolate a point $q$, moreover range $f_i\circ z(t)\supset$range $f_i$ for each nonconstant $f_i\circ z(t)$, $i=1, 2, \cdots, n-1$ then $f_1, \cdots, f_n$ isolate any point $y$. The space is homeomorphic to $\boldsymbol{R}^n$.*

For $n=2$ this follows from (47). Let $n\geqq 3$, $f_i(q)=\alpha_i$, $f_i(y)=\beta_i$. Put $[f_i=\alpha_i]=F_i$, $[f_i=\beta_i]=F_i'$. By (47) $\dim F_1\cap F_2=\dim F_1'\cap F_2'=n-2$ and the hypothesis guarantees $\dim F_1\cap F_2\cap F_3=n-3$. The general theory gives $\dim F_1'\cap F_2'\cap F_3'\geqq n-3$. We want to prove equality. Without restriction we assume $F_3'\neq F_3$ and work in $F_3'$ as space; denoting the restriction to $F_3'$ of a function $f$ in $R$ by $\bar{f}$. Then $F_3'\cap F_1=[\bar{f}_1=\alpha_1]\neq F_3'\cap F_2=[\bar{f}_2=\alpha_2]$, by (47) and $F_3'\cap F_3=\emptyset$ as $\beta_3\neq\alpha_3$ since $F_3'\neq F_3$.

Now $\dim F_1'\cap F_2'\cap F_3'=n-2$ would imply $F_3'\cap F_1'=F_3'\cap F_2'$, hence $[\bar{f}_1=\beta_1]=[\bar{f}_2=\beta_2]$ contradicting (47).

This settles the case $n=3$ and shows at the same time that a repeated application of (48) yields the theorem, except for the last statement. Putting $f_i(y)=y^i$ we see that $(y^1, \cdots, y^n)$ and $y$ determine each other uniquely (so that the $y^i$ may be considered as cöordinates of $y$) and proves that $R$ is homeomorphic to $\boldsymbol{R}^n$.

We notice that if under hypothesis of (48) the space is Desarguesian, then with suitable affine coordinates $x^1, x^2, \cdots, x^n$ the space has the form $R=I_1\times I_2\times$

$\cdots \times I_n$ where the $I_i$ are the ranges of $f_i$. This shows that very few Desarguesian spaces possess monotone functions isolating a point.

A good example is the strip $-\pi/2 < x_1 < \pi/2$ of an $(x_1, x_2)$-plane metrized as a Desarguesian space by $xy = [(x_1-y_1)^2+(x_2-y_2)^2]^{1/2} + |\tan x_1 - \tan y_1|$. The natural functions are here $f_1(x)=x_1$, $f_2(x)=x_2$. For a geodesic $z(t)=(z_1(t), z_2(t))$ ($t$ is arclength) we have $f_i \circ z(t) = z_i(t)$. For all geodesics other than $(a, t)$ on which $f_1$ is constant, we have range $f_1 \circ z(t) = (-\pi/2, \pi/2)$. Thus the conditions for the ranges in (48) are satisfied.

If $f(x)$ is linear then for any $z(t)$ either $f \circ z(t)$ is constant or range $f \circ z(t) = (-\infty, \infty)$. This and (48) enable us to prove the promised improvement of (45).

(49) THEOREM. *Let* $\dim R = n \geqq 2$ *and let linear functions* $f_1, f_2, \cdots, f_n$ *exist which isolate a point* $q$. *Then* $R$ *is Minkowskian.*

The condition on the ranges in (48) holds because the $f_i$ are linear. Indroduce coordinates $x^1, x^2, \cdots, x^n$ as above.

We show that the homeomorphism $\phi$ with $\phi(z(t)) = (z^1(t), \cdots, z^n(t))$, $z^i(t) = f_i(z(t)) = k_i t + \lambda_i$ sends each geodesic onto an affine line. For this we simply have to observe that the equations $z^i(t) = k_i t + \lambda_i$ define in $A^n$ a line through $(\lambda_1, \lambda_2, \cdots, \lambda_n)$ parallel to the cartesian vector $(k_1, k_2, \cdots, k_n)$. From this one can see that $\phi$ is an incidence preserving bijective homeomorphism of $R$ onto $A^n$. In particular, therefore, $R$ is straight.

We may thus identify $A^n$ and $R$ taking $x^1, x^2, \cdots, x^n$ as affine co-ordinates for $R$. Because of linearity we know that for any $t'$, $t''$ the midpoint of $z(t')$ and $z(t'')$ is $z(t_m)$ where $t_m = (t'+t'')/2$. Affine midpoints and metric midpoints therefore coincode and hence the space is Minkowskian, see $G$ p. 94.

## 4. Monotone $h(\tilde{r}, x)$.

Whereas requiring $h(\tilde{r}, x)$ to be nearly peakless (in a straight space) proved a relatively mild condition, see (18), the $h(\tilde{r}, x)$ are monotone in only very special spaces.

The loci $h(\tilde{r}, x) = \alpha$ (or $[h(\tilde{r}) = \alpha]$) are the limit spheres with central ray $\tilde{r}$. Thus:

(50) *In a straight space the* $h(\tilde{r}, x)$ *are monotone iff the limit spheres are flat.*

As corollaries of (50) and $G$ (24.13) and (24.14) we have:

(51) *In a straight space the functions* $h(\tilde{r}, x)$ *are monotone iff a set on which*

*each point has exactly one foot is convex and closed.*

(52) *A straight space of dimension greater than* 2 *is Minkowskian with differentiable spheres iff the* $h(\tilde{r}, x)$ *are monotone.*

Whether (52) holds for $n=2$ has been an open question since 50 years. The case $n=2$ can be covered by strengthening the hypothesis of (52).

(53) *A straight space is Minkowskian with differentiable spheres iff the* $h(\tilde{r}, x)$ *are linear.*

For if the $h(\tilde{r}, x)$ are linear then the condition $(D)$ of [8] holds and hence $R$ is Minkowskian with differentiable spheres. The converse is obvious.

We give an example of the nonexistance of a monotone $h(\tilde{r}, x)$ in a straight space.

A simply connected space with strictly convex capsules is straight, therefore it suffices by (50) to show:

(54) *In a simply connected space with strictly convex capsules no limit sphere is flat.*

Assume that $H_0$ given by $h(\tilde{r}_0, x)=0$ is flat. Let $\tilde{r}$ be a proper subray of $\tilde{r}_0$ and consider $H=h(\tilde{r}, x)=0$. $H$ is flat or convex, its interior is convex and does not contain $H_0$.

Let $p\neq q$ lie on $H$ and $(puq)$. Denote their (unique, see $G$ (22.18), (23.2)) feet on $H_0$ by $p_0$, $u_0$, $q_0$. Then $T(u, u_0)$ intersects $H$ in a point $v$ and $\sigma=pp_0=qq_0=u_0v\leqq uu_0$.

If $u'$ is the foot of $u$ on $T(p_0, q_0)$ then $uu_0\leqq uu'$ because $T(p_0, q_0)\subset H_0$. Thus $u'$ does not lie in the interior of $C=C(T(p, q), \sigma)$ although $p_0\in C$ and $q_0\in C$, so that $C$ is not strictly convex.

## References


[1] E. Artin, Ein mechanisches System mit quasiergodischen Bahnen, Hamburger Abh. **3** (1924), 170-175.

[2] V. Bangert, Totally convex sets in complete Riemannian manifolds, to appear in J. Diff. Geom.

[3] V.N. Berestovskiĭ, On the finite dimensionlity problem for Busemann's $G$-spaces, (Russian), Siberian J. Math. **XVIII** (1977), 219-221.

[4] R.L. Bishop and B. O'neill, Manifolds of negative curvature, Trans. A.M.S. **145** (1969), 1-49.

[5] H. Busemann, The geometry of geodesics, New York 1955.

[6] H. Busemann, Recent synthetic differential geometry, New York-Heidelberg-Berlin 1970.

[7] H. Busemann, Remark on "Planes with analogues to euclidean angular bisectors", Math. Scand. **38** (1976), 81-82.

[8] H. Busemann and B. B. Phadke, Minkowskian geometry, convexity conditions and the parallel axiom, J. Geometry **12** (1979), 17-33.

[9] G. Darboux, Leçons sur la théorie générale des surfaces, 3e partie, Paris, 1894.

[10] W. Hurewicz, Sur la dimension des produits carteésiens, Ann. Math. **36** (1935), 194-197.

[11] P. Kelly and E. G. Straus, Curvature in Hilbert geometry, Pacific J. Math. **8** (1958), 119-125.

[12] B. B. Phadke, The theorem of Desargues in planes with analogues to euclidean angular bisectors, Math. Scand. **39** (1976), 191-194.

[13] E. Soetens, Convexity in Busemann spaces, Bull. Soc. Math. Belg. **XIX** (1967), 194-213.

[14] R. Walter, Konvexität in riemannschen Mannigfaltigkeiten, Jahresber. d. deut. Math. Verein **83** (1981), 1-31.

Department of Mathematics,
University of Southern California,
Los Angeles, California 90007, U.S.A.
School of Mathematical Sciences,
The Flinders University of South Australia,
Bedford Park, South Australia 5042, Australia