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INDECOMPOSABLE MODULES OVER ONE-SIDED SERIAL LOCAL RINGS AND RIGHT PURE SEMISIMPLE PI-RINGS

By

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Introduction

Let R be a ring with an identity element and let $J=J(R)$ be the Jacobson radical. We denote by $\mathrm{Mod}(R)$ and by $\mathrm{mod}(R)$ the categories of all right R-modules and finitely generated R-modules, respectively. We recall that a local ring R is said to be *right serial* (resp. *left serial*) if the right (resp. left) ideals in R are linearly ordered by the inclusion. We call R *one-sided serial* if R is either left or right serial.

Following ideas of Nazarova [6] and Nazarova and Rojter [7] we describe in the present paper a method allowing us to reduce the study of modules over one-sided serial local rings R to the study of finitely generated modules over triangular matrix rings of the form $\begin{pmatrix} G & {}_GN_F \\ 0 & F \end{pmatrix}$ where G, F are division rings and ${}_GN_F$ is an $G-F$-bimodule (comp. [2]). In the paper the method is mainly used in constructing large indecomposable modules.

In Section 1 we prove that if R is a right serial local ring with $J(R)^2\neq 0$ which is not left serial then there are subdivision rings $G\subset H$ of $F=R/J(R)$ both isomorphic to F such that $\dim_G F=(\dim_H F)^2\leqq 4$ and the category consisting of such finitely generated right R-modules M that $M/\mathrm{soc}(M)$ is a direct sum of copies of $R/J(R)^2$ is representation-equivalent to the category $\imath_H({}_GF_F)$ consisting of those modules X over the ring $\begin{pmatrix} G & {}_GF_F \\ 0 & F \end{pmatrix}=A$ for which the module $X\otimes_A\begin{pmatrix} H & {}_HF_F \\ 0 & F \end{pmatrix}$ over the ring $\begin{pmatrix} H & {}_HF_F \\ 0 & F \end{pmatrix}$ has no simple injective summands. A counterpart of this result for right modules over a left serial local ring is also proved. Hence we conclude that if R is a one-sided serial local ring with $J(R)^2\neq 0$ which is not both left and right serial then there exists an indecomposable right R-module which is not finitely generated.

In Sections 2 and 3 we discuss the following open problem (see [10, 11, 12]):

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($\mathbf{pss}_R$) *If every right R-module is a direct sum of finitely presented modules, does R is of finite representation type?*

Unfortunately we are not able to solve the problem in the general case. However, using the positive solution of ($\mathbf{pss}_R$) for hereditary PI-rings given in [11] together with the result mentioned above we prove in Section 2 that ($\mathbf{pss}_R$) has a positive answer for local PI-rings. Furthermore, we show that the solution of ($\mathbf{pss}_R$) for one-sided serial local rings R can be reduced to ($\mathbf{pss}_S$) for hereditary rings S discussed in [11, Sec. 3].

In Section 3 we prove that the problem ($\mathbf{pss}_R$) has a positive solution for schurian factors of hereditary artinian PI-rings. This was done by applying the results obtained recently in [5, 13, 14] on vector space categories and associated right peak rings. The method presented in Section 3 can be also applied to the non-schurian right pure semisimple rings. It reduces the problem to rather difficult questions concerning subspaces of non-schurian vector space PI-categories (see [13, Theorem 1.1]).

Indecomposable modules over a one-sided serial local ring R with $R/J(R)$ commutative were studied by Dlab and Ringel in [2]. The main results obtained there can be also deduced from our results in Sections 1 and 2 by using the diagammatic characterization of hereditary PI-rings of finite representation type obtained in [3].

Throughout this paper $\mathrm{soc}(X)$ denotes the socle of the module X and X^t denotes the direct sum of t copies of X.

1. Modules over one-sided serial local rings.

Throughout this section we fix the following notation. R is a one-sided serial local ring, $B=R/J(R)^2$, $F=R/J(R)$ and $\varepsilon: R\rightarrow F$ denotes the natural ring epimorphism. We fix $z\in J(R)$ such that $J(R)=zR$ provided R is right serial and $J(R)=Rz$ provided R is left serial. If R is right serial (resp. left serial) we define a ring homomorphism

$$\sigma: F \longrightarrow F \qquad (\text{resp. } \tau: F\rightarrow F)$$

by the formula $\varepsilon(r)\bar{z}=\bar{z}\sigma\varepsilon(r)$ (resp. $\bar{z}\varepsilon(r)=\tau\varepsilon(r)\bar{z}$) where $r\in R$ and $\bar{z}=z+J(R)^2\in J(R)/J(R)^2$.

We start with the following simple lemma.

LEMMA 1.1 *Suppose that* $J(R)^2\neq 0$ *and* $J(R)^3=0$. *If* R *is right serial with* $J(R)=zR$ *and* $r, s\in R$ *then* $rz^2=z^2s$ *if and only if* $\varepsilon(s)=\sigma^2\varepsilon(r)$. *If* R *is left* *serial with* $J(R)=Rz$ *then* $rz^2=z^2s$ *if and only if* $\varepsilon(r)=\tau^2\varepsilon(s)$.

PROOF. Let $\sigma\varepsilon(r)=\varepsilon(r')$ and $\sigma^2\varepsilon(r)=\varepsilon(r'')$. Then $rz+J(R)^2=\varepsilon(r)\bar{z}=\bar{z}\sigma\varepsilon(r)=zr'+J(R)^2$. Hence $rz-zr'=t\in J(R)^2$ and similarly $r'z-zr''=t'\in J(R)^2$. By our assumption we have

$$z^2\varepsilon(s)=z^2s=rz^2=(zr'+t)z=zr'z=z(zr''+t')=z^2\sigma^2\varepsilon(r).$$

Since $z^2\neq 0$ then $\varepsilon(s)=\sigma^2\varepsilon(r)$ as we required. The converse implication as well as the second equivalence can be proved similarly.

In order to formulate the main result of this section we need some terminology and notation.

A *corepresentation* of an $F-G$-bimodule ${}_FN_G$ is a triple (U_G, V_F, i) where U_G and V_F are finitely generated modules over the ring G and F, respectively, and $i: U_G\to V\otimes_F N_G$ is an G-homomorphism. A map form (U_G, V_F, i) into (U'_G, V'_F, i') is pair (g, f) with $g\in\mathrm{Hom}_G(U, U')$, $f\in\mathrm{Hom}_F(V, V')$ such that $(f\otimes 1)i=i'g$. The category of corepresentations of ${}_FN_G$ is denoted by $cr({}_FN_G)$.

If ${}_FN'_H$ is an $F-H$-bimodule, ${}_GK_H$ is an $G-H$-bimodule, $\boldsymbol{c}: {}_FN_G\otimes_G K_H\to {}_FN'_H$ is an $F-H$-bilinear map and V_F is a right F-module then elements $a_1, \cdots, a_q$ in $V\otimes_F N_G$ are called ${}_GK_H$*-independent* if the equality $\boldsymbol{c}(a_1\otimes k_1)+\cdots+\boldsymbol{c}(a_q\otimes k_q)=0$ with $k_j\in {}_GK_H$ implies that $k_1=\cdots=k_q=0$. If in addition F, G, H are division rings then a corepresentation (U_G, V_F, i) is said to be ${}_GK_H$-independent if given a basis $e_1, \cdots, e_q$ of U_G the elements $i(e_1), \cdots, i(e_q)\in V\otimes_F N_G$ are ${}_GK_H$-independent.

We denote by $cr({}_FN_G)^K_{N'}$ the full subcategory of $cr({}_FN_G)$ consisting of ${}_GK_H$-independent corepresentations.

Finally, we denote by $\mathcal{E}(B, F)$ the full subcategory of $\mathrm{mod}(R)$ consisting of modules M such that $M/\mathrm{soc}(M)\cong B^t$ for some t.

We recall that an additive functor between two additive categories is said to be a *representation equivalence* if it is full, dense and reflects isomorphisms.

Now we are able to prove the main result of this section.

THEOREM 1.2. *Let R be a right noetherian one-sided serial local ring and let $F=R/J(R)$. If $J(R)^2\neq 0$, $J(R)^3=0$, and R is not both left and right serial then:*

(i) *There exist division rings G and H both isomorphic to F, bimodules ${}_FN_G$, ${}_FN'_H$, ${}_GK_H$, and $F-H$-bimodule map $\boldsymbol{c}: {}_FN_G\otimes_G K_H\to {}_FN'_H$ and G-linearly independent elements e^*, x^*, y^* in ${}_FN_G$ such that e^*, y^* are ${}_GK_H$-independent.*

(ii) *There exists an additive functor*

$$T: \mathcal{E}(B, F)\longrightarrow cr({}_FN_G)^K_{N'}$$

which is a representation equivalence.

PROOF. First we define two subdivision rings $G^\circ \subset H^\circ$ of F. We put $G^\circ = \sigma^2(F)$, $H^\circ = \sigma(F)$ if R is right serial and $G^\circ = \tau^2(F)$, $H^\circ = \tau(F)$ if F is left serial. Next we fix an element $x \in H^\circ \setminus G^\circ$ and an element $y \in F \setminus H^\circ$. The existence of such elements follows from our assumption that R is not both left and right serial.

Now we define the division rings and bimodules required in (i) by the following formulas:

$$G^{op} = \{f^* \in \mathrm{Ext}_R^1(B, F);\ f^* = \mathrm{Ext}_R^1(f, F) \text{ with } f \in \mathrm{End}(B_R)\},$$

$$H^{op} = \{f^* \in \mathrm{Ext}_R^1(F, F);\ f^* = \mathrm{Ext}_R^1(f, F) \text{ with } f \in \mathrm{End}(F_R)\},$$

$${}_F N_G = \mathrm{Ext}_R^1(B, {}_F F) \text{ and } nf^* = f^*(n) \text{ for } n \in {}_F N_G \text{ and } f^* \in G,$$

$${}_F N'_H = \mathrm{Ext}_R^1(F, {}_F F) \text{ and } mf^* = f^*(m) \text{ for } m \in {}_F N'_H \text{ and } f^* \in H,$$

$${}_G K_H = \mathrm{Im}[\mathrm{Ext}_R^1(-, F) : \mathrm{Hom}_R(F, B) \longrightarrow \mathrm{Hom}_Z(\mathrm{Ext}_R^1(B, F), \mathrm{Ext}_R^1(F, F))],$$

$$\boldsymbol{c}(n \otimes q^*) = q^*(n) \text{ for } n \in {}_F N_G,\ q^* = \mathrm{Ext}_R^1(q, F),\ q \in \mathrm{Hom}_R(F, B).$$

It is easy to see that G and H are factor rings of F and therefore they are isomorphic to F.

Now we are going to describe matrix representations of G, H and of the bimodule ${}_G K_H$ which will be useful in our further calculations. For this purpose we fix a basis $\gamma'_1 = \bar{z}, \gamma'_2, \cdots, \gamma'_c$ of the right vector space $J(R)/J(R)^2$ over F and a basis $\beta'_1 = z^2, \beta'_2, \cdots, \beta'_d$ of the right vector space $J(R)^2$ over F. Next we define elements $\gamma_1, \cdots, \gamma_c$ and $\beta_1, \cdots \beta_d$ as follows. If R is right serial then $c = d = 1$ and we put $\gamma_1 = \beta_1 = 1$. Now suppose that R is left serial. Then for any $j \geqq 1$ there are $\gamma''_j, \beta''_j \in R$ such that $\gamma'_j = \gamma''_j z$ and $\beta'_j = \beta''_j z^2$. We put $\gamma_j = \varepsilon(\gamma''_j)$ and $\beta_j = \varepsilon(\beta''_j)$. It is easy to check that $\gamma_1, \cdots, \gamma_c \in F_{H^\circ}$ are linearly independent over H° and $\beta_1, \cdots, \beta_d \in F_{G^\circ}$ are linearly independent over G°.

Now we denote by H^* the subring of the full matrix ring $\boldsymbol{M}_c(H^\circ)$ consisting of all matrices

$$h = \begin{bmatrix} h^{11}, & h^{12}, & \cdots, & h^{1c} \\ \vdots & \vdots & & \vdots \\ h^{c1}, & h^{c2}, & \cdots, & h^{cc} \end{bmatrix}, \qquad h^{ij} \in H^\circ,$$

whose coefficients satisfy the following equalities

$$(h^{11} + \gamma_2 h^{21} + \cdots + \gamma_c h^{c1})\gamma_j = h^{1j} + \gamma_2 h^{2j} + \cdots + \gamma_c h^{cj},$$

for $j = 2, \cdots, c$. We denote by G^* the subring of $\boldsymbol{M}_d(G^\circ)$ consisting of all matrices $g = (g^{ij})$, $1 \leqq i, j \leqq d$, satisfying the following equalities

$$(g^{11} + \beta_2 g^{21} + \cdots + \beta_d g^{d1})\beta_j = g^{1j} + \beta_2 g^{2j} + \cdots + \beta_d g^{dj}$$

for $j = 2, \cdots, d$. Finally, we denote by K^* the set of matrices

$$k=\begin{bmatrix} k^{11}, & \cdots, & k^{1c} \\ \vdots & & \vdots \\ k^{d1}, & \cdots, & k^{dc} \end{bmatrix}, \qquad k^{ij}\in G^{\circ},$$

whose coefficients satisfy the following equalities

$$(k^{11}+\beta_2 k^{21}+\cdots+\beta_d k^{d1})\tau(\gamma_j)=k^{1j}+\beta_2 k^{2j}+\cdots+\beta_d k^{dj}$$

for $j=2, \cdots, c$.

We will show that there are ring isomorphisms

$$\xi: G\longrightarrow G^*, \qquad \zeta: H\longrightarrow H^*.$$

Given $f^*\in G$ with $f\in\mathrm{End}(B_R)$ we consider the projective resolution of f in mod(R)

$$\begin{array}{ccccccc} \cdots\longrightarrow & P_2 & \xrightarrow{p_2} & R^d & \xrightarrow{p_1} & R \\ & \downarrow f_2 & & \downarrow f_1 & & \downarrow f_0 \\ \cdots\longrightarrow & P_1 & \xrightarrow{p_2} & R^d & \xrightarrow{p_1} & R \end{array}$$

where $p_1(x_1, \cdots, x_d)=\beta_1' x_1+\cdots+\beta_d' x_d$. Let $f_1=(r_{ij})$ with $r_{ij}\in R$ and let $f_0(1)=r$.

First suppose that R is right serial. Then $d=1$, $G^*=G^{\circ}$ and we put $\xi(f^*)=\varepsilon(r_{11})$. Since $z^2 r_{11}=rz^2$ then by Lemma 1.1 $\varepsilon(r_{11})\in G^{\circ}$ and ξ is obviously a ring isomorphism.

Next suppose that R is left serial. Let $\xi(f^*)=(\tau^2\varepsilon(r_{ij}))$. It is easy to see that ξ does not depend on the choice of f_0 and f_1. Moreover, since R is left serial then $z^2 r_{ij}=s_{ij}z^2$ for some $s_{ij}\in R$ and it follows from Lemma 1.1 that $\tau^2\varepsilon(r_{ij})=\varepsilon(s_{ij})$ for all i, j. Then $f_0 p_1=p_1 f_1$ if and only if

$$\begin{aligned} r\beta_j'' z^2 &= \beta_1'' z^2 r_{1j}+\cdots+\beta_d'' z^2 r_{dj} \\ &= [\beta_1''\varepsilon(s_{1j})+\cdots+\beta_d''\varepsilon(s_{dj})]z^2 \end{aligned}$$

for $j=1, \cdots, d$. Since $z^2\neq 0$ the equalities hold if and only if

$$\varepsilon(r)=\varepsilon(s_{11})+\beta_2\varepsilon(s_{21})+\cdots+\beta_d\varepsilon(s_{d1})$$

and

$$\varepsilon(r)\beta_j=\varepsilon(s_{1j})+\beta_2\varepsilon(s_{2j})+\cdots+\beta_d\varepsilon(s_{dj}) \qquad \text{for } j\geqq 2.$$

It follows that $\xi(f^*)\in G^*$. Conversely, suppose that $g=(g_{ij})\in G^*$. Let r, $r_{ij}\in R$ be such that

$$\varepsilon(r)=g^{11}+\beta_2 g^{21}+\cdots+\beta_d g^{d1} \quad \text{and} \quad \tau^2\varepsilon(r_{ij})=g_{ij}.$$

It follows from the discussion above that the formulas $f_0(1)=r$ and $f_1=(r_{ij})$ define R-homomorphisms f_0: $R\to R$ and f_1: $R^d\to R^d$ such that $f_0 p_1=p_1 f_1$ and therefore there exits $f\in\mathrm{End}(B_R)$ such that $\xi(f^*)=g$. Since ξ preserves the addition and the multiplication then it is a ring isomorphism. The isomorphism ζ

is defined in a similar way. The details are left to the reader.

It is easy to see that a matrix which belongs either to G^* or to H^* is equal zero if and only if one of its rows or columns is zero.

Now we will define a group isomorphism $\omega: {}_GK_H \to K^*$. Let $t^* = \mathrm{Ext}^1_R(t, F)$ where $t \in \mathrm{Hom}_R(F, B)$. Then we have a commutative diagram with exact rows

$$\begin{array}{ccccccc} R^c & \xrightarrow{p_1'} & R & \longrightarrow & F & \longrightarrow & 0 \\ \downarrow t_1 & & \downarrow t_0 & & \downarrow t & & \\ R^d & \xrightarrow{p_1} & R & \longrightarrow & B & \longrightarrow & 0 \end{array}$$

where $p_1'(y_1, \cdots, y_c) = \gamma_1'' z y_1 + \cdots + \gamma_c'' z y_c$. Let $t_0(1) = s$ and $t_1 = (t_{ij})$ where $t_{ij} \in R$.

First suppose that R is right serial. Then $c = d = 1$, $K^* = G^\circ$ and we put $\omega(t^*) = \sigma\varepsilon(t_{11}) \in \sigma^2(F) = G^\circ$. It is clear that ω is an isomorphism.

Next suppose that R is left serial. Let $\omega(t^*) = (\tau^2\varepsilon(t_{ij}))$. It is clear that ω does not depend on the choice of t_0 and t_1. In order to show that $\omega(t^*) \in K^*$ we can suppose that $t \neq 0$. Then $s = s'z$ where s' is an invertible element. Now if we put $k^{ij} = \tau^2\varepsilon(t_{ij})$ then $t_0 p_1' = p_1 t_1$ if and only if

$$\begin{aligned} s'\tau(\gamma_j)z^2 = s'z\gamma_j'' z &= z^2 t_{1j} + \beta_2'' z^2 t_{2j} + \cdots + \beta_d'' z^2 t_{dj} \\ &= (k^{1j} + \beta_2 k^{2j} + \cdots + \beta_d k^{dj}) z^2 \end{aligned}$$

for $j = 1, \cdots, c$. Since $z^2 \neq 0$ then the equalities hold if and only if

$$\varepsilon(s') = k^{11} + \beta_2 k^{21} + \cdots + \beta_d k^{d1}$$

and

$$\varepsilon(s')\tau(\gamma_j) = k^{1j} + \beta_2 k^{2j} + \cdots + \beta_d k^{dj} \qquad \text{for} \quad j = 2, \cdots, c.$$

It follows that $\omega(t^*) \in K^*$ and that ω is surjective. Since ω is obviously injective it is an isomorphism.

Now K^* can be considered as an G^*-H^*-bimodule via the isomorphisms ξ, ζ, ω. It is easy to check that $\omega(gkh)$ is the multiplication of matrices $\xi(g)\omega(k)\tau\zeta(h)$ for any $g \in G$, $k \in K$, $h \in H$ where $\tau(\bar{h}) = (\tau(\bar{h}^{ij}))$. Moreover, the right G-module action on ${}_FN_G$ corresponds to the usual right matrix action of G^* on F^d via the natural composed isomorphism

$$(*) \qquad {}_FN_G = \mathrm{Ext}^1_R(B, F) \cong \mathrm{Hom}_R(R^d, F) \cong F^d.$$

Similarly ${}_FN_H' \cong F^c$ and the right action of H on N' corresponds to the right matrix action of H^* on F^c. Finally the bilinear map $\boldsymbol{c}: {}_FN_G \otimes_G K_H \to {}_FN_H'$ corresponds to the map $\boldsymbol{c}': {}_F(F^d) \otimes_{G^*} K^*_{H^*} \to {}_F(F^c)_{H^*}$ defined by the formula $\boldsymbol{c}'(v \otimes (k^{ij})) = v \cdot (\tau^{-1}(k^{ij}))$.

Now let $e^* = (1, 0, \cdots, 0)$, $x^* = (x, 0, \cdots, 0)$, $y^* = (y, 0, \cdots, 0) \in F^d \cong {}_FN_G$. Then the equality $e^*g_0+x^*g_1+y^*g_2=0$ with $g_i\in G^*$ implies that each of the matrices g_i has the first row equal zero. Hence they are zero matrices because they belons to G^*. Now suppose that $\boldsymbol{c}'(e^*\otimes k_0)+\boldsymbol{c}'(y^*\otimes k_1)=0$ where $k_s=(k_s^{ij})\in K^*$. Then $\tau(k_0^{1j})+y\tau(k_1^{1j})=0$ for $j=1, \cdots, c$. Since $y\notin H^\circ$, $\tau(k_s^{1j})\in H^\circ$ then $k_0^{1j}=k_1^{1j}=0$ for $j=1, \cdots, c$ and therefore $\boldsymbol{c}'(e^*\otimes k_s)=0$ for $s=0, 1$. Then (i) will be proved in the case R is left serial if we show that e^* is ${}_{G^*}K^*_{H^*}$-independent. We will do it later after the proof of the statement (ii). If R is right serial then it is easy to see that $G\cong G^\circ$, $H\cong H^\circ$, ${}_FN_G\cong {}_FF_{G^\circ}$, ${}_FN'_H\cong {}_FF_{H^\circ}$, ${}_GK_H\cong {}_{G^\circ}H^\circ_{H^\circ}$ and $\boldsymbol{c}$ is induced by the multiplication ${}_FF\otimes_{G^\circ}H^\circ_{H^\circ}\to {}_FF_{H^\circ}$. If we put $e^*=1$, $x^*=x$ and $y^*=y$ then (i) follows.

In order to proof (ii) we define group isomorphisms

$$\mathrm{Ext}^1_R(B^t, F^n)\xrightarrow{a_{tn}}\mathrm{Hom}_R(P^t_1, F^n)\xrightarrow{b_{tn}}\mathrm{Hom}_G(G^t, F^n\otimes_F N_G)$$

for any positive integers t and n, where $P_1=R^d$. For this purpose we consider a projective resolution

$$\cdots\longrightarrow P^t_2\xrightarrow{p^t_2}P^t_1\xrightarrow{p^t_1}R^t$$

of B^t in mod(R). Since $\mathrm{Ext}^1_R(B^t, F^n)$ is the first cohomology group of the complex

$$\mathrm{Hom}_R(R^t, F^n)\xrightarrow{(p^t_1)^*}\mathrm{Hom}_R(P^t_1, P^n)\xrightarrow{(p^t_2)^*}\mathrm{Hom}_R(P^t_2, F^n)\longrightarrow\cdots$$

and $(p^t_1)^*=(p^t_2)^*=0$ then there is a natural isomorphism $\mathrm{Ext}^1_R(B^t, F^n)\cong\mathrm{Hom}_R(P^t_1, F^n)$ and we take it for a_{tn}.

In order to define b_{tn} we denote by $e'_1, \cdots, e'_t$ the standard basis in G^t and by $e_1, \cdots, e_n$ the standard basis in F^n. Now given $h\in\mathrm{Hom}_R(P^t_1, F^n)$ we put

$$b_{tn}(h)e'_i=e_1\otimes a_{11}^{-1}(h_{1i})+\cdots+e_n\otimes a_{11}^{-1}(h_{ni})$$

for $i=1, \cdots, t$, where $h_{ji}=\pi_j h\nu_i$, $\nu_i: P_1\to P^t_1$ is the injection into the i^{th} coordinate and $\pi_j: F^n\to F$ is the projection on the j^{th} coordinate. It is clear that b_{tn} is an isomorphism.

Now we will prove that:

1°. a_{tn} and b_{tn} are natural maps with respect to R-homomorphisms $F^n\to F^m$.

2°. If $\bar{f}: B^r\to B^t$ is an R-homomorphism given by the matrix $(\bar{r}_{ij})$, $r_{ij}\in R$ and

$$\begin{array}{ccccc}\cdots\longrightarrow & P^r_1 & \xrightarrow{p^r_1} & R^r \\ & \downarrow f' & & \downarrow f \\ \cdots\longrightarrow & P^t_1 & \xrightarrow{p^t_1} & R^t\end{array}$$

is a projective resolution of $\bar{f}$ then we have a commutative diagram

$$\begin{array}{ccccc} \mathrm{Ext}_R^1(B^t, F^n) & \xrightarrow{a_{tn}} & \mathrm{Hom}_R(P_1^t, F^n) & \xrightarrow{b_{tn}} & \mathrm{Hom}_G(G^t, F^n \otimes_F N_G) \\ \downarrow \bar{f}^* & & \downarrow (f')^* & & \downarrow w(\bar{f})^* \\ \mathrm{Ext}_R^1(B^r, F^n) & \xrightarrow{a_{rn}} & \mathrm{Hom}_R(R^r, F^n) & \xrightarrow{b_{rn}} & \mathrm{Hom}_G(G^r, F^n \otimes_F N_G) \end{array}$$

where h^* means the map induced by h and the G-linear map $w(\bar{f})$ is defined by the formula

$$w(\bar{f})e_i' = e_1'\bar{r}_{i1}^* + \cdots + e_t'\bar{r}_{it}^*$$

where $\bar{r}_{ij}^* = \mathrm{Ext}_R^1(\bar{r}_{ij}, F) \in G$.

The property 1° as well as the commutativity of the left hand square in the diagram above are obvious. In order to complete the proof of 2° we suppose that f' has the form $f' = (f_{ij}')$ where $f_{ij}' \in \mathrm{Hom}_R(P_1, P_1)$. Then for any $h = (h_{ij}) \in \mathrm{Hom}_R(P_1^t, F^n)$ and any s we have

$$\begin{aligned} w(\bar{f})^* b_{tn}(h)e_s' &= b_{tn}(h)w(\bar{f})e_s' = \sum_{j=1}^{t} [b_{tn}(h)e_j']\bar{r}_{js}^* \\ &= \sum_{j=1}^{t} \sum_{k=1}^{n} e_k \otimes \bar{r}_{js}^* a_{11}^{-1}(h_{kj}) \\ &= \sum_j \sum_k e_k \otimes a_{11}^{-1}(f_{js}')^* h_{kj} \\ &= \sum_j \sum_k e_k \otimes a_{11}^{-1}(h_{kj} f_{js}') \\ &= \sum_{k=1}^{n} e_k \otimes a_{11}^{-1}(hf')_{ks} \\ &= [b_{rn}(f'^*)](h)e_s' \end{aligned}$$

and (ii) follows.

Now we define a functor $T : \mathcal{E}(B, F) \to c\imath(_F N_G)$. Given a module M in $\mathcal{E}(B, F)$ we consider the exact sequence

$$\boldsymbol{e}_M : 0 \longrightarrow \mathrm{soc}\,(M) \longrightarrow M \longrightarrow M/\mathrm{soc}\,(M) \longrightarrow 0\,.$$

Since $\mathrm{soc}\,(M) \cong F^n$ and $M/\mathrm{soc}\,(M) \cong B^t$ for some n and t then $\boldsymbol{e}_M \in \mathrm{Ext}_R^1(B^t, F^n)$ and we put $T(M) = (G^t, F^n, u_{tn}^M)$ where $u_{tn}^M = b_{tn}a_{tn}(\boldsymbol{e}_M)$. If $t : L \to M$ is a homomorphism in $\mathcal{E}(B, F)$ and $\mathrm{soc}\,(L) \cong F^m$, $L/\mathrm{soc}\,(L) \cong B^r$ then we have a commutative diagram

$$\begin{array}{ccccccccc} \boldsymbol{e}_L : 0 & \longrightarrow & F^m & \longrightarrow & L & \longrightarrow & B^r & \longrightarrow & 0 \\ & & \downarrow t' & & \downarrow t & & \downarrow t'' & & \\ \boldsymbol{e}_M : 0 & \longrightarrow & F_n & \longrightarrow & M & \longrightarrow & B^t & \longrightarrow & 0 \end{array}$$

and by 1° and 2° the diagram

$$\begin{array}{ccc} G^r & \xrightarrow{u^L_{rm}} & F^m \otimes_F N_G \\ \downarrow{\scriptstyle w(t'')} & & \downarrow{\scriptstyle t' \otimes id} \\ G^t & \xrightarrow{u^M_{tn}} & F^n \otimes_F N_G \end{array}$$

is also commutative. If we put $T(t)=(w(t''), t')$ then T becomes an additive functor.

In order to show that T is full we take a map $(q, q'): T(L) \to T(M)$. If $q=(q_{ij})$ with $q_{ij}=\bar{r}^*_{ij}$ where $\bar{r}_{ij} \in \mathrm{End}(B)$ and if we define $\bar{f} \in \mathrm{Hom}_R(B^r, B^t)$ by the matrix $\bar{f}=(\bar{r}_{ij})$ then obviously $w(\bar{f})=q$. Next if we denote by p the map $(q' \otimes id)u^L_{rm}=u^M_{tn}w(\bar{f}) \in \mathrm{Hom}_G(G^r, F^n \otimes_F N_G)$ and if

$$\boldsymbol{e}: 0 \longrightarrow F^n \longrightarrow Z \longrightarrow B^r \longrightarrow 0$$

is an exact sequence in $\mathrm{mod}(R)$ such that $b_{rn}a_{rn}(\boldsymbol{e})=p$ then 1° and 2° yield $\mathrm{Ext}^1_R(\bar{f}, id)\boldsymbol{e}_M=\boldsymbol{e}=\mathrm{Ext}^1_R(id, q')\boldsymbol{e}_L$ and hence there is a commutative diagram

$$\begin{array}{ccccccccc} \boldsymbol{e}_L: 0 & \longrightarrow & F^m & \longrightarrow & L & \longrightarrow & B^r & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle q'} & & \downarrow{\scriptstyle h'} & & \downarrow{\scriptstyle id} & & \\ \boldsymbol{e}: 0 & \longrightarrow & F^n & \longrightarrow & Z & \longrightarrow & B^r & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle id} & & \downarrow{\scriptstyle h''} & & \downarrow{\scriptstyle \bar{f}} & & \\ \boldsymbol{e}_M: 0 & \longrightarrow & F^n & \longrightarrow & M & \longrightarrow & B^t & \longrightarrow & 0. \end{array}$$

It follows that $T(h''h')=(q, q')$. Moreover, if (q, q') is an isomorphism in $\mathcal{ct}({}_FN_G)$ then q and q' are isomorphisms. We claim that $\bar{f}$ is an isomorphism, too. In order to prove it we can suppose (without loss of generality) that $q=id$. It follows from the definition of $w(\bar{f})$ that $\bar{r}_{ij}=id$ and $\bar{r}_{ij} \in BJ(R)$ for $i \neq j$, $i, j =1, \cdots, t$. Then there is a commutative diagram

$$\begin{array}{ccc} B^t & \xrightarrow{\bar{\varepsilon}^t} & F^t \\ \downarrow{\scriptstyle \bar{f}} & & \downarrow{\scriptstyle id} \\ B^t & \xrightarrow{\bar{\varepsilon}^t} & F^n \end{array}$$

where $\bar{\varepsilon}^t$ is a minimal epimorphism. Hence $\bar{f}$ is an isomorphism as we claimed. Consequenly, $h''h'$ is an isomorphism and therefore T reflects isomorphisms.

Since $b_{tn}a_{tn}$ is an isomorphism then in order to finish the proof of (ii) it is sufficient to show that given an exact sequence in $\mathrm{mod}(R)$

$$\boldsymbol{e}: 0 \longrightarrow F^n \xrightarrow{j} X \longrightarrow B^t \longrightarrow 0$$

the corepresentation $(G^t, F^n, b_{tn}a_{tn}(\boldsymbol{e}))$ is ${}_GK_H$-independent if and only if $\operatorname{Im} j = \operatorname{soc}(X)$. For this purpose given $i=(i^1, \cdots, i^t)\in \operatorname{Hom}_R(F, B^t)$ we consider its projective resolution in $\operatorname{mod}(R)$

$$\begin{array}{ccccc} \cdots \longrightarrow & P_1' & \xrightarrow{p_1'} & R \\ & \downarrow i_1 & & \downarrow i_0 \\ \cdots \longrightarrow & P_1^t & \xrightarrow{p_1^t} & R^t \end{array}$$

and the induced commutative diagram

$$\begin{array}{ccccc} \operatorname{Ext}^1_R(B^t, F^n) & \xrightarrow{a_{tn}} & \operatorname{Hom}_R(P_1^t, F^n) & & \\ \downarrow i^* & & \downarrow i_1^* & & \\ \operatorname{Ext}^1_R(F, F^n) & \xrightarrow{a_t'^n} & \operatorname{Hom}_R(P_1', F^n) & \xrightarrow{\lambda} & F^n\otimes_F N_H' \end{array}$$

where $\lambda(h_1', \cdots, h_n')=e_1\otimes a_{11}'^{-1}(h_1')+\cdots+e_n\otimes a_{11}'^{-1}(h_n')$. If $a_{tn}(\boldsymbol{e})=(h_{ij})$ and $i_1=(i_1^1, \cdots, i_1^t)$ where $h_{ij}\in\operatorname{Hom}_R(P_1, F)$, $i_1^s\in\operatorname{Hom}_R(P_1', P_1)$ then $a_{11}'(i^s)^*=(i_1^s)^*a_{11}$ for every s and therefore

$$\lambda a_{tn}' i^*(\boldsymbol{e})=\lambda(hi_1)=\sum_{j=1}^n e_j\otimes a_{11}'^{-1}\Big(\sum_{s=1}^t h_{js}i_1^s\Big)$$

$$=\sum_j\sum_s e_j\otimes a_{11}'^{-1}(i_1^s)^*(h_{js})$$

$$=\sum_j\sum_s e_j\otimes (i^s)^* a_{11}^{-1}(h_{js})$$

$$=\sum_s \boldsymbol{c}\Big[\sum_j e_j\otimes a_{11}^{-1}(h_{js})\otimes(i^s)^*\Big]$$

$$=\sum_s \boldsymbol{c}[b_{tn}a_{tn}(\boldsymbol{e})e_s'\otimes(i^s)^*]\,.$$

Since $\lambda a_{tn}'$ is an isomorphism then from the above equality follows that the corepresentation $(G^t, F^n, b_{tn}a_{tn}(\boldsymbol{e}))$ is ${}_GK_H$-independent if and only if there is no nonzero maps $i: F\to B^t$ such that $i^*(\boldsymbol{e})=0$. On the other hand it is easy to see that $\operatorname{Im} j\neq\operatorname{soc}(X)$ if and only if there is a nonzero map $i: F\to B^t$ such that $i^*(\boldsymbol{e})=0$ (cf. [7, Sec. 12]). Consequently, $\operatorname{Im} j=\operatorname{soc}(X)$ if and only if $(G^t, F^n, b_{tn}a_{tn}(\boldsymbol{e}))$ is ${}_GK_H$-independent. Hence T is a representation equivalence and (ii) follows.

To finish the proof of (i) it remains to show that the element e^* is ${}_{G^*}K^*_{H^*}$-independent. For this purpose we consider the exact sequence

$$\boldsymbol{e}: 0\longrightarrow J(R)^2\longrightarrow R\longrightarrow B\longrightarrow 0\,.$$

Since $J(R)^2=\mathrm{soc}(R_R)\cong F^d$ then according to the statement proved above the G-linear map $b_{1d}a_{1d}(\boldsymbol{e})\colon G\to F^d\otimes_F N_G$ defines an ${}_GK_H$-independent corepresentation. Since obviously $a_{1d}(\boldsymbol{e})\colon R^d\to F^d$ is the natural epimorphism ε^d then

$$b_{1d}a_{1d}(\boldsymbol{e})1=e_1\otimes a_{11}^{-1}(\pi_1\varepsilon^d)+\cdots+e_d\otimes a_{11}^{-1}(\pi_d\varepsilon^d).$$

It follows that the element $a_{11}^{-1}(\pi_1\varepsilon^d)$ is ${}_GK_H$-independent. Since $e^*\in F^d$ corresponds to $a_{11}^{-1}(\pi_1\varepsilon^d)$ under the isomorphism (*) then e^* is ${}_{G^*}K^*_{H^*}$-independent as we required. Now the proof of the theorem is complete.

REMARK. Another useful (but not functorial) method for the study of indecomposable modules over one sided serial local rings R with $J(R)^2\neq 0$ can be found in [2, Section 6].

2. Right pure semisimple local rings.

We recall from [9] that a ring R is *right pure semisimple* if every right R-module is a direct sum of finitely presented modules. We keep the terminology and notation introduced in [11] where the reader is also referred for a background of right pure semisimple rings.

We start with the following technical lemma.

LEMMA 2.1. *Let F, G, H be division rings and ${}_FN_G$, ${}_FN'_H$, ${}_GK_H$ be bimodules defined in the proof of Theorem* 1.2. *Then there exists a sequence*

$$L_1\xrightarrow{u_1^*}L_2\longrightarrow\cdots\longrightarrow L_s\xrightarrow{u_s^*}L_{s+1}\longrightarrow\cdots$$

in the category $\mathcal{r}({}_FN_G)_N^K$, such that L_t is indecomposable, u_t^* is a proper monomorphism for all t and $L=\mathrm{colim}\,L_t$ is indecomposable.

PROOF. Let e^*, x^*, y^* be the elements defined in the proof of Theorem 1.2 and let $L_s=(U_s, F^s, i_s)$ where F^s is the standard s-dimensional vector space over F, $e_1,\cdots,e_s$ is the standard basis of F^s, U_s is the G-subspace of $V_s=F^s\otimes_F N_G$ generated by elements $e_i^*=e_i\otimes e^*$, $i=1,\cdots,s$, and $v_j=e_j\otimes xe^*+e_{j+1}\otimes ye^*$, $j=1,\cdots,s-1$, and $i_s\colon U_s\to V_s$ is the inclusion map (comp. [1, Sec. 5]).

By the property (i) in Theorem 1.2 the elements $e_1^*,\cdots,e_s^*$ '$v_1,\cdots,v_{s-1}$ are ${}_GK_H$-independent and therefore L_s is an object of $\mathcal{r}({}_FN_G)_{N'}^K$. The F-linear injection $u_s\colon F^s\to F^{s+1}$ given by $u_s(e_i)=e_{i+1}$ for $i=1,\cdots,s$ defines a map $u_s^*\colon L_s\to L_{s+1}$ because $(u_s\otimes 1)(e_i^*)=e_{i+1}^*$ and $(u_s\otimes 1)(v_j)=v_{j+1}$ for $1\leqq i\leqq s$ and $1\leqq j\leqq s-1$.

We recall from the proof of Theorem 1.2 that ${}_FN=F\oplus\cdots\oplus F$. Then for every $n\in N$ and $f\in F$ we have defined $nf\in N$ and hence we have also defined $vf\in V_s$ for all $v\in V_s$ and $f\in F$.

Now we prove by induction on s that L_s is indecomposable by showing that any nonzero idempotent in the ring $\mathrm{End}(L_s)$ is the identity map. For this purpose we note first that $L_1=(e^*G,\ F,\ i_1)$ and $L_{s+1}/L_s\cong(e^*G+x^*G,\ F,\ i)$ are indecomposable. Next suppose that L_s is indecomposable and let $f^*\in\mathrm{End}(L_{s+1})$ be a nonzero idempotent. Since $1-f^*$ is also an idempotent and $f^*u_s\neq 0$ or $(1-f^*)u_s\neq 0$ then we may suppose that $f^*u_s\neq 0$. Since f^* is given by an F-linear map $f: F^{s+1}\to F^{s+1}$ such that $(f\otimes 1)U_{s+1}\subset U_{s+1}$ then

$$(f\otimes 1)e_j^*=e_1^*g_{1j}+\cdots+e_{s+1}^*g_{s+1j}+v_1q_{1j}+\cdots+v_sq_{sj}$$

for $j=1,\cdots,s+1$, where $g_{ij}=(g_{ij}^{tu})$, $q_{ij}=(q_{ij}^{tu})$ are matrices in G^* with g_{ij}^{tu}, $g_{ij}^{tu}\in G^\circ$. Since $(f\otimes 1)v_i\in U_{s+1}$ for $i=1,\cdots,s$ then

$$(*)\qquad (f\otimes 1)v_i=e_1^*h_{1i}+\cdots+e_{s+1}^*h_{s+1i}+v_1k_{1i}+\cdots+v_sk_{si}$$

for some elements $h_{ji}=(h_{ji}^{tu})$, $k_{ji}=(k_{ji}^{tu})$ in G^* with h_{ji}^{tu}, $k_{ji}^{tu}\in G^\circ$. On the other hand we have

$$(**)\qquad (f\otimes 1)v_i=[(f\otimes 1)e_i^*]x+[(f\otimes 1)e_{i+1}^*]y\,.$$

Now from the comparison of the right side terms in (*) and (**) we easily conclude that

$$h_{1i}^{1t}+xk_{1i}^{1t}-g_{1i}^{1t}x-xq_{1i}^{1t}x=g_{1i+1}^{1t}y+xq_{1i+1}^{1t}y$$

for $i=1,\cdots,s$ and $t\geqq 1$. Since $x\in H^\circ$ then the left side of the above equality belongs to H°. It follows that $g_{1i+1}^{1t}+xq_{1i+1}^{1t}=0$ because otherwise $y\in H^\circ$ which is a contradiction. Hence $g_{1i+1}^{1t}=q_{1i+1}^{1t}=0$ for $i=1,\cdots,s$, $t\geqq 1$, and $h_{1i}^{1t}=k_{1i}^{1t}=0$ for $i=2,\cdots,s$, $t\geqq 1$. It follows that the matrices g_{1i+1}, q_{1i+1}, h_{1i}, k_{1i} for $i\geqq 1$ have their first rows equal zero and therefore they are zero matrices because they belongs to G^*. Consequently $(f\otimes 1)e_2^*,\cdots,(f\otimes 1)e_{s+1}^*$, $(f\otimes 1)v_2,\cdots,(f\otimes 1)v_s$ belongs to $(u_s\otimes 1)U_s$ and therefore there is a commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & L_s & \xrightarrow{u_s^*} & L_{s+1} & \longrightarrow & L_{s+1}/L_s & \longrightarrow & 0\\
 & & \downarrow{\bar f^*} & & \downarrow{f^*} & & \downarrow{\tilde f^*} & & \\
0 & \longrightarrow & L_s & \xrightarrow{u_s^*} & L_{s+1} & \longrightarrow & L_{s+1}/L_s & \longrightarrow & 0\,.
\end{array}$$

Since L_s is indecomposable and $\bar f^*$ is a nonzero idempotent in $\mathrm{End}(L_s)$ then $\bar f^*$ is the identity map. It follows that $g_{ii}=1$ for $i=2,\cdots,s$, $g_{ij}=0$ for $i\neq j$, $2\leqq j\leqq s+1$, $1\leqq i\leqq s+1$ and $q_{ij}=0$ for $i=1,\cdots,s$, $j=2,\cdots,s+1$. Note also that from the equality $(f\otimes 1)e_1^*=e_1p_1e^*+\cdots+e_{s+1}p_{s+1}\otimes e^*$ with some $p_j\in F$ we easily conclude that $g_{j1}^{1t}=q_{j1}^{1t}=0$ for $j\geqq 1$ and $t\geqq 2$. Then the equality $f^{*2}=f^*$ yields $(g_{11}^{11}+xq_{11}^{11})^2=g_{11}^{11}+xq_{11}^{11}$ and therefore $q_{11}^{11}=0$. Hence $q_{11}=0$ because its first row is equal zero. Now from the comparison of the right side terms in (*) and (**) for $i=1$

we easily conclude that

$$g_{j1}^{1t}x+yq_{j-11}^{1t}x+xq_{j1}^{1t}x=h_{j1}^{1t}+yk_{j-11}^{1t}+xk_{j1}^{1t}$$

for $t\geqq 1$, $j=3, \cdots, s+1$ (we put $k_{s+11}^{1t}=q_{s+11}^{1t}=0$) and

$$g_{21}^{1t}x+xq_{21}^{1t}x=h_{21}^{1t}+yk_{11}^{1t}+xk_{21}^{1t} \qquad \text{for} \quad t\geqq 2,$$

$$g_{21}^{11}x+xq_{21}^{11}x+y=h_{21}^{11}+yk_{11}^{11}+xk_{21}^{11}$$

$$g_{11}^{1t}x=h_{11}^{1t}+xk_{11}^{1t} \qquad \text{for} \quad t\geqq 1.$$

Hence we inductively conclude that $g_{j1}=q_{j-11}=h_{j1}=k_{j-11}=0$ for $j=s+1, s, \cdots, 3$, $g_{21}=h_{21}=0$, $k_{11}^{1t}=0$ for $t\geq 2$ and $k_{11}^{11}=1$. Now from the last equality above follows $h_{11}=0$, $g_{11}^{1t}=0$ for $t\geq 2$ and $g_{11}^{11}=1$. It follows that the idempotent $\tilde{f}^*$ is nonzero. Since L_{s+1}/L_s is indecompossable then $\tilde{f}^*$ is the identity map and therefore f^* is also the identity map, as we required. Consequently L_{s+1} is indecomposable. The indecomposability of L can be proved in a similar way and we leave it to the reader. Then the lemma is proved.

Now we are able to prove a result which shows that the open question (**pss**$_R$) for a one-sided serial local ring R can be reduced to (**pss**$_S$) for a hereditary ring S discussed in [11, Section 3].

THEOREM 2.2. *If R is a one-sided serial local right pure semisimple ring then either R is both left and right serial or $J(R)^2=0$.*

PROOF. Suppose that R is not both left and right serial and that $J(R)^2\neq 0$. Then by Theorem 1.2 there is a representation equivalence $T: \mathcal{E}(B, F)\to c\imath({}_F N_G)_{N'}^K$. It follows from Lemma 2.1 that there exists a sequence

$$D_1 \xrightarrow{d_1} D_2 \longrightarrow \cdots \longrightarrow D_s \xrightarrow{d_s} D_{s+1} \longrightarrow \cdots$$

where D_i are indecomposable modules in $\mathcal{E}(B, F)$, d_i is not bijective for $i\geq 1$ and $d_j d_{j-1}\cdots d_1\neq 0$ for any j. On the other hand R is right pure semisimple. Then by [8, Theorem 6.3] there is an integer m such that $d_m d_{m-1}\cdots d_1=0$ and we get a contradiction. Then the theorem is proved.

Now we are able to prove the following result announced in [11, Note Added in Proof] which answers the question (**pss**$_R$) in affirmative for any local PI-ring R.

COROLLARY 2.3. *Let R be a local ring such that the division ring $F=R/J(R)$ is finite dimensional over its center and let $d=\dim_F(J(R)/J(R)^2)$, $d'=$* $\dim(J(R)/J(R)^2)_F$. *Then the following conditions are equivalent*:

(a) *R is of finite representation type,*

(b) *R is right pure semisimple,*

(c) *R is artinian and either $dd'=1$ or $J(R)^2=0$ and $2\leq dd'\leq 3$.*

PROOF. (a)→(b) follows from [8, Theorem 6.3] and (a)↔(c) was proved in [3]. In order to prove that (b) implies (c) we note that by the right pure semisimplicity of R and [11, Corollary 3.4] R is right artinian and $R/J(R)^2$ is of finite representation type. It follows that R is one-sided serial because we know from [3] that $dd'\leq 3$. Now (c) is a consequence of Theorem 2.2 and the proof is complete.

REMARK 1. An explicite description of indecomposable modules over a local ring R satisfying the conditions in Corollary 2.3 can be found in [3, Section 2] (see also [2]).

REMARK 2. The problem ($\mathbf{pss}_R$) remains open for arbitrary local rings R as well as for arbitrary hereditary rings R (see [11]).

3. Right pure semisimple factors of hereditary PI-rings.

In this section we give a positive solution of the problem ($\mathbf{pss}_R$) for schurian factors of hereditary PI-rings by applying the results on vector space categories and right peak rings obtained in [5, 13, 14]. We use the terminology and notation introduced in [14] where the reader is referred for details.

THEOREM 3.1. *If R is a right pure semisimple schurian factor of a hereditary PI-ring then R is of finite representation type.*

PROOF. Suppose that R is an indecomposable right pure semisimple schurian PI-ring. Then R is right artinian. We will prove by induction on the length $l(R_R)$ that R is of finite representation type and that the endomorphism ring of any indecomposable right R-module in $\mathrm{mod}(R)$ is a division PI-ring.

The case $l(R_R)=0$ is obvious. Suppose $l(R_R)>0$. Since R is a factor of a hereditary ring then R has a simple injective right module and therefore there is a ring isomorphism

$$R\cong\begin{pmatrix} F & {}_FM_S \\ 0 & S \end{pmatrix}$$

where F is a division PI-ring and ${}_FM_S$ is an $F-S$-bimodule. Since $l(S_S)<l(R_R)$ then by the inductive assumption S is of finite representation type and $\mathrm{End}(X_S)$ is a division PI-ring for any indecomposable module X_S in $\mathrm{mod}(S)$.

First we will show that the dimension of $U_F^X=\mathrm{Hom}_S({}_FM_S, X_S)$ is finite for any indecomposable module X_S in $\mathrm{mod}(S)$. Assume the contrary (i.e. that

$\dim U_F^X$ is infinite) and consider the right R-module C_R^X defined by the triple (U_F^X, X_S, v) where $v: U_F^X \otimes_F M_S \to X_S$ is the S-homomorphism adjoint to the identity map on U_F^X. Since X_S is indecomposable then C_R^X is indecomposable. Moreover, C_R^X is not finitely generated because $\dim U_F^X$ is infinite. Hence R is not right pure semisimple and we get a contradiction.

It follows that the category $\mathbf{K}_F = \mathrm{Hom}_S({}_F M_S, \mathrm{mod}(S))$ together with the embedding functor $|-|: \mathbf{K}_F \to \mathrm{mod}(F)$ is a vector space category. Then by the inductive assumption on S and by [14, Lemma 4.6] the ring $\boldsymbol{R}_{\mathbf{K}}$ associated to $\mathbf{K}_F$ (see [14, Sec. 3]) is a schurian artinian right peak PI-ring. Hence the assumptions required in [14, Theorem 4.1] are satisfied and therefore there is an equivalence of categories

$$(*) \qquad \mathrm{mod}(R)/[\mathrm{mod}(S)] \cong \mathrm{mod}_{sp}(D)$$

where $D = (\boldsymbol{R}_{\mathbf{K}})_*$. Now it follows from the formula following the definition of the functor G in [14, Sec. 3] that D is an artinian schurian right peak PI-ring. Moreover, since R is right pure semisimple then by [8, Theorem 6.3] the Jacobson radical of the category $\mathrm{mod}_{sp}(D)$ is indecomposably right T-nilpotent in the sense that for any sequence of D-homomorphisms

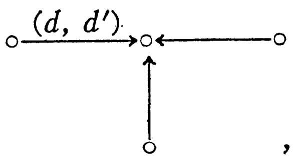
$$X_1 \xrightarrow{f_1} X_2 \longrightarrow \cdots \longrightarrow X_n \xrightarrow{f_n} X_{n+1} \longrightarrow \cdots$$

where X_i are pairwise nonisomorphic indecomposable modules in $\mathrm{mod}_{sp}(D)$ there is an integer m such that $f_m \cdots f_2 f_1 = 0$.

Now we are going to prove that the value scheme $(\boldsymbol{I}_D, \boldsymbol{d})$ of D does not contain as an upper value subscheme one of the value schemes

(a) $\circ \xrightarrow{(d,\, d')} \circ$, $dd' \geq 4$,

(b) $\circ \xrightarrow{(d,\, d')} \circ \longleftarrow \circ$, $dd' = 3$,

(c)  , $dd' = 2$,

(d) $\circ \xrightarrow{(d,\, d')} \circ \xleftarrow{(e,\, e')} \circ$, $2 \leq dd'$, $ee' \leq 3$,

(e) $\circ \longrightarrow \circ \overset{(d,\, d')}{\text{———}} \circ \longleftarrow \circ \longleftarrow \circ$, $dd' = 2$ and $\circ\text{———}\circ$ means either $\circ \longrightarrow \circ$ or $\circ \longleftarrow \circ$,

(f) $\circ \longrightarrow \circ \xrightarrow{(d,\, d')} \circ$, $dd' = 3$,

(g) partially ordered sets of the form K^* where K is one of the Kleiner's

[4] critical posets (1, 1, 1, 1), (2, 2, 2), (1, 3, 3), (1, 2, 5), $(N, 4)$ and K^* denotes the enlargement of K by a unique maximal element.

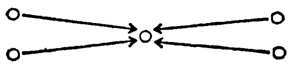
For the definition of the upper value subscheme of a given one the reader is referred to [5, 13].

Let $D=P_1\oplus\cdots\oplus P_n\oplus P_{n+1}$ where P_j are indecomposable right ideals of D and P_{n+1} is the right peak of D. We recall that given an upper value subscheme $(\boldsymbol{L}, \boldsymbol{d})$ of $(\boldsymbol{I}_D, \boldsymbol{d})$ there is a pair of adjoint functors

$$\mathrm{mod}_{sp}(D_L) \underset{r_L}{\overset{T_L}{\rightleftarrows}} \mathrm{mod}_{sp}(D)$$

where $D_L=\mathrm{End}(\oplus_{j\in L}P_j)$, $r_L(-)=\mathrm{Hom}_D(\oplus_{j\in L}P_j, -)$ and

$$T_L(-)=\mathrm{Hom}_{D_L}(\mathrm{Hom}_D(\oplus_{j\in L}P_j, D), -)\,.$$

The functor T_L is full, faithful, reflects isomorphisms and carries over indecomposable modules into indecomposable ones. Then, in view of the observation above, it follows that the Jacobson radical of $\mathrm{mod}_{sp}(D_L)$ is indecomposably right T-nilpotent for any upper value subscheme $(\boldsymbol{L}, \boldsymbol{d})$ of $(\boldsymbol{I}_D, \boldsymbol{d})$. On the other hand if $(\boldsymbol{L}, \boldsymbol{d})$ is of one of the forms (a)—(g) except the poset $(N, 4)^*$ then by [3] and [11, Corollary 3.3] the hereditary artinian PI-ring D_L is not right pure semisimple. Hence by [8, Theorem 6.3] theJacobson radical of $\mathrm{mod}(D_L)$ is not indecomposably right T-nilpotent. This is a contradiction because one can easily show that for any such value scheme $(\boldsymbol{L}, \boldsymbol{d})$ the category $\mathrm{mod}_{sp}(D_L)$ is cofinite in $\mathrm{mod}(D_L)$ in the sense that all but a finite number of indecomposable modules in $\mathrm{mod}(D_L)$ belongs to $\mathrm{mod}_{sp}(D_L)$. Now suppose that $(\boldsymbol{L}, \boldsymbol{d})$ is the poset $(N, 4)^*$. Applying the Nazarova-Rojter differentiation procedure to $(N, 4)^*$ in a finite number of steps we get a poset of width ≥ 4. This means that there exists a representation equivalence $\mathbf{A}\to\mathrm{mod}_{sp}(\Lambda)$ where $\mathbf{A}$ is a full additive subcategory of $\mathrm{mod}_{sp}(D_L)$ and Λ is a hereditary PI-ring of the extended Dynkin type . It follows that the Jacobson radical of the category $\mathrm{mod}_{sp}(\Lambda)$ is indecomposably right T-nilpotent. On the other hand we know from [3, 11] that Λ is not right pure semisimple and therefore by [8, Theorem 6.3] the Jacobson radical of $\mathrm{mod}(\Lambda)$ is not indecomposably right T-nilpotent. This is a contradiction because $\mathrm{mod}_{sp}(\Lambda)$ is obviously cofinite in $\mathrm{mod}(\Lambda)$.

Consequently, $(\boldsymbol{I}_D, \boldsymbol{d})$ does not contain as an upper value subscheme one of the value schemes (a)—(g) and it follows from [5, Theorem 2] (see also [13, Theorem 3.1]) that $\mathrm{mod}_{sp}(D)$ is of finite representation type. Moreover, it follows from the proof of [14, Theorem 4.4] that $\mathrm{End}(Z)$ is a division PI-ring for any indecomposable module Z in $\mathrm{mod}_{sp}(D)$. Hence, using the equivalence (*) and the inductive assumption we easily conclude that R is of finite representation type and that the endomorphism ring of any indecomposable module in $\mathrm{mod}(R)$ is a division PI-ring. Then the proof is complete.

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