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CHARACTERIZATIONS OF HOMOGENEOUS BOUNDED DOMAINS

By

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Introduction.

This paper is a continuation of our previous one [9] and we retain the terminology and notations there.

Let M be a Kähler manifold and let $\mathrm{Aut}(M)$ be the group of holomorphic isometries of M. M is said to be *homogeneous* if the group $\mathrm{Aut}(M)$ acts transitively on M. In [9], the second named author investigated the structures of homogeneous Kähler manifolds admitting simply transitive solvable Lie groups. In the present paper, by using the same methods as in [9] we show the following

MAIN THEOREM. *Let M be a connected homogeneous Kähler manifold of complex dimension n on which a solvable Lie group G acts transitively as a group of holomorphic isometries. Assume that one of the following conditions is satisfied:*

(C.1) *The canonical hermitian form h of M is non-degenerate.*

(C.2) *M contains no complex line, that is, there is no holomorphic map of $\boldsymbol{C}$ into M except constant maps.*

Then M is holomorphically equivalent to a homogeneous bounded domain in $\boldsymbol{C}^n$.

As an immediate consequence of this theorem, we obtain the following

COROLLARY. *Let M be a homogeneous Kähler manifold which is homeomorphic to an Euclidean space. We assume that the identity component of* $\mathrm{Aut}(M)$ *has finite center and that one of the conditions* (C.1) *and* (C.2) *in the Main Theorem is satisfied. Then M is holomorphically equivalent to a homogeneous bounded domain.*

Recall that for a homogeneous complex manifold M the condition (C.2) is equivalent to the hyperbolicity of M [8]. Therefore, our Corollary provides a partial affirmative answer to a problem posed by Kobayashi [6, Problem 12, p. 133]. In a recent paper [10], Shimizu investigated the structures of homogeneous

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Kähler manifolds of complex dimension two and proved that the "fundamental conjecture" due to Vinberg and Gindikin [3], [11] is true in this case. Using this, he also classified complex two dimensional homogeneous Kähler manifolds and resolved affirmatively the Kobayashi's problem cited above for complex two dimensional Kähler manifolds.

Recently, Dorfmeister informed us that he succeeded in making clear the structures of homogeneous Kähler manifolds admitting a transitive solvable Lie group and gave in such a case an affirmative answer for Vinberg-Gindikin's fundamental conjecture [2].

This paper is organized as follows. In section 1 we prove our Main Theorem in the case where G acts simply transitively on M. In section 2 we deal with the general case and reduce the problem to the special case in section 1. In section 3 we give a proof of Corollary.

1. Proof of Theorem: Simply transitive case.

In this section we always assume that G acts simply transitively on M. We denote by $(\mathfrak{g}, J, \rho)$ the Kähler algebra corresponding to G. We set

$$\psi(X)=\mathrm{Tr}_{\mathfrak{g}}(\mathrm{ad}\,(JX)-J\cdot\mathrm{ad}\,(X))\,.$$

$$\eta(X, Y)=(1/2)\cdot\psi([JX, Y]) \qquad \text{for } X, Y\in\mathfrak{g}\,,$$

and call ψ and η the Koszul form and the canonical hermitian form of $(\mathfrak{g}, J, \rho)$ respectively. If M satisfies the condition (C.2), then $(\mathfrak{g}, J, \rho)$ satisfies the condition

(C.2)$'$ If $[JX, X]=0$, then $X=0$.

In fact, if there exists a non-zero element X in $\mathfrak{g}$ such that $[JX, X]=0$, then $\{JX\}+\{X\}$ is a J-invariant commutative subalgebra and so it corresponds to a locally flat homogeneous Kähler submanifold of M, which contradicts (C.2).

In order to prove our Main Theorem it is enough to verify the following theorem by the same reasoning as in [9].

THEOREM 1. *Assume that one of the following conditions is satisfied:*

(C.1)$'$ *The canonical hermitian form η is non-degenerate.*

(C.2)$'$ *If $[JX, X]=0$, then $X=0$.*

Then the canonical hermitian form η is positive definite, and hence $(\mathfrak{g}, J, \psi)$ is a proper J-algebra.

In [9] we already proved Theorem 1 under the condition (C.1)$'$. For the purpose of later use, we shall now recall the proof. We showed by induction on n that there exists decomposition

$$\mathfrak{g}=\sum_{k=1}^{n-1}(\{JE_k\}+\{E_k\}+\mathfrak{p}_k)+\mathfrak{g}^n \tag{1.1}$$

of $\mathfrak{g}$ into direct sum of vector spaces with the following properties:

1) Putting $\mathfrak{g}_k=\{JE_k\}+\{E_k\}+\mathfrak{p}_k$, $\mathfrak{g}_k$ is a J-invariant subalgebra such that

$$[JE_k, E_k]=E_k, \qquad [JE_k, \mathfrak{p}_k]\subset\mathfrak{p}_k$$

$$[E_k, \mathfrak{p}_k]=\{0\}, \qquad [\mathfrak{p}_k, \mathfrak{p}_k]\subset\{E_k\},$$

$$J\mathfrak{p}_k=\mathfrak{p}_k$$

and the real parts of the eigenvalues of $\mathrm{ad}(JE_k)$ on $\mathfrak{p}_k$ are equal to 1/2.

2) If we put $\mathfrak{g}^1=\mathfrak{g}$ and $\mathfrak{g}^{k+1}=\sum_{i=k+1}^{n-1}\mathfrak{g}_i+\mathfrak{g}^n$, then $\mathfrak{g}^{k+1}$ is a J-invariant subalgebra such that

$$[JE_k, \mathfrak{g}^{k+1}]\subset\mathfrak{g}^{k+1}, \qquad [E_k, \mathfrak{g}^{k+1}]=\{0\},$$

$$[\mathfrak{p}_k, \mathfrak{g}^{k+1}]\subset\mathfrak{p}_k$$

and the real parts of the eigenvalues of $\mathrm{ad}(JE_k)$ on $\mathfrak{g}^{k+1}$ are equal to 0.

3) η is positive definite on $\sum_{k=1}^{n-1}(\{JE_k\}+\{E_k\}+\mathfrak{p}_k)$ and the factors of the decomposition (1.1) of $\mathfrak{g}$ are mutually orthogonal with respect to η.

For the proof of the above facts we used the following two propositions successively.

PROPOSITION 2. *Suppose we have a decomposition* (1.1) *with the properties given there for an integer* $n\geqq 1$. *Then there exists* $E_n\neq 0$ *in* $\mathfrak{g}^n$ *such that*

$$[JE_n, E_n]=E_n \quad \text{and} \quad [E_n, \mathfrak{g}^n]\subset\{E_n\}.$$

PROPOSITION 3. *Let* E_n *be an element in* $\mathfrak{g}^n$ *as in Proposition* 2. *Then we obtain a decomposition*

$$\mathfrak{g}^n=\{JE_n\}+\{E_n\}+\mathfrak{p}_n+\mathfrak{g}^{n+1} \tag{1.2}$$

of $\mathfrak{g}^n$ *into direct sum of vector spaces such that:*

1) $\mathfrak{g}_n=\{JE_n\}+\{E_n\}+\mathfrak{p}_n$ *and* $\mathfrak{g}^{n+1}$ *has the properties stated as above.*

2) η *is positive definite on* $\mathfrak{g}_n$ *and the factors of the decomposition* (1.2) *are mutually orthogonal with respect to* η.

The condition (C.1)′ was only used for the proof of Proposition 2 and the results of Proposition 3 are valid only if $\mathfrak{g}^n$ admits an element E_n as in Proposition 2. Therefore, for the proof of Theorem 1 it suffices to show Proposition 2 under the condition (C.2)$'$.

Now we shall prove Proposition 2 under the assumption (C.2)$'$. We first show that $\mathfrak{g}^n$ contains a one dimensional ideal. Since $\mathfrak{g}^n$ is solvable, by Lie's theorem there exist $E, F \in \mathfrak{g}^n$ such that $E \neq 0$ or $F \neq 0$ and

$$[X, E] = \lambda(X)E + \mu(X)F,$$
$$[X, F] = -\mu(X)E + \lambda(X)F$$

for all $X \in \mathfrak{g}^n$, where λ and μ are linear functions on $\mathfrak{g}^n$. Let $\mathfrak{r}$ be the ideal of $\mathfrak{g}^n$ generated by E and F. Assume that E, F are linearly independent. Setting $\mathfrak{g}' = J\mathfrak{r} + \mathfrak{r}$, $(\mathfrak{g}', J, \rho)$ is a Kähler subalgebra of $\mathfrak{g}^n$. Let η' and ψ' denote the canonical hermitian form and the Koszul form of $(\mathfrak{g}', J, \rho)$ respectively. We have now two cases to consider. We consider the first

Case 1: η' is non-degenerate on $\mathfrak{r}$.

If we suppose $\psi' = 0$ on $\mathfrak{r}$, then

$$2\eta'(X, Y) = \psi'([JX, Y]) = 0 \qquad \text{for all } X, Y \in \mathfrak{r},$$

which contradicts our assumption. Thus $\psi' \neq 0$ on $\mathfrak{r}$ and so there exists a unique non-zero element $A \in \mathfrak{r}$ such that

$$2\eta'(A, X) = \psi'(X) \qquad \text{for all } X \in \mathfrak{r}.$$

We have then

$$\begin{aligned} 2\eta'([JA, A], X) &= \psi'([JX, [JA, A]]) \\ &= \psi'([[JX, JA], A]) + \psi'([JA, [JX, A]]) \\ &= \psi'([J[JX, A] + J[X, JA], A]) + \psi'([JA, [JX, A]]) \\ &= \psi'([JX, A]) + \psi'([X, JA]) + \psi'([JX, A]) \\ &= \psi'([JX, A]) = 2\eta'(A, X) \end{aligned}$$

for all $X \in \mathfrak{r}$. This implies

$$[JA, A] = A. \tag{1.3}$$

Choose a non-zero element $B \in \mathfrak{r}$ in such a way that A, B is a basis of $\mathfrak{r}$ and

$$\begin{aligned} [X, A] &= \lambda'(X)A + \mu'(X)B, \\ [X, B] &= -\mu'(X)A + \lambda'(X)B \end{aligned} \tag{1.4}$$

for all $X \in \mathfrak{g}^n$, where λ' and μ' are linear functions on $\mathfrak{g}^n$ (Cf. [9, § 3]). By a routine calculation, we have then by using (1.4) that

$$\psi'(C)=4\lambda'(JC) \qquad \text{for } C\in\mathfrak{r}\,. \tag{1.5}$$

Moreover, it follows from (1.3) and (1.4)

$$\lambda'(JA)=1 \quad \text{and} \quad \mu'(JA)=0\,. \tag{1.6}$$

By using (1.4), (1.5) and (1.6), we obtain

$$\det\begin{pmatrix} \eta'(A,\,A), & \eta'(A,\,B) \\ \eta'(B,\,A), & \eta'(B,\,B) \end{pmatrix}=-4\mu'(JB)\,,$$

which yields $\mu'(JB)\neq 0$, because η' is non-degenerate. On the other hand, it follows from Jacobi identity that

$$\begin{aligned} 0&=[[JA,\,JB],\,A]+[[JB,\,A],\,JA]+[[A,\,JA],\,JB] \\ &=-\lambda'(JB)\mu'(JB)A+(1-\mu'(JB))\mu'(JB)B\,. \end{aligned}$$

Therefore we conclude that $\lambda'(JB)=0$, $\mu'(JB)=1$ and so

$$\begin{aligned} &[JA,\,A]=A\,, \qquad [JB,\,A]=B\,, \\ &[JA,\,B]=B\,, \qquad [JB,\,B]=-A\,. \end{aligned}$$

This combined with the relation (K.5) in [9, § 1] yields a contradiction:

$$\begin{aligned} 0&=\rho([JA,\,B],\,JB)+\rho([B,\,JB],\,JA)+\rho([JB,\,JA],\,B) \\ &=\rho(B,\,JB)+\rho(A,\,JA)<0\,. \end{aligned}$$

Next we consider the second

Case 2: η' is degenerate on $\mathfrak{r}$.

Let A be a non-zero element in $\mathfrak{r}$ such that $\eta'(A,\,X)=0$ for all $X\in\mathfrak{r}$. Then we have

$$\begin{aligned} 0&=\psi'([JA,\,E])=\lambda(JA)\psi'(E)+\mu(JA)\psi'(F)\,, \\ 0&=\psi'([JA,\,F])=-\mu(JA)\psi'(E)+\lambda(JA)\psi'(F)\,. \end{aligned}$$

This implies $\lambda(JA)=\mu(JA)=0$ or $\psi'(E)=\psi'(F)=0$. If $\lambda(JA)=\mu(JA)=0$, then we obtain $[JA,\,E]=[JA,\,F]=0$ and so $[JA,\,A]=0$. This contradicts the condition (C.2)$'$. Suppose $\psi'(E)=\psi'(F)=0$. Since $\psi'(C)=4\lambda(JC)$ for all $C\in\mathfrak{r}$, we have

$$\lambda(JE)=\lambda(JF)=0\,,$$

and hence

$$\begin{aligned} &[JE,\,E]=\mu(JE)F\,, \qquad [JE,\,F]=-\mu(JE)E\,, \\ &[JF,\,E]=\mu(JF)F\,, \qquad [JF,\,F]=-\mu(JF)E \end{aligned}$$

This combined with Jacobi identity yields

$$0=[[JE, E], JF]+[[E, JF], JE]+[JF, JE], E]$$
$$=(\mu(JE)^2+\mu(JF)^2)\cdot F,$$

and therefore $\mu(JE)=\mu(JF)=0$. Thus we conclude

$$[J\mathfrak{r}, \mathfrak{r}]=\{0\},$$

which contradicts the condition (C.2)$'$.

Therefore, in any cases, it is impossible that E, F are linearly independent. Consequently, $\mathfrak{r}$ is a one dimensional ideal of $\mathfrak{g}^n$. Let E be a non-zero element in $\mathfrak{r}$. By the condition (C.2)$'$ we have then

$$[JE, E]=\lambda E \qquad \text{for some } \lambda,\ \lambda\neq 0\,.$$

If we put $E_n=(1/\lambda)E$, E_n has the desired property in Proposition 2, completing the proof.

2. Proof of Theorem (continued): General case.

Let G be a connected solvable Lie group acting transitively on M and let K be the isotropy subgroup of G at a point o of M. We may identify M with the quotient space G/K. We denote by $\mathfrak{g}$ the Lie algebra of G and by $\mathfrak{k}$ the subalgebra of $\mathfrak{g}$ corresponding to K.

We first prove the theorem under the condition (C.2). Since G is solvable, the topological closure $\bar{G}$ of G in $\mathrm{Aut}\,(M)$ is also a solvable Lie group. Therefore, without loss of generality we may assume that G is closed in $\mathrm{Aut}\,(M)$ and accordingly K is compact in G [6]. Now, putting $\mathfrak{n}=[\mathfrak{g}, \mathfrak{g}]$, we here claim $\mathfrak{n}\cap\mathfrak{k}=\{0\}$. Indeed, let X be an arbitrary element in $\mathfrak{n}\cap\mathfrak{k}$. Then, since $\mathfrak{k}$ is the subalgebra corresponding to the compact Lie subgroup K and $\mathfrak{n}=[\mathfrak{g}, \mathfrak{g}]$ is a nilpotent Lie subalgebra, $\mathrm{ad}\,(X)$ is a semi-simple and nilpotent endomorphism of $\mathfrak{g}$. Hence $\mathrm{ad}\,(X)=0$. On the other hand, we know that the center of $\mathfrak{g}$ is trivial [6, p. 133]. Therefore we have $X=0$, as desired. Next, choose any vector subspace $\mathfrak{a}$ of $\mathfrak{g}$ in such a way that $\mathfrak{g}=\mathfrak{k}+\mathfrak{n}+\mathfrak{a}$ is a direct sum of vector subspaces. Put $\mathfrak{s}=\mathfrak{n}+\mathfrak{a}$. Then $\mathfrak{s}$ is an ideal of $\mathfrak{g}$. Let S be the analytic subgroup of G corresponding to $\mathfrak{s}$. Since $S\cap K$ is discrete (and hence finite) and $\dim S=\dim\mathfrak{s}=\dim\mathfrak{g}/\mathfrak{k}=\dim M$, S acts transitively on M [7, Corollary 4.8, p. 178], so that $M=S/(S\cap K)$ and $\pi: S=S/\{e\}\to M=S/(S\cap K)$ is a covering map, where π is the natural projection and e is the identity element of S. Therefore, S admits an S-invariant Kähler structure such that $\pi: S\to M$ is holomorphic. Being a covering manifold of the hyperbolic manifold M, S is also hyperbolic [6]. As an immediate consequence of section 1, we now conclude that S is holomorphically equivalent to a homogeneous bounded domain, and hence so is M [5, Proposition 6.3, p. 44].

It remains to prove the Main Theorem under the condition (C.1). But the proof can be done along the same line as in the first case as follows. Replacing G by the closure $\bar{G}$ if necessarily, we may assume that G is closed in $\mathrm{Aut}(M)$. So K is compact. Since the canonical hermitian form h of M is non-degenerate by our assumption, we know that the center of $\mathfrak{g}$ is trivial [4]. Therefore, we obtain a holomorphic covering space $\pi: S=S/\{e\}\to M=S/(S\cap K)$ with the same properties as in the first case. It is obvious that $S=S/\{e\}$ has the non-degenerate canonical hermitian form. Repeating the same arguments as in the first case, we therefore obtain the Main Theorem, completing the proof.

3. Proof of Corollary.

Let G be the identity component of $\mathrm{Aut}(M)$ and K the isotropy subgroup of G at a point o of M. Then K is a maximal compact subgroup of G, since $M=G/K$ is homeomorphic to an Euclidean space. Therefore, denoting by Z the center of G, we can see by our assumption that Z is contained in K. Hence Z is trivial, because G acts effectively on M. Form now on, we identify G with the matrix group $\mathrm{Ad}(G)$. Let $G=G_1G_2$ be a Levi decomposition of G, where G_1 is a connected semi-simple Lie subgroup of G with finite center and G_2 is the radical of G. Let $G_1=K_1S_1$ be a Iwasawa decomposition of G_1, where K_1 is a maximal compact subgroup in G_1 and S_1 is a closed solvable Lie subgroup of G_1. Then, since K_1 is a compact subgroup and K is a maximal one in G, there exists an element $g_0\in G$ such that $g_0^{-1}\cdot K_1\cdot g_0\subset K$. Thus, putting $\tilde{o}=g_0\cdot o$ and $H=G_2S_1$, we have $K_1\cdot\tilde{o}=\tilde{o}$ and accordingly

$$M=G\cdot\tilde{o}=G_2S_1K_1\cdot\tilde{o}=H\cdot\tilde{o}\,,$$

which says that the solvable Lie subgroup H of $\mathrm{Aut}(M)$ acts transitively on M. Our Corollary is therefore an immediate consequence of the Main Theorem, completing the proof.

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