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ON CONNECTION ALGEBRAS OF HOMOGENEOUS CONVEX CONES

By

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§1. Introduction.

Let V be a homogeneous convex cone in an n-dimensional vector space X over the real number field $\boldsymbol{R}$. If the dual cone of V with respect to a suitable inner product on X coincides with V, then V is said to be *self-dual*. By using the characteristic function of V, we can define a canonical $G(V)$-invariant Riemannian metric g_V on V, where $G(V)$ is the Lie group of all linear automorphisms of X leaving V invariant. Let us take a point $e\in V$ and a system of linear coordinates $(x^1, x^2, \cdots, x^n)$ on X. Then, a commutative multiplication $\square$ is defined in X by

$$x^i(a\square b)=-\sum_{j,k}\Gamma^i_{jk}(e)x^j(a)x^k(b) \qquad (1\leqq i\leqq n)$$

for every $a, b\in X$, where Γ^i_{jk} means the Christoffel symbols for the canonical metric g_V with respect to $(x^1, x^2, \cdots, x^n)$. The structure of the algebra $(X, \square)$ is independent of choosing the point e and the system of linear coordinates $(x^1, x^2, \cdots, x^n)$. This algebra $(X, \square)$ is called the *connection algebra* of V (cf. [13], [14]). A commutative (but not necessarily associative) algebra A over $\boldsymbol{R}$ is said to be *power-associative* if the subalgebra $\boldsymbol{R}[a]$ of A generated by any element $a\in A$ is associative.

The aim of the present note is to prove the following assertion: *If the connection algebra of a homogeneous convex cone V is power-associative, then V is self-dual* (Theorem 1).

It is known that any Jordan algebra over $\boldsymbol{R}$ is power-associative (cf. e.g. [3] or [7]). So, from this, we have the known result by Dorfmeister [2]: A homogeneous convex cone V is self-dual if the connection algebra of V is Jordan. On the other hand, it is known that a commutative power-associative algebra over $\boldsymbol{R}$ having no nilpotent element is Jordan (cf. chap. 5 of [7]). From this, we can see that a power-associative connection algebra is necessarily Jordan. Therefore, the above assertion is contained in [2], but our method used here is

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elementary and quite different from that of [2]. In fact, we will start out from the theory of T-algebras developed by E. B. Vinberg and use an identity for a power-associativity condition on a connection algebra. And also, we will make use of the results on the invariant Riemannian connection for the canonical metric obtained in the previous papers [9], [10], [11].

Throughout this note, the same terminologies and notation as those in the author's previous papers will be employed.

§ 2. Preliminaries.

In this section, we will recall the fundamental results on homogeneous convex cones and T-algebras due to Vinberg. Detailed description for them may be found in [12], [13], [14].

Let $\mathfrak{A}=\sum_{1\leq i,j\leq r}\mathfrak{A}_{ij}$ be a T-algebra of rank r provided with an involutive anti-automorphism $*$. A general element of $\mathfrak{A}_{ij}$ will be denoted as a_{ij}, and also an element of $\mathfrak{A}$ will be denoted like as a matrix $a=(a_{ij})$, where a_{ij} is the $\mathfrak{A}_{ij}$-component of $a\in\mathfrak{A}$. From now on, the following notation will be used:

$$n_{ij}=\dim\mathfrak{A}_{ij}\qquad(1\leqq i,\ j\leqq r),$$

$$n_i=1+\frac{1}{2}\sum_{1\leq k<i}n_{ki}+\frac{1}{2}\sum_{i<k\leq r}n_{ik}\qquad(1\leqq i\leqq r),$$

$$\mathrm{Sp}\,a=\sum_{1\leq i\leq r}n_i a_{ii}\qquad(a=(a_{ij})\in\mathfrak{A}),$$

$$(a,\ b)=\mathrm{Sp}\,ab^*\qquad(a,\ b\in\mathfrak{A})\,. \tag{2.1}$$

From the axiom of T-algebra (cf. p. 380 in [13]), it follows that the scalar product $(\ ,\ )$ defined by (2.1) is positive definite and the numbers $\{n_{ij}\}_{1\leq i,j\leq r}$ satisfy the following condition:

$$\max\{n_{ij},\ n_{jk}\}\leqq n_{ik} \tag{2.2}$$

for every triple $(i,\ j,\ k)$ of indices $i<j<k$ satisfying $n_{ij}n_{jk}\neq 0$.

Let us define subsets $T=T(\mathfrak{A})$, $V=V(\mathfrak{A})$ and $X=X(\mathfrak{A})$ of $\mathfrak{A}$ by

$$T=\{t=(t_{ij})\in\mathfrak{A}\,;\ t_{ii}>0\quad(1\leqq i\leqq r),\qquad t_{ij}=0\quad(1\leqq j<i\leqq r)\}$$

and

$$V=\{tt^*\,;\ t\in T\}\subset X=\{x\in\mathfrak{A}\,;\ x^*=x\}\,.$$

Then $V=V(\mathfrak{A})$ is a homogeneous convex cone in the real vector space X and T is a connected Lie group which acts linearly and simply transitively on V. Conversely, every homogeneous convex cone is realized in this form up to linear equivalence.

Let $e=(e_{ij})$ be the unit element of the Lie group T. Then $e_{ij}=\delta_{ij}$ (Kronecker delta) and $e\in V$. The tangent space $T_e(V)$ of V at the point e may be naturally identified with the ambient space X and also with the Lie algebra $\mathfrak{t}$ of T. On the other hand, the Lie algebra $\mathfrak{t}$ may be identified with the subspace $\sum_{1\leq i\leq j\leq r}\mathfrak{A}_{ij}$ of $\mathfrak{A}$ provided with the bracket product: $[a, b]=ab-ba$. A canonical linear isomorphism between $\mathfrak{t}$ and X is given by

$$\xi: a\in\mathfrak{t}=\sum_{1\leq i\leq j\leq r}\mathfrak{A}_{ij}\longrightarrow a+a^*\in X=T_e(V). \tag{2.3}$$

The canonical Riemannian metric g_V at the point e determines an inner product $\langle,\rangle$ on $\mathfrak{t}$ via the isomorphism ξ by

$$\langle a, b\rangle=g_V(e)(\xi(a), \xi(b))$$

for every $a, b\in\mathfrak{t}$. Concerning two inner products $(,)$ and $\langle,\rangle$, we have the following relations (cf. p. 389, p. 391 and p. 392 in [13]):

$$\begin{aligned}&\langle a_{ij}, b_{ij}\rangle=2(a_{ij}, b_{ij})=2(a_{ij}^*, b_{ij}^*) \qquad (1\leqq i<j\leqq r),\\ &\langle a_{ii}, b_{ii}\rangle=4(a_{ii}, b_{ii}) \qquad (1\leqq i\leqq r).\end{aligned} \tag{2.4}$$

$$\langle a_{ij}b_{jk}, c_{ik}\rangle=\langle a_{ij}^*c_{ik}, b_{jk}\rangle=\langle a_{ij}, c_{ik}b_{jk}^*\rangle \qquad (1\leqq i<j<k\leqq r). \tag{2.5}$$

$$\langle\mathfrak{A}_{ij}, \mathfrak{A}_{kl}\rangle=0 \qquad ((i, j)\neq(k, l)). \tag{2.6}$$

We now put

$$e_i=\frac{1}{2\sqrt{n_i}}e_{ii}\in\mathfrak{A}_{ii} \qquad (1\leqq i\leqq r). \tag{2.7}$$

Then

$$\|e_i\|=1. \tag{2.8}$$

Here, $\|a\|$ denotes the norm of an arbitrary element $a\in\mathfrak{t}$ with respect to the inner product $\langle,\rangle$.

The connection function α and the curvature tensor R for the canonical Riemannian metric g_V are described in terms of the Lie algebra $\mathfrak{t}$ and the inner product $\langle,\rangle$ as follows (cf. Nomizu [4]):

$$\alpha: \mathfrak{t}\times\mathfrak{t}\longrightarrow\mathfrak{t},$$

$$2\langle\alpha(a, b), c\rangle=\langle[c, a], b\rangle+\langle a, [c, b]\rangle+\langle[a, b], c\rangle$$

and

$$R: \mathfrak{t}\times\mathfrak{t}\times\mathfrak{t}\longrightarrow\mathfrak{t},$$

$$R(a, b, c)=R(a, b)c=\alpha(a, \alpha(b, c))-\alpha(b, \alpha(a, c))-\alpha([a, b], c) \tag{2.9}$$

for every $a, b, c \in \mathfrak{t}$. The multiplication $\square$ in X defined in § 1 determines a multiplication $\circ$ in $\mathfrak{t}$ via the isomorphism ξ (cf. (2.3)) as follows:

$$a \circ b = \xi^{-1}(\xi(a) \square \xi(b))$$

for every $a, b \in \mathfrak{t}$. Then it is known that the identity

$$a \circ b = \frac{1}{2}(\xi^{-1}(\xi(a)\xi(b) + \xi(b)\xi(a))) \tag{2.10}$$

holds for every $a, b \in \mathfrak{t}$ (cf. Theorem 3 in p. 389 of [13]). In the present note, the algebra $(\mathfrak{t}, \circ)$ thus obtained is called the *connection algebra* of $V = V(\mathfrak{A})$. It is known in Proposition 1 of Shima [8] that the curvature tensor R has the following expression:

$$R(a, b, c) = b \circ (a \circ c) - a \circ (b \circ c) \tag{2.11}$$

for every $a, b, c \in \mathfrak{t}$.

§ 3. Power-associativity.

In this section, $(\mathfrak{t}, \circ)$ always denotes the connection algebra of a homogeneous convex cone $V = V(\mathfrak{A})$ in $X(\mathfrak{A})$ given in § 2. By making use of the results obtained in [9], [10] and [11], we will calculate a condition for the connection algebra $(\mathfrak{t}, \circ)$ to be power-associative in terms of the curvature tensor R.

It is known in Albert [1] that a commutative algebra $(A, \circ)$ over $\boldsymbol{R}$ is power-associative if and only if the identity

$$(a \circ a) \circ (a \circ a) = a \circ (a \circ (a \circ a)) \tag{3.1}$$

holds for every $a \in A$. Therefore, by (2.11) and (3.1), the connection algebra $(\mathfrak{t}, \circ)$ is power-associative if and only if the identity

$$R(a \circ a, a, a) = 0 \tag{3.2}$$

holds for every $a \in \mathfrak{t}$.

From now on, we will prove two lemmas on the necessary conditions for the connection algebra to be power-associative. We first prove the following

LEMMA 1. *If the connection algebra $(\mathfrak{t}, \circ)$ is power-associative, then the equality $n_i = n_j$ holds for every pair (i, j) of indices $i < j$ satisfying $n_{ij} \neq 0$.*

PROOF. By (2.3) and (2.10), we have

$$a \circ a = \xi^{-1}(\xi(a)\xi(a)) = \xi^{-1}((a + a^*)(a + a^*)) \tag{3.3}$$

for every $a \in \mathfrak{t}$. Putting $a = a_{ij}$ $(\neq 0)$ in (3.3), we have

$$a \circ a = \frac{1}{2}(a_{ij}a_{ij}^* + a_{ij}^* a_{ij}) \in \mathfrak{A}_{ii} + \mathfrak{A}_{jj} .$$

By (2.4) and (2.7), we have

$$\langle a_{ij}a_{ij}^*, e_i \rangle = 4\,\mathrm{Sp}((a_{ij}a_{ij}^*)e_i) = \frac{2}{\sqrt{n_i}}\mathrm{Sp}(a_{ij}a_{ij}^*) = \frac{1}{\sqrt{n_i}}\|a_{ij}\|^2$$

and

$$\langle a_{ij}^* a_{ij}, e_j \rangle = \frac{1}{\sqrt{n_j}}\|a_{ij}\|^2 .$$

By using the formulas (1) in Lemmas 3.1 or 3.2 of [10] and the formula (2.9), we get

$$R(e_i, a_{ij}, a_{ij}) = \frac{1}{4}\|a_{ij}\|^2\Big(\frac{1}{\sqrt{n_i n_j}}e_j - \frac{1}{n_i}e_i\Big) \tag{3.4}$$

and

$$R(e_j, a_{ij}, a_{ij}) = \frac{1}{4}\|a_{ij}\|^2\Big(\frac{1}{\sqrt{n_i n_j}}e_i - \frac{1}{n_j}e_j\Big) .$$

Therefore, by the condition (2.8), we have

$$\begin{aligned} R(a \circ a, a, a) &= \frac{1}{2}(\langle a_{ij}a_{ij}^*, e_i \rangle R(e_i, a, a) + \langle a_{ij}^* a_{ij}, e_j \rangle R(e_j, a, a)) \\ &= \frac{1}{2}\|a\|^2\Big(\frac{1}{\sqrt{n_i}}R(e_i, a_{ij}, a_{ij}) + \frac{1}{\sqrt{n_j}}R(e_j, a_{ij}, a_{ij})\Big) . \end{aligned}$$

From this and (3.2), we get

$$R(a \circ a, a, a) = \frac{1}{8}\|a\|^4\Big(\frac{1}{n_j} - \frac{1}{n_i}\Big)\Big(\frac{1}{\sqrt{n_i}}e_i - \frac{1}{\sqrt{n_j}}e_j\Big) = 0 ,$$

which means $n_i = n_j$. q.e.d.

We next show the following

LEMMA 2. *If the connection algebra* $(\mathfrak{t}, \circ)$ *is power-associative, then the following two identities hold*:

(1) $\|a_{ij}^* a_{ik}\|^2 = \dfrac{1}{2n_i}\|a_{ij}\|^2 \|a_{ik}\|^2$

and

(2) $\|a_{ik}a_{jk}^*\|^2 = \dfrac{1}{2n_k}\|a_{jk}\|^2 \|a_{ik}\|^2$

for every $a_{ij} \in \mathfrak{A}_{ij}$, $a_{jk} \in \mathfrak{A}_{jk}$ *and* $a_{ik} \in \mathfrak{A}_{ik}$ $(i<j<k)$.

PROOF. We first show the identity (1). Since the equality in (1) holds trivially for the case of $n_{ij}n_{ik} = 0$, we may assume that $n_{ij}n_{ik} \neq 0$. By Lemma 1, we can put $n_i=n_j=n_k=m$. Let us put $a=a_{ij}+a_{ik}$ in (3.3). Then, by (2.3) and (2.10), we have

$$a\circ a=x_{ii}+x_{jj}+x_{kk}+x_{jk}\,,$$

where

$$x_{ii}=\frac{1}{2}(a_{ij}a_{ij}^*+a_{ik}a_{ik}^*)\,,\qquad x_{jj}=\frac{1}{2}a_{ij}^*a_{ij}\,,$$

$$x_{kk}=\frac{1}{2}a_{ik}^*a_{ik}\quad\text{and}\quad x_{jk}=a_{ij}^*a_{ik}\,.$$

Similarly as in the proof of Lemma 1, we have

$$x_{ii}=\frac{1}{2\sqrt{m}}\|a\|^2e_i\quad\text{and}\quad x_{pp}=\frac{1}{2\sqrt{m}}\|a_{ip}\|^2e_p\quad(p=j,\ k)\,. \tag{3.5}$$

We now consider the $\mathfrak{A}_{ii}$-component of $R(a\circ a,\ a,\ a)$. Using a well-known identity on the curvature tensor (cf. the formula (1.14) of [11]), we get

$$\langle R(a\circ a,\ a,\ a),\ e_i\rangle=-\langle R(a\circ a,\ a,\ e_i),\ a\rangle\,.$$

From the condition (1.12) of [11] and the formula (2.9), it follows that the identity

$$\begin{aligned}R(a\circ a,\ a,\ e_i)=&R(x_{ii},\ a_{ij},\ e_i)+R(x_{ii},\ a_{ik},\ e_i)+R(x_{jj},\ a_{ij},\ e_i)\\&+R(x_{kk},\ a_{ik},\ e_i)+R(x_{jk},\ a_{ij},\ e_i)+R(x_{jk},\ a_{ik},\ e_i)\end{aligned}$$

holds. On the other hand, by using Lemmas 1.1 and 2.2 of [9], the formulas (2.9) and (3.5), we obtain the following formulas:

$$R(x_{ii},\ a_{ip},\ e_i)=\frac{1}{8m\sqrt{m}}\|a\|^2a_{ip}$$

and

$$R(x_{pp},\ a_{ip},\ e_i)=\frac{-1}{8m\sqrt{m}}\|a_{ip}\|^2a_{ip}\qquad(p=j,\ k)\,.$$

Furthermore, we have

$$R(x_{jk},\ a_{ij},\ e_i)=\frac{-1}{4\sqrt{m}}a_{ij}x_{jk}=\frac{-1}{4\sqrt{m}}a_{ij}(a_{ij}^*a_{ik})$$

and

$$R(x_{jk},\ a_{ik},\ e_i)=\frac{-1}{4\sqrt{m}}a_{ik}x_{jk}^*=\frac{-1}{4\sqrt{m}}a_{ik}(a_{ik}^*a_{ij})$$

(cf. the condition (1.14) of [11] and the formula used in the proof of Proposition 5.1 of [11]). Hence, from the conditions (2.5) and (2.6), it follows that

$$\langle R(a\circ a,\ a,\ a),\ e_i\rangle=\frac{1}{2\sqrt{m}}\left(\|a_{ij}^*a_{ik}\|^2-\frac{1}{2m}\|a_{ij}\|^2\|a_{ik}\|^2\right)$$

holds. From this, we have the equality (1).

We proceed to showing the equality (2). Similarly as in the above case, we may assume that $n_{jk}n_{ik}\neq 0$ and also we may put $n_i=n_j=n_k=m$. By putting $a=a_{jk}+a_{ik}$, we have

$$a\circ a=x_{ii}+x_{jj}+x_{kk}+x_{ij},$$

where

$$x_{ii}=\frac{1}{2\sqrt{m}}\|a_{ik}\|^2e_i, \qquad x_{jj}=\frac{1}{2\sqrt{m}}\|a_{jk}\|^2e_j,$$

$$x_{kk}=\frac{1}{2\sqrt{m}}\|a\|^2e_k \quad\text{and}\quad x_{ij}=a_{ik}a_{jk}^*.$$

Similarly as in the above case, we have

$$R(a\circ a,\ a,\ e_k)=\frac{1}{2\sqrt{m}}\|a_{ik}\|^2R(e_i,\ a_{ik},\ e_k)+\frac{1}{2\sqrt{m}}\|a_{jk}\|^2R(e_j,\ a_{jk},\ e_k)$$
$$+\frac{1}{2\sqrt{m}}\|a\|^2(R(e_k,\ a_{jk},\ e_k)+R(e_k,\ a_{ik},\ e_k))$$
$$+R(x_{ij},\ a_{jk},\ e_k)+R(x_{ij},\ a_{ik},\ e_k)\,.$$

By using the following formulas (cf. Lemmas 1.1 and 2.2 of [9] and the condition (2.9)):

$$R(e_i,\ a_{ik},\ e_k)=-R(e_k,\ a_{ik},\ e_k)=\frac{-1}{4m}a_{ik},$$

$$R(x_{ij},\ a_{jk},\ e_k)=\frac{-1}{4\sqrt{m}}x_{ij}a_{jk}=\frac{-1}{4\sqrt{m}}(a_{ik}a_{jk}^*)a_{jk}$$

and

$$R(x_{ij},\ a_{ik},\ e_k)=\frac{-1}{4\sqrt{m}}x_{ij}^*a_{ik}=\frac{-1}{4\sqrt{m}}(a_{jk}a_{ik}^*)a_{ik},$$

we have

$$\langle R(a\circ a,\ a,\ a),\ e_k\rangle=\frac{1}{2\sqrt{m}}\left(\|a_{ik}a_{jk}^*\|^2-\frac{1}{2m}\|a_{ik}\|^2\|a_{jk}\|^2\right).$$

Therefore, by (3.2), $\|a_{ik}a_{jk}^*\|^2=(1/2m)\|a_{ik}\|^2\|a_{jk}\|^2$ holds. q.e.d.

§4. Main result.

In this section, we prove the theorem stated in §1 by making use of the lemmas obtained in §3.

We now have the following

THEOREM 1. *If the connection algebra of a homogeneous convex cone V is power-associative, then V is self-dual.*

Proof. By the result of Vinberg [13] recalled in § 2, we can assume that V is realized as the cone $V(\mathfrak{A})$ in terms of a T-algebra $\mathfrak{A}=\sum_{1\leq i,j\leq r}\mathfrak{A}_{ij}$. We first show that the equality $n_{ik}=n_{jk}$ holds for every triple (i, j, k) of indices $i<j\neq k\neq i$ satisfying the condition $n_{ij}\neq 0$. In fact, let us consider the case of $i<j<k$. Then, by (1) of Lemma 2, the linear mapping: $x\in\mathfrak{A}_{ik}\rightarrow a_{ij}^{*}x\in\mathfrak{A}_{jk}$ is injective for an arbitrary non-zero element $a_{ij}\in\mathfrak{A}_{ij}$. Hence, we have $n_{ik}\leqq n_{jk}$. Combining this with the condition (2.2), we get the equality $n_{ik}=n_{jk}$. We proceed to the case of $i<k<j$. By (1) and (2) of Lemma 2, we can see that both of the linear mappings:

$$x\in\mathfrak{A}_{ik}\longrightarrow x^{*}a_{ij}\in\mathfrak{A}_{kj} \quad \text{and} \quad y\in\mathfrak{A}_{kj}\longrightarrow a_{ij}y^{*}\in\mathfrak{A}_{ik}$$

are injective for every non-zero element $a_{ij}\in\mathfrak{A}_{ij}$. Therefore, we have the equality $n_{ik}=n_{jk}$. Finally, we consider the case of $k<i<j$. Similarly as in the above cases, by using (2) of Lemma 2, we can easily see that the equality $n_{ik}=n_{jk}$ holds in this case. Therefore, the kernel of the T-algebra $\mathfrak{A}$ coincides with $\mathfrak{A}$ (cf. p. 69 of Vinberg [14] or Lemma 2.2 of [11]). On the other hand, it is known in [14] that $V=V(\mathfrak{A})$ is self-dual if and only if the kernel of $\mathfrak{A}$ coincides with $\mathfrak{A}$. Hence, V is self-dual. q.e.d.

Several characterizations of homogeneous self-dual cones are known. Combining the result obtained above with them, we can state the following

Theorem 2. *For a homogeneous convex cone* V *in* $X=\boldsymbol{R}^{n}$, *the following six conditions are equivalent:*

(1) *The connection algebra of* V *is power-associative.*

(2) V *is self-dual.*

(3) *The connection algebra of* V *is Jordan.*

(4) V *is Riemannian symmetric with respect to the canonical metric* g_{V}.

(5) *The tube domain* $D(V)=\{z\in\boldsymbol{C}^{n};\ \mathrm{Im}\, z\in V\}$ *is Hermitian symmetric with respect to the Bergman metric of* $D(V)$.

(6) *The level surface of the characteristic function of* V *is Riemannian symmetric with respect to the metric induced from* (V, g_{V}).

In fact, the implications (2)→(3)→(1) have been proved by [3] and (4)→(2) has been obtained in [8], [9] or [11]. It is known in [5], [6] that the conditions (2) and (5) are equivalent and the condition (2) implies the condition (4). The implications (4)↔(6) are found in [10]. By Theorem 1, we have the implication (1)→(2) (For (3)→(2), see also [2].), and so the conditions stated above are mutually equivalent.

THEOREM 3. *For a homogeneous convex cone V in $\boldsymbol{R}^n$ $(n\geqq 2)$, the following three conditions are equivalent.*

(1) *The connection algebra of V is associative.*

(2) *The curvature tensor for the canonical metric g_V is identically zero.*

(3) *V is linearly isomorphic to the product cone of the half-lines of positive real numbers.*

PROOF. As was stated in § 2, we can assume that V is realized as the cone $V(\mathfrak{A})$ by means of a T-algebra $\mathfrak{A}=\sum_{1\leq i,j\leq r}\mathfrak{A}_{ij}$ of rank r. The implications (1)↔(2) follow from the formula due to Shima [8] recalled by (2.11). The condition (3) implies that V is isometric to the product Riemannian manifold of the half-lines of positive real numbers. Hence, we get (3)→(2). By the formula (3.4) in the proof of Lemma 1, we can see that the condition (2) implies $n_{ij}=0$ for every pair (i, j) of indices $1\leqq i<j\leqq r$. Hence, $\mathfrak{A}=\mathfrak{A}_{11}+\mathfrak{A}_{22}+\cdots+\mathfrak{A}_{rr}$. From this and the construction theorem of homogeneous convex cones due to Vinberg [13] recalled in § 2, it follows that the implication (2)→(3) holds. q.e.d.

References

[1] Albert, A. A., On the power-associativity of rings, Summa Brasil. Math. **2** (1948), 21-32.

[2] Dorfmeister, J., Inductive construction of homogeneous cones, Trans. Amer. Math. Soc. **252** (1979), 321-349.

[3] Koecher, M., Jordan Algebras and their Applications, Lect. Notes, Univ. of Minnesota, 1962.

[4] Nomizu, K., Invariant affine connections on homogeneous spaces, Amer. J. Math. **76** (1954), 33-65.

[5] Rothaus, O. S., Domains of positivity, Abh. Math. Sem. Univ. Hamburg **24** (1960), 189-235.

[6] Rothaus, O. S., The construction of homogeneous convex cones, Ann. of Math. **83** (1966), 358-376.

[7] Schafer, R. D., An Introduction to Nonassociative Algebras, Acad. Press, New York, 1966.

[8] Shima, H., A differential geometric characterization of homogeneous self-dual cones, to appear.

[9] Tsuji, T., A characterization of homogeneous self-dual cones, to appear in Tokyo J. Math. **5** (1982).

[10] Tsuji, T., On homogeneous convex cones of non-positive curvature, to appear.

[11] Tsuji, T., On infinitesimal isometries of homogeneous convex cones, to appear.

[12] Vinberg, E. B., Homogeneous convex cones, Soviet Math. Dokl. **1** (1961), 787-790.

[13] Vinberg, E. B., The theory of convex homogeous cones, Trans. Moscow Math. Soc. **12** (1963), 340-403.

[14] Vinberg, E. B., The structure of the group of automorphisms of a homogeneous convex cone, Trans. Moscow Math. Soc. **13** (1965), 63-93.

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