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VARIOUS COMPACT MULTI-RETRACTS AND SHAPE THEORY

By

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1. Introduction.

Recently Suszycki [22] defined the notion of multi-retractions on compact metric spaces and considered interesting properties. The author [15] extended that notion to the case of metric spaces and announced some properties related to shape theory. First the notion of multi-retractions resulted from inverses of CE-maps. But in shape theory we studied various kinds of Vietoris-type maps. Then in this paper we shall define notions of various multi-valued functions and consider related topics.

Throughout this paper we assume that all spaces are metrizable and all maps are continuous. AR and ANR mean those for metric spaces. Dimension means covering dimension and by $\dim X$ we denote the covering dimension of a space X.

Let X and Y be spaces. By a *multi-valued function* $\varphi: Y\rightarrow Y$ we mean a function assigning to each point $x\in X$ a non-empty closed subset $\varphi(x)$ of Y. A multi-valued function $\varphi: X\rightarrow Y$ is *compact* if $\varphi(x)$ is compact for every $x\in X$. A multi-valued function $\varphi: X\rightarrow Y$ is said to be *upper semi-continuous* (shortly u. s. c.) provided for each point $x\in X$ and for each neighborhood V of $\varphi(x)$ in Y there exists a neighborhood U of x in X such that $\varphi(U)=\cup\{\varphi(z)\mid z\in U\}\subset V$. For a multi-valued function $\varphi: X\rightarrow Y$, the *graph* of φ is defined as follows

$$\Phi=\{(x, y)\in X\times Y\mid y\in\varphi(x),\ x\in X\}.$$

And let $p:\Phi\rightarrow X$ and $q:\Phi\rightarrow Y$ be the natural projections. Then if a multi-valued function $\varphi: X\rightarrow Y$ is u. s. c., the graph Φ of φ is closed in $X\times Y$. Moreover if φ is compact, then the natural projection $p:\Phi\rightarrow X$ is a proper map.

For each $n=0, 1, 2, 3, \cdots, \infty$ we say that an u. s. c. compact multi-valued function $\varphi: X\rightarrow Y$ is a *compact n-multi-map* (shortly a *c-n-multi-map*) if $\varphi(x)$ is AC^{n} (see [3] or [7]) for every $x\in X$. Moreover if $\varphi(x)$ has the trivial shape (see [3] or [7]) for every $x\in X$, then we simply call a *compact multi-map* shortly a *c-multi-map*.

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It is clear that on compact metric spaces our definition of a *c-multi-map* agrees with Suszycki's one of a *multi-map* [22].

A space X is said to be *countable dimensional* if X can be represented as the union of a countable number of zero-dimensional subspaces. A space X is said to have the *property C* (to be a *C-space*) if for every sequence $\{\mathfrak{U}_i\}_{i\geq 1}$ of open covers of X there is a sequence $\{\mathfrak{V}_i\}_{i\geq 1}$ of collections of pairwise disjoint open subsets of X such that family $\bigcup_{i\geq 1}\mathfrak{V}_i$ is a cover of X and $\mathfrak{V}_i$ refines $\mathfrak{U}_i$ for each $i\geqq 1$. The notion of C-spaces was originally defined by Haver [11] and studied further by Addis and Gresham [1]. It is well-known that a countable dimensional space is a C-space (see [1] Corollary 2.10 or [2] Lemma 3.3). Hence it seems to us that the class of all C-spaces is sufficiently wide. But we remark that by the example of Pol [21] the converse of the assertion is not valid (see [9] Example 8.18). The property C plays an important part in ANR theory and shape theory.

We refer readers to [3] and [7] for shape theory.

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2. Shape morphisms induced by c-multi-maps.

Let $\varphi: X\to Y$ be a c-multi-map from a C-space X to a space Y. Let Φ be the graph of φ and let $p: \Phi\to X$ and $q: \Phi\to Y$ be the natural projections. Now p is a CE-map, because φ is a c-multi-map. Since X has the property C, by [2] Corollary 5.3, and remarks below the Main Theorem 3.2, p is a hereditary shape equivalence (see [7] or [17]). Hence we can define a shape morphism $S(q)\circ S(p)^{-1}: X\to Y$, where $S(f)$ is the shape morphism induced by a map f. Then we shall call $S(q)\circ S(p)^{-1}$ the *shape morphism induced by* φ and denote by $S(\varphi): X\to Y$ (cf. [13]).

2.1. THEOREM. *Let $\varphi: X\to Y$ be a c-multi-map from a C-space X to a space Y. If there exists a map $g: Y\to X$ such that $y\in\varphi(g(y))$ for every $y\in Y$, then $S(\varphi): X\to Y$ is a shape domination. Therefore $Sh(X)\geqq Sh(Y)$.*

PROOF. Let Φ be the graph of φ and let $p: \Phi\to X$ and $q: \Phi\to Y$ be the natural projections. Define the map $h: Y\to\Phi$ by $h(y)=(g(y), y)$ for each $y\in Y$. Then $q\circ h=id_Y$. Hence $S(\varphi)\circ(S(p)\circ S(h))=S(q)\circ S(p)^{-1}\circ S(p)\circ S(h)=S(q)\circ S(h)=S(id_Y)$. Therefore $S(\varphi)$ is a shape domination.

2.2. COROLLARY. *Under the hypothesis of Theorem* 2.1 *if X satisfies a here-*

ditary shape preperty (P), for example, MAR, MANR, movability, $Sd(X)\leqq n, \cdots$, *etc, then Y also satisfies (P).*

We shall show that the property C of X is essential in Theorem 2.1 and Corollary 2.2.

2.3. EXAMPLE. Let $f: Y\rightarrow Q$ be the Taylor's cell-like map from a non-movable continuum Y onto the Hilbert cube Q [23]. Then let X be the mapping cylinder $(Y\times[0, 1]\cup Q)/\sim$ of f, where $\sim$ identifies $(y, 1)$ with $f(y)$ for each point $y\in Y$. It is clear that X is an FAR. Since X contains Q, by [1] Corollary 3.3, X is not a C-space. Moreover we define a c-multi-map $\varphi: X\rightarrow Y$ as follows

$$\varphi([y, t])=\{y\} \qquad \text{for every } (y, t)\in Y\times[0, 1), \text{ and}$$

$$\varphi([z])=f^{-1}(z) \qquad \text{for every } z\in Q .$$

Defining the map $g: Y\rightarrow X$ by $g(y)=[y, 0]$ for every $y\in Y$, we have that $y\in \varphi(g(y))$ for every $y\in Y$. But $Sh(X)\ngeqq Sh(Y)$, because Y is non-movable.

Let (X, x_0) and (Y, y_0) be pointed spaces with given base points x_0 and y_0, respectively. Then we write $\varphi: (X, x_0)\rightarrow(Y, y_0)$ if φ is a c-multi-map and $y_0\in \varphi(x_0)$. For two c-multi-maps $\varphi_0, \varphi_1: (X, x_0)\rightarrow(Y, y_0)$ if there exists a c-multi-map $\chi: X\times[0, 1]\rightarrow Y$ such that $\chi|X\times\{0\}=\varphi_0$, $\chi|X\times\{1\}=\varphi_1$ and $y_0\in\chi(x_0, t)$ for every $t\in[0, 1]$, we say that φ_0 and φ_1 are *compact multi-homotopic* (shortly *c-multi-homotopic*) and we denote $\varphi_0 \overset{m_c}{\simeq} \varphi_1$. Then we call χ the *compact multi-homotopy* (shortly *c-multi-homotopy*) connecting φ_0 and φ_1.

It is clear that the relation of the c-multi-homotopy is an equivalence relation on the set of all c-multi-maps from (X, x_0) to (Y, y_0). We write $[\varphi]$ the equivalence class of a c-multi-map φ. By $M((X, x_0), (Y, y_0))$ we denote the set of all those equivalence classes.

On unpointed spaces we do not require the condition of base point preserving, thus we can define the notation of unpointed c-multi-homotopy and the set $M(X, Y)$ of unpointed classes. On compact metric spaces our definition of c-multi-homotopy agrees with Suszycki's definition of multi-homotopy [22].

We remark that every two homotopic maps from (X, x_0) to (Y, y_0) are c-multi-homotopic but the converse is not valid (see [22] Example 3.2).

For each $n=0, 1, 2, \cdots, \infty$ we can similarly define the relation of *compact n-multi-homotopy* (shortly *c-n-multi-homotopy*) of pointed and unpointed c-n-multi-maps.

2.4. THEOREM. *Let* φ_0 *and* φ_1 *be c-multi-maps from a C-space X to a space*

Y. *If* $\varphi_0 \overset{m_c}{\simeq} \varphi_1$, *then* $S(\varphi_0)=S(\varphi_1)$.

PROOF. Let $\chi: X\times[0, 1]\to Y$ be a c-multi-homotopy connecting φ_0 and φ_1. Let Φ be the graph of χ and let $p: \Phi\to X\times[0, 1]$ and $q: \Phi\to Y$ be the natural projections. Then by [1] Corollary 2.24 $X\times[0, 1]$ is a C-space. Hence we can define the shape morphism $S(\chi)=S(q)\circ S(p)^{-1}: X\times[0, 1]\to Y$. For $k=0, 1$ let $e_k: X\to X\times[0, 1]$ be the embedding defined by $e_k(x)=(x, k)$ for each $x\in X$. Defining $\Phi_k=\Phi\cap(X\times\{k\}\times Y)=p^{-1}(X\times\{k\})$, we can identify the graph of $\varphi_k=\chi\circ e_k$ with Φ_k. Since p is a hereditary shape equivalence, $p_k=p|\Phi_k: \Phi_k|X\times\{k\}$ is a shape equivalence and by the definition $S(\varphi_k)=S(q_k)\circ S(p_k)^{-1}\circ S(e_k): X\to Y$, where $q_k=q|\Phi_k: \Phi_k\to Y$. Let $i_k: X\times\{k\}\to X\times[0, 1]$ and $j_k: \Phi_k\to\Phi$ be the inclusion maps. Since $i_k\circ p_k=p\circ j_k$ and i_k is a shape equivalence, j_k is a shape equivalence. Hence $S(\varphi_k)=S(q_k)\circ S(p_k)^{-1}\circ S(e_k)=S(q)\circ S(j_k)\circ S(j_k)^{-1}\circ S(p)^{-1}\circ S(i_k)\circ S(e_k)=S(q)\circ S(p)^{-1}\circ S(i_k\circ e_k)=S(\chi)\circ S(i_k\circ e_k)$ for each $k=0, 1$. Since $i_0\circ e_0\simeq i_1\circ e_1$, $S(i_0\circ e_0)=S(i_1\circ e_1)$. Therefore $S(\varphi_0)=S(\varphi_1)$. We complete the proof of Theorem 2.4.

$$\begin{array}{ccccccc}
 & & X\times[0,1] & \xleftarrow{\;p\;} & \Phi & \xrightarrow{\;q\;} & Y \\
 & & \uparrow{\scriptstyle i_k} & & \uparrow{\scriptstyle j_k} & & \Vert \\
X & \xrightarrow{\;e_k\;} & X\times\{k\} & \xleftarrow{\;p_k\;} & \Phi_k & \xrightarrow{\;q_k\;} & Y.
\end{array}$$

For spaces X and Y we denote the set of all shape morphisms from X to Y by $Sh(X, Y)$. If Y is an ANR, every shape morphism from X to Y is generated by a map from X to Y. Hence we have the following.

2.5. COROLLARY. *If X is a C-space, for an arbitrary space Y the correspondence S induces a function from $M(X, Y)$ to $Sh(X, Y)$. Moreover if Y is an ANR, S is surjective.*

Let $O_{x_0}: (X, x_0)\to(X, x_0)$ be the constant map to x_0. We say that (X, x_0) is *compact multi-contractible* (shortly *c-multi-contractible*) if $O_{x_0} \overset{m_c}{\simeq} id_{(X, x_0)}$. If (X, x_0) is c-multi-contractible for every $x_0\in X$, X is simply said to be *compact multi-contractible* (shortly *c-multi-contractible*). For each $n=1, 2, \cdots, \infty$ we can similarly define the notation of *compact n-multi-contractibility* (shortly *c-n-multi-contractibility*). In the case of compact metric spaces our definition of c-multi-contractibility agrees with Suszycki's definition of multi-contractibility (see [22]).

2.6. COROLLARY. *If C-space X is c-multi-contractible, then X has the trivial shape. Therefore X is an MAR.*

Since there is a c-multi contractible compact space which is not an FAR (see Remark 4.16 and 4.18), the property C of X is essential in Corollary 2.6. But it is unknown whether the converse of Corollary 2.6 is valid. We remark that every FAR c-multi-contractible (see [22] 3.9).

PROBLEM 1. *Is every MAR c-multi-contractible?*

Next we shall consider the pointed version. Let $\varphi:(X, x_0)\to(Y, y_0)$ be a pointed c-multi-map from a compact C-space X to a compact space Y. Then the graph Φ of φ is compact and $(x_0, y_0)\in\Phi$. Let $p:\Phi\to X$ and $q:\Phi\to Y$ be the natural projections. Then $p(x_0, y_0)=x_0$ and $q(x_0, y_0)=y_0$. Since p is a hereditary shape equivalence, by [8] Theorem 7.10 and Corollary 4.6, $p:(\Phi,(x, y_0))\to(X, x_0)$ is a fine shape equivalence.*) Hence we can define the fine shape morphism $S_f(q)\circ S_f(p)^{-1}:(X, x_0)\to(Y, y_0)$, where $S_f(g)$ is the fine shape morphism induced by a map g. Then we shall call $S_f(q)\circ S_f(p)^{-1}$ the *fine shape morphism induced by* and denoted by $S_f(\varphi):(X, x_0)\to(Y, y_0)$. By the same way as Theorem 2.1 we can prove the following.

2.7. THEOREM. *Let $\varphi:(X, x_0)\to(Y, y_0)$ by a c-multi-map from a compact C-space X to a compact space Y. If there exists a map $g:(Y, y_0)\to(X, x_0)$ such that $y\in\varphi(g(y))$ for every $y\in Y$, then $S_f(\varphi):(X, x_0)\to(Y, y_0)$ is a fine shape domination. Therefore $Sh_f(X, x_0)\geqq Sh_f(Y, y_0)$, especially $Sh(X, x)\geqq Sh(Y, y)$.*

2.8. COROLLARY. *Under the hypothesis of Theorem 2.7 if X satisfies a pointed hereditary (fine) shape property (P), for example, pointed FANR, pointed (n-) movability, fine (n-) movability, ..., etc, then Y also satisfies (P).*

By Example 2.3 the property C of X is essential in Theorem 2.7 and Corollary 2.8. By slight modifications using the result of [4], we can prove the pointed version of Theorem 2.4 and Corollary 2.5. Here we leave readers the detail of proofs.

2.9. THEOREM. *Let (X, x_0) be a pointed compact C-space and (Y, y_0) a*

*) Fine shape theory defined in [14] is equivalent to strong shape theory defined in [8]. In this paper we shall use the terminology "fine shape."

pointed compact space. Then the correspondence S_f induces a function from $M((X, x_0), (Y, y_0))$ to $Sh_f((X, x_0), (Y, y_0))$, where $Sh_f((X, x_0), (Y, y_0))$ is the set of all fine shape morphisms from (X, x_0) to (Y, y_0).

Moreover if Y is an ANR, S_f is surjective.

For a c-multi-map $\varphi: (S^n, s_0)\rightarrow(X, x_0)$, where X is compact, Suszycki also defined the induced shape morphism $[\underline{a}^{\varphi}]: (S^n, s_0)\rightarrow(X, x_0)$ (see [22] Theorem 4.4). His method was essentially depend on the infinite-dimensional manifold theory. But we can easily see that $[\underline{a}^{\varphi}]=S(\varphi)$. Our method is simpler than his one and can be applied to the non-compact case.

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3. Algebraic properties of *c-n*-multi-maps.

In the section 1 under the hypothesis that X is a C-space we considered (fine) shape morphisms induced by a c-multi-map $\varphi: X\rightarrow Y$. In this section without dimension-theoritic assumptions we shall consider some algebraic properties of c-n-multi-maps. Let $\mathfrak{G}$ be the category of groups and homomorphisms.

Let $\varphi: X\rightarrow Y$ be a c-n-multi-map, where $n=0, 1, 2, \cdots, \infty$. Let Φ be the graph of φ and let $p: \Phi\rightarrow X$ and $q: \Phi\rightarrow Y$ be the natural projections. Then p is a proper map and $p^{-1}(x)\in AC^n$ for every $x\in X$. For every integer k, $0\leqq k\leqq n$, by Vietoris theorem in shape theory (see [6] or [20]) p induces isomorphisms

$$\text{pro-}\pi_k(p): \text{pro-}\pi_k(\Phi, (x, y))\cong\text{pro-}\pi_k(X, x),$$

$$\text{pro-}H_k(p): \text{pro-}H_k(\Phi; G)\cong\text{pro-}H_k(X; G) \quad\text{and}$$

$$\check{H}^k(p): \check{H}^k(X: G)\cong\check{H}^k(\Phi; G),$$

where $(x, y)\in\Phi$ and G is an abelian groups.

Here for each integer k, $0\leqq k\leqq n$, we can define homomorphisms

$$\text{pro-}\pi_k(q)\circ\text{pro-}\pi_k(p)^{-1}: \text{pro-}\pi_k(X, x)\longrightarrow\text{pro-}\pi_k(Y, y),$$

$$\text{pro-}H_k(q)\circ\text{pro-}H_k(p)^{-1}: \text{pro-}H_k(X; G)\longrightarrow\text{pro-}H_k(Y; G) \quad\text{and}$$

$$\check{H}^k(p)^{-1}\circ\check{H}^k(q): \check{H}^k(Y; G)\longrightarrow\check{H}^k(X, G),$$

where $(x, y)\in\Phi$ and G is an abelian groups.

We shall call those compositions *homomorphisms induced by* φ and denote by pro-$\pi_k(\varphi)$, pro-$H_k(\varphi)$ and $\check{H}^k(\varphi)$, respectively.

3.1. THEOREM. *Let* $\varphi: X \to Y$ *be a c-n-multi-map, where* $n=0, 1, 2, \cdots, \infty$. *If there exists a map* $g: Y \to X$ *such that* $y \in \varphi(g(y))$ *for all* $y \in Y$, *then homomorphisms* pro-$\pi_k(\varphi)$, pro-$H_k(\varphi)$ *and* $\check{H}^k(\varphi)$ *induced by* φ *are dominations in suitable categories.*

PROOF. Let Φ be the graph of φ and let $p: \Phi \to X$ and $q: \Phi \to Y$ be natural projections. Difine the map $h: Y \to \Phi$ by $h(y)=(g(y), y)$ for each $y \in Y$. Then $q \circ h = id_Y$. Hence for every non-negative integer k pro-$\pi_k(q)$, pro-$H_k(q)$ and $\check{H}^k(q)$ are dominations in suitable categories. Therefore for every integer k, $0 \leqq k \leqq n$, pro-$\pi_k(\varphi)$, pro-$H_k(\varphi)$ and $\check{H}^k(\varphi)$ are dominations in suitable categories.

At the latter part of this section, unless the contrary is specifically indicated, we assume that for spaces X and Y there exist a c-n-multi-map $\varphi: X \to Y$, where $n=0, 1, 2, \cdots, \infty$, and a map $g: Y \to X$ such that $y \in \varphi(g(y))$ for every $y \in Y$.

3.2. COROLLARY. *For an integer* k, $0 \leqq k \leqq n$, *if* $X \in AC^k$, *then* $Y \in AC^k$. *And if* φ *is a c-*∞*-multi-map and* X *is acyclic, then so is* Y.

3.3. COROLLARY. *For an integer* k, $1 \leqq k \leqq n$, *if* X *is a pointes* S^k*-movable continuum, then so is* Y.

PROOF. Since p is a proper map and q is a surjective map, Y is compact. Moreover by Corollary 3.1 Y is connected. That is, Y is a continuum. Now we fix any point $y \in Y$. Then by Theorem 3.1 pro-$\pi_m(Y, y) \cong$ pro-$\pi_m(X, g(y))$ in pro-$\mathfrak{G}$ for every $m=1, 2, \cdots, n$ and $m < \infty$. Since X is pointed S^k-movable, pro-$\pi_k(X, g(y))$ satisfies the Mittag-Leffler condition. Since $k \leqq n$, pro-$\pi_k(Y, y)$ also satisfies the Mittag-Leffler condition. Then by [16] (Y, y) is pointed S^k-movable. Therefore Y is a pointed S^k-movable continuum.

3.4. COROLLARY. *For an integer* k, $1 \leqq k \leqq n$, *if* pro-$\pi_k(X, x)$ *is stable in* pro-$\mathfrak{G}$, *then so is* pro-$\pi_k(Y, y)$ *for every* $y \in \varphi(x)$.

3.5. COROLLARY. *Let* X *be an FAR. If* φ *is a c-*∞*-multi-map and* Y *is movable, then* Y *is also an FAR.*

PROOF. Since X is FAR, X is an AC^∞ continuum. Hence by Corollary 3.2 and the proof of Corollary 3.3 Y is also an AC^∞ continuum. Then if Y is movable, by [18] Y is an FAR.

3.6. COROLLARY. *Let* X *be an FANR. If* $Fd(Y) < \infty$ *and* φ *is a c-*∞*-multi-map, then* Y *is also an FANR.*

PROOF. By the proof of Corollary 3.3 and Theorem 3.1 Y is compact and the number of all components of Y is finite. Hence we may assume that X and Y are continua. Let us fix a point $y \in Y$. Since $(X, g(y))$ is a pointed $FANR$ by [10], for every $k=1, 2, \cdots$ pro-$\pi_k(X, g(y))$ is stable in pro-$\mathfrak{G}$ and $\check{\pi}_k(X, g(y))$ is a countable group. They by Corollary 3.4 and Theorem 3.1 pro-$\pi_k(Y, y)$ is stable in pro-$\mathfrak{G}$ and $\check{\pi}_k(Y, y)$ is a countable group for every $k=1, 2, \cdots$. Hence since $Fd(Y)<\infty$, (Y, y) is a pointed $FANR$ (see [5] or [24]). Therefore Y is an $FANR$.

3.7. REMARK. By Example 2.3 the movability of Y and the being $Fd(Y)<\infty$ are essential in Corollary 3.5 and Corollary 3.6, respectively.

4. m_c^n-ANR, m_c-ANR, m_c^n-AR and m_c-AR.

Let Y be a subset of a space X. Then a c-n-multi-map $\varphi: X \to Y$, where $n=0, 1, 2, \cdots, \infty$, is said to be a *compact n-multi-retraction* (shortly a *c-n-multi-retraction*) of X onto Y provided $y \in \varphi(y)$ for every $y \in Y$. Similarly we call a c-multi-map $\varphi: X \to Y$ a *compact multi-retraction* (shortly a *c-multi-retraction*) of X onto Y provided $y \in \varphi(y)$ for every $y \in Y$. If there exists a c-n-multi-retraction (resp. c-multi-retraction) of X onto Y, then we say that Y is a *compact n-muli-retract* (resp. *compact multi-retract*) (shortly *c-n-multi-retract* (resp. *c-multi-retract*)) of X.

Obviously for every $0 \leqq n \leqq m \leqq \infty$ every m-multi-retraction of X onto Y is a c-n-multi-retraction. Every retraction of X onto Y is a c-multi-retraction. If there exists au u.s.c. compact multi-function $\varphi: X \to Y$ such that $y \in \varphi(y)$ for every $y \in Y$, Y is a closed subset of X. Therefore if Y is a c-0-multi-retract of X, Y is a closed subset of X.

Let Y be a subset of X. If there exist a neighborhood U of Y in X and c-n-multi-retraction (resp. c-multi-retraction) $\varphi: U \to Y$, then we way that Y is a *neighborhood compact n-multi-retract* (resp. *neighborhood compact multi-retract*) of X.

For $n=0, 1, 2, \cdots, \infty$ a space Y is said to be an *absolute neighborhood compact n-multi-retract* (shortly m_c^n-ANR) provided for every space M containing Y as a closed subset Y is a neighborhood compact n-multi-retract of M. If for every space M containing Y as a closed subset Y is a c-n-multi-retract of M, we say that Y is an *absolute compact multi-retract* (shortly m_c^n-ANR). Similarly by using notions of a neighborhood compact multi-retract and a compact multi-retract we can define notions of an *absolute neighborhood compact multi-retract* (shortly m_c-ANR) and an *absolute compact multi-retract* (shortly m_c-AR).

It is easily seen that our definitions are topological invariants. By definitions it is clear that for every $0 \leqq k \leqq n \leqq \infty$ every m_c^n-AR (resp. m_c^n-ANR) is an m_c^k-AR (resp. m_c^k-ANR) and every m_c-AR (resp. m_c-ANR) is an m_c^∞-AR (resp. m_c^∞-ANR). In the case of compact metric spaces our definitions of m_c-AR and m_c-ANR agree with Suszycki's definitions of m-AR and m-ANR (see [22]).

We easily have following properties, where $n=0, 1, 2, \cdots, \infty$ (see [22] 2.5–2.8).

4.1. *A space Y is an m_c^n-AR (resp. m_c-AR) if and only if Y is a c-n-multi-retract (resp. c-multi-retract) of every (equivalently some) AR-space N containing Y as a closed subset.*

4.2. *A space Y is an m_c^n-ANR (resp. m_c-ANR) if and only if Y is a neighborhood compact n-multi-retract (resp. neighborhood compact multi-retract) of every (equivalently some) ANR-space N containing Y as a closed subset.*

4.3. *A space Y is an m_c^n-AR (resp. m_c-AR) if and only if for every closed subset X of a space M and for every map $f: X \to Y$ there exists a c-n-multi-map (resp. c-multi-map) $\varphi: M \to Y$ such that $f(x) \in \varphi(x)$ for every $x \in X$.*

4.4. *A space Y is an m_c^n-ANR (resp. m_c-ANR) if and only if for every closed subset X of a space M and for every map $f: X \to Y$ there exist a neighborhood U of X in M and a c-n-multi-map (resp. c-multi-map) $\varphi: U \to Y$ such that $f(x) \in \varphi(x)$ for every $x \in X$.*

4.5. Remark. Every AR (resp. ANR) is clearly an m_c-AR (resp. m_c-ANR). In [22] 2.9 Suszycki essentially proved that every c-1-mulri-retract of a locally connected space is also locally connected. Hence for every $n \geqq 1$ every m_c^n-ANR is locally connected. On the other hand every continuum is an m_n^0-AR. Indeed, for every continuum Y and for every space M containing Y we can define a c-0-multi-retraction $\varphi: M \to Y$ by $\varphi(z)=Y$ for every $z \in M$. Similarly every FAR is an m_c-AR. But Suszycki [22] 2.27 showed that there is a 1-dimensional planar $FANR$ which is not an m_c-ANR. Indeed, his example is not an m_c^1-ANR and has the shape of the 1-sphere. Therefore notions of m_c^n-ANR and m_c-ANR is not shape invariants.

In the case of non-compact spaces the next problem is still open.

Problem 2. *Is it valid that every MAR is an m_c-AR?*

Using results of sections 1 and 2 we can easily point out properties of m_c^n-AR, m_c^n-ANR, m_c-AR and m_c-ANR.

4.6. *If Y is an m_c^n-AR, then $Y \in AC^n$,* pro-$H_k(Y)=0$ *in* pro-$\mathfrak{G}$ *and* $\check{H}^k(Y)=0$ *in* $\mathfrak{G}$ *for every integer* k, $0 \leqq k \leqq n$.

4.7. *If Y is an m_c^n-ANR, then* pro-$\pi_k(Y, y)$ *and* pro-$H_k(Y)$ *are stable in* pro-$\mathfrak{G}$ *for every* $y \in Y$ *and every integer* k, $1 \leqq k \leqq n$.

4.8. *If Y is a compact m_c^n-ANR, $\check{\pi}_k(Y, y)$ is countable, and $H_k(Y)$ and $\check{H}^k(Y)$ are finitely generated for every $y \in Y$ and every integer k, $0 \leqq k \leqq n$. Moreover if Y is an m_c^∞-ANR, $H_k(Y)=0=\check{H}^k(Y)$ for almost all $k \geqq 1$.*

4.9. *Every compact connected m_c^n-ANR is pointed S^k-movable for every integer k, $1 \leqq k \leqq n$. In particular, every compact connected m_c^n-ANR ($n \geqq 1$) is pointed 1-movale.*

4.10. *If Y is a compact m_c^n-AR and $Fd(Y) \leqq n < \infty$, then Y is an FAR. Therefore for a compactum Y with $Fd(Y) < \infty$ Y is an m_c-AR if and only if Y is an FAR.*

4.11. *Every compact movable m_c-AR is an FAR.*

4.12. *If Y is a compact m_c-ANR and $Fd(Y) < \infty$, then Y is an FANR.*

Related to above proparties following problems remain open.

PROBLEM 3. *Does every compact m_c-ANR Y with $Fd(Y) < \infty$ have a shape of a finite polyhedron?*

PROBLEM 4. *If Y is an m_c-AR (resp. m_c-ANR) and $Sd(Y) < \infty$, then is it valid that Y is an MAR (resp. MANR)?*

We remark that by Theorem 2.1, Corollary 2.2 and [12] Corollary 1 above problems in the case $\dim Y < \infty$ are valid.

By the same way as [22] 2.10 we can prove the following.

4.13. LEMMA. *Let $\varphi: X \to Y$ be a c-n-multi-map, where $n=0, 1, 2, \cdots, \infty$. Let $g: Y \to X$ be a map such that $y \in \varphi(g(y))$ for every $y \in Y$. Then if X is an AR (resp. ANR), Y is an m_c^n-AR (resp. m_c^n-ANR). In particular, if φ is a c-multi-map, then Y is an m_c-AR (resp. m_c-ANR).*

4.14. EXAMPLE. For $n=0, 1, 2, \cdots$ let S^{n+1} be the $(n+1)$-sphere and let $f: S^{n+1} \to S^{n+1}$ be a map with $\deg f=2$. Then let us define $X_i=S^{n+1}$ and $f_i=f$: $X_{i+1} \to X_i$ for every $i=1, 2, \cdots$. Then the inverse limit $X(n)=\varprojlim \{X_i, f_i\}$ is the

$(n+1)$-dimensional dyadic solenoid. Since $X(n)\in AC^n$, then by Lemma 4.12 $X(n)$ is an m_c^n-AR. But $X(n)$ is not an m_c^{n+1}-ANR because $X(n)$ is not S^{n+1}-movable. Therefore an m_c^n-AR does not always imply an m_c^{n+1}-AR.

4.15. EXAMPLE. In the Hilbert cube Q for each $k=1, 2, \cdots$ let us define the k-dimensional sphere

$$X_k=\left\{(x_i)_{i\geq 1}\in Q \,\middle|\, \left\{x_1-\frac{2k+1}{2k(k+1)}\right\}^2+x_2^2+\cdots+x_{k+1}^2=\left\{\frac{1}{2k(k+1)}\right\}^2,\right.$$
$$\left. x_i=0 \quad \text{if } i>k+1\right\}.$$

Now let us define a continuum X as follws

$$X=\{(0, 0, \cdots)\}\cup(\bigcup_{k\geq 1} X_k).$$

Then for each $n=1, 2, \cdots$, X_n is an ANR and $\{(0, 0, \cdots)\}\cup(\bigcup_{k\geq n+1} X_k)$ is an AC^n continuum. Hence by Lemma 4.13 X is an m_c^n-ANR for every $n=0, 1, 2, \cdots$. But $\check{H}_n(X)\neq 0$ for every $n\geqq 1$. Therefore by 4.8 X is not an m_c^∞-ANR.

By Example 4.14 and Example 4.15 there are gaps between m_c^n-ANR and m_c^{n+1}-ANR and between m_c^n-ANR for every $n\geqq 0$ and m_c^∞-ANR. But the following is open.

PROBLEM 5. *Is there an m_c^∞-ANR which is not an m_c-ANR?*

4.16. REMARK (Suszycki [22]). Let $f: Y\rightarrow Q$ be the Taylor's CE-map [23] (see Example 2.3). Then by Lemma 4.13 Y is an m_c-AR. Therefore on properties 3.10–3.12 our assumptions are essential.

4.17. REMARK. The continuum X in Example 4.15 is an approximative polyhedron (see [19]). Therefore we have an approximative polyhedron which is not an m_c-ANR. Conversely the continuum in Remark 4.16 is an m_c-AR but not an approximative polyhedron.

In the proof of [22] 3.8 by using Kuratowski-Wajdysławski theorem instead of the embedding theorem of compacta into the Hilbert cube, we have the following.

4.18. *Every m_c^n-AR is c-n-multi-contractible. Every m_c-AR is c-multi-contractible.*

4.19. *Every FAR is c-multi-contractible. Therefore every compact connected m_c^n-AR Y with $Fd(Y)\leqq n<\infty$ is c-multi-contractible.*

The converse of 4.19 is partially held by Corollary 2.6 but in general, it is not valid by Remark 4.16. We notice that the continuum $X(n)$ in Example 4.14 is a $(n+1)$-dimensional m_c^n-AR which is not c-multi-contractible.

By the same way as [22] 3.12 we have the next result.

4.20. *Every c-n-multi-contractible ANR is an* m_c^n-AR. *Every c-multi-contractible ANR is an* m_c-AR.

4.21. *Every n-dimensional c-n-multi-contractible ANR, where n is finite, is an AR. If a c-multi-contractible ANR has the property C, then it is an AR.*

5. Topological operations of m_c^n-AR, m_c^n-ANR, m_c-AR and m_c-ANR.

In [22] Suszycki asked the following problem: *Do* m_c-AR (*resp.* m_c-ANR)-*spaces are invariant under CE-maps*? We do not know whether his problem is valid. But by the same way as [22] 2.12 we have its non-compact version.

5.1. THEOREM. *Let* $g: Y \to X$ *be a CE-map. Let* M *be an AR containing* X *as a closed subset. If there exist a neighborhood* U *of* X *in* M *and a c-multi-retraction* $\varphi: U \to X$ *such that* $\dim \varphi(z) < \infty$ *for every* $z \in U$, *then* Y *is an* m_c-ANR. *Moreover if* $U = M$, *then* Y *is an* m_c-AR.

5.2. REMARK. On Theorem 5.1 the assumption "$\dim \varphi(z) < \infty$ for every $z \in U$" is necessary to show that

$$(*) \quad Sh(g^{-1}(\varphi(z))) = Sh(\varphi(z)) \qquad \text{for every } z \in U.$$

Then if we added some assumption for holding (*), by the same way we have following results.

5.3. COROLLARY. *Let* $g: Y \to X$ *be a hereditary shape equivalence. If* X *is an* m_c-AR (*resp.* m_c-ANR), *then* Y *is also an* m_c-AR (*resp.* m_c-ANR).

5.4. COROLLARY. *Let* $g: Y \to X$ *be a CE-map. If* X *is a C-space and an* m_c-AR (*resp.* m_c-ANR), *then* Y *is also an* m_c-AR (*resp.* m_c-ANR).

On the other hand for m_c^n-AR and m_c^n-ANR we have the following theorem.

5.5. THEOREM. *Let* $g: Y \to X$ *be a proper map such that* $g^{-1}(x) \in AC^n$ *for every* $x \in X$, *where* $n = 0, 1, 2, \cdots, \infty$. *If* X *is an* m_c^n-AR (*resp.* m_c^n-ANR), *then* Y *is also an* m_c^n-AR (*resp.* m_c^n-ANR).

PROOF. Let M and N be ARs' containing X and Y as closed subsets, respectively. Then g has a continuous extension $\tilde{g}: N \to M$. Then if X is an m_c^n-ANR, there are a neighborhood U of X in M and a c-n-multi-retraction $\varphi: U \to X$. Define a neighborhood $V = \tilde{g}^{-1}(U)$ of Y in N and a u.s.c. compact multi-valued function $\psi: V \to Y$ as follows

$$\psi(z) = g^{-1}(\varphi \circ \tilde{g}(z)) \qquad \text{for every } z \in V .$$

Then $\varphi \circ \tilde{g}(z) \in AC^n$ for every $z \in V$. Hence applying Vietoris theorem in shape theory (see [6] or [20]) to the restriction $g|\psi(z): \varphi(z) \to \varphi \circ \tilde{g}(z)$, we have that $\psi(z) \in AC^n$ for every $z \in V$. Moreover it is clear that $y \in \psi(y)$ for every $y \in Y$. That is, ψ is a c-n-multi-retraction of V onto Y. Therefore, by 4.2, Y is an m_c^n-ANR. Similarly we can prove the case X is an m_c^n-AR.

It is unknow whether the converse of Theorem 5.5 is valid. That is,

PROBLEM 6. *Let $g: Y \to X$ be a proper surjective map such that $g^{-1}(x) \in AC^n$ for every $x \in X$. Then if Y is an m_c^n-AR (resp. m_c^n-ANR), is X an m_c^n-AR (resp. m_c^n-ANR)?*

Next by using the standard way we can easily prove following.

5.6. THEOREM *If X_i is an m_c^n-AR (resp. m_c-AR) for every $i=1, 2, \cdots$, then the preduct space $\prod_{i \geq 1} X_i$ is also an m_c^n-AR (resp. m_c-AR).*

5.7. THEOREM *If X_1 and X_2 are m_c^n-$ANRs$' (resp. m_c-$ANRs$'), then the product space $X_1 \times X_2$ is also an m_c^n-ANR (resp. m_c-ANR).*

Since every single-valued u.s.c. function is continuous, every totally disconnected m_c^0-ANR is an ANR. Hence the Cantor set is not an m_c^0-ANR. Therefore we can not generally extend Theorem 5.7 to infinite products.

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