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AN APPLICATION OF WEIGHTED NORM INEQUALITIES FOR MAXIMAL FUNCTIONS TO SEMIGROUPS OF CONVOLUTION TRANSFORMS ON $L_w^p(R^n)$

By

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Abstract. By applying weighted norm inequalities for maximal functions it is shown that the convolution transforms with kernels

$$p(\alpha; t, x)=(2\pi)^{-n}\int_{R^n}\exp\left(ixy-\frac{t}{2}|y|^{\alpha}\right)dy, \qquad (t>0)$$

on $L_w^p(R^n)$ to itself form a semigroup of class (C_0).

Introduction. E. Hille showed in [3] that the Poisson transforms

$$(P(t)f)(x)=\int_{-\infty}^{\infty}\frac{t}{\pi[t^2+(x-y)^2]}f(y)dy$$

for f in $L^p(R)$ $(p>1)$ form a semigroup of class (C_0) with the infinitesimal generator $-(d/dx)\cdot C=-C\cdot(d/dx)$, where the operator C denotes the Hilbert transform. For multi-dimensional case we can show by the results in [12] that the Poisson transforms

$$(P(t)f)(x)=\int_{R^n}\frac{c_n t}{[t^2+|x-y|^2]^{(n+1)/2}}f(y)dy$$

for f in $L^p(R^n)$ $(p>1)$ form a semigroup of class (C_0) with the infinitesimal generator of the closed extension of $-\sum_{j=1}^n(\partial/\partial x_j)\cdot R_j=-\sum_{j=1}^n R_j\cdot(\partial/\partial x_j)$, where the operators R_j denote the Riesz transforms. In this note by using the weighted norm inequalities for maximal functions and singular integrals obtained by B. Muckenhoupt and R. Wheeden [9], [10], B. Muckenhoupt [8], R. Hunt, B. Muckenhoupt and R. Wheeden [5], R. Coifman and C. Fefferman [1] we obtain the one-parameter semigroups of the convolution transforms with the infinitesimal generators of fractional powers of the Laplacean $-\Delta$ on $L_w^p(R^n)$ $(p>1)$ and in particular we obtain the semigroups of the Poisson transforms with the infinitesimal generators of $-(1/2)(d/dx)\cdot C=-(1/2)C\cdot(d/dx)$ on $L_w^p(R)$ and the closed extension of $-(1/2)\sum_{j=1}^n(\partial/\partial x_j)\cdot R_j=-(1/2)\sum_{j=1}^n R_j\cdot(\partial/\partial x_j)$ on $L_w^p(R^n)$, respectively.

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These results are the general extensions of the result obtained by E. Hille [3] and the semi-groups with the infinitesimal generators of fractional powers of the Laplacean $-\Delta$ on $L^p(R^n)$. In this note we suppose that the weight $w(x)$ is nonnegative and $w(x)$, $[w(x)]^{-1/(p-1)}$ are locally integrable and $w(x)$ satisfies an A_p condition in [1]; i.e., $w\in A_p$ if there is a constant C such that

$$\left(\frac{1}{|Q|}\int_Q w(x)dx\right)\left(\frac{1}{|Q|}\int_Q [w(x)]^{-1/(p-1)}dx\right)^{p-1}\leqq C,$$

for all cube $Q\subset R^n$. It is known [7] that $w(x)=|x|^{\beta}\in A_p$ if $-n<\beta<n(p-1)$. We say $f\in L_w^p(R^n)$, $(p>1)$, if

$$\|f\|_{p,w}=\left[\int_{R^n}|f(x)|^p w(x)dx\right]^{1/p}<\infty.$$

We use p' to denote the index conjugate to p; $1/p+1/p'=1$. It is known [5, 10] that

$$\int_{R^n}\frac{w(x)}{1+|x|^{np}}dx<\infty,\quad \int_{R^n}\frac{[w(x)]^{-1/(p-1)}}{1+|x|^{np'}}dx<\infty. \tag{0.1}$$

From these facts it is seen that the totality of continuous functions with compact support, say $C_0(R^n)$, is contained in $L_w^p(R^n)$ and $L_{w^{-1/(p-1)}}^{p'}(R^n)$. Since the space $C_0(R^n)$ is dense in $L_w^p(R^n)$ and $L_{w^{-1/(p-1)}}^{p}(R^n)$, the totality of infinitely differentiable functions with compact support, say $D(R^n)$, is also dense. We will make use of the Hardy-Littlewood maximal function m_f for f in $L_w^p(R^n)$ (cf. [12]).

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§1. The semigroups of the convolution transforms on $L_w^p(R^n)$.

Let

$$p(\alpha;t,x)=(2\pi)^{-n}\int_{R^n}\exp\left(ixy-\frac{t}{2}|y|^{\alpha}\right)dy \tag{1.1}$$

for $0<\alpha<\infty$ and $0<t<\infty$. When $0<\alpha\leqq 2$, $p(\alpha;t,x)$ is known as the symmetric stable density with exponent α (cf. [6]). In particular

$$p(2;2t,x)=(4\pi t)^{-n/2}\exp\left(-\frac{|x|^2}{4t}\right)$$

and

$$p(1;2t,x)=\frac{c_n t}{[t^2+|x|^2]^{(n+1)/2}},$$

where $c_n=\Gamma[(n+1)/2]\pi^{-(n+1)/2}$. Let us consider the fractional powers of the

Laplacean $-\Delta$, say $(-\Delta)^{\alpha/2}$ $(0<\alpha<\infty)$, to be

$$((-\Delta)^{\alpha/2}f)(x))=\int_{R^n}(2\pi)^{-n/2}e^{ixy}|y|^{\alpha}\hat{f}(y)dy \tag{1.2}$$

for $f\in D[(-\Delta)^{\alpha}]=\Big\{f\in L_w^p(R^n): f\in L^2(R^n),\ |y|^{\alpha}\hat{f}\in L^1(R^n)\cap L^2(R^n)$ and

$$(2\pi)^{-n/2}\int_{R^n}e^{ixy}|y|^{\alpha}\hat{f}(y)dy\in L_w^p(R^n)\Big\}.$$

where $\hat{f}$ denotes the Fourier transform of f. Let us denote the operator $-(1/2)(-\Delta)^{\alpha/2}$ by A_α.

LEMMA. *The operator A_α is closable in $L_w^p(R^n)$.*

PROOF. When f belongs to $D(R^n)$ let

$$g(x)=\int_{R^n}(2\pi)^{-n/2}e^{ixy}|y|^{\alpha}\hat{f}(y)dy\,.$$

By the fact that $|x|^n g(x)$ is bounded and by (0.1) we obtain

$$\int_{R^n}|g(x)|^p w(x)dx\leqq \sup_{x\in R^n}[(1+|x|^{np})|g(x)|^p]\int_{R^n}\frac{w(x)}{1+|x|^{np}}dx<\infty\,.$$

Also we can show $g\in L^{p'}_{w^{-1/(p-1)}}(R^n)$. Consequently, if f_n belongs to $D(A_\alpha)$ and $f_n\to 0$, $A_\alpha f_n\to h$ as $n\to\infty$ in the L_w^p norm we obtain

$$(A_\alpha f_n, \phi)=\int_{R^n}f_n(x)\overline{(A_\alpha\phi)(x)}dx \longrightarrow \int_{R^n}h(x)\overline{\phi(x)}dx=0$$

as $n\to\infty$ for all ϕ in $D(R^n)$. Therefore $h(x)=0$ for almost all x and A_α is closable in $L_w^p(R^n)$. Q.E.D.

Let us denote the smallest closed extension of A_α by $\bar{A}_\alpha$ and its domain by $D(\bar{A}_\alpha)$.

THEOREM. *Let*

$$(T_\alpha(0)f)(x)=f(x)\,,$$

$$(T_\alpha(t)f)(x)=\int_{R^n}p(\alpha\,;t,\ x-y)f(y)dy\,,$$

for f in $L_w^p(R^n)$. Then the family $[T(t): 0\leqq t<\infty]$ forms a one-parameter semigroup of class (C_0) with the infinitesimal generator $\bar{A}_\alpha$ and the domain $D(\bar{A}_\alpha)$.

PROOF. $T_\alpha(t)$ is bounded uniformly in t: Suppose $0<\alpha\leqq 2$. It is known [12] that $p(\alpha\,;t,\ x)$ is a radial function for $n\geqq 2$ and it is seen from Theorem XX in [13] that $p(\alpha\,;t,\ x)$ is a decreasing function of $|x|$. By making use of the maximal function and by [1] we obtain

$$\int_{R^n}|(T_\alpha(t)f)(x)|^p w(x)dx \leqq \int_{R^n}[m_f(x)]^p w(x)dx \leqq C\|f\|_{p,w}^p, \tag{1.3}$$

where C is a constant number not depending on f (cf. [12. p. 59]).

If $\alpha>2$ we can obtain

$$|(T_\alpha(t)f)(x)| \leqq \left[\sup_{y\in R^n}\frac{|p(\alpha;1,y)|}{p(1;1,y)}\right]m_f(x)$$

and since

$$\sup_{y\in R^n}\frac{|p(\alpha;1,y)|}{p(1;1,y)}$$

is bounded the inequality (1.3) holds.

Semigroup property and strong continuity: These properties follow from the facts that $D(R^n)$ is dense in $L_w^p(R^n)$ and $T_\alpha(t)$ is uniformly bounded in t.

Infinitesimal generator and its domain: Let us denote the infinitesimal generator of the semigroup of the family $[T_\alpha(t): 0\leqq t<\infty]$ by C_α and its domain by $D(C_\alpha)$. It is seen from (1.3) that

$$\lim_{t\to\infty}\frac{1}{t}\log\|T(t)\|=\omega\leqq 0.$$

The resolvent $R(\lambda, C_\alpha)$ of C_α is given by

$$R(\lambda, C_\alpha)f(x)=(B)\int_0^\infty e^{-\lambda t}T_\alpha(t)f(x)dt \tag{1.4}$$

for f in $L_w^p(R^n)$ and for $\lambda>0$, where (B) denotes the Bochner integral, and $D(C_\alpha)=\{g: g=R(1, C_\alpha)f$ for f in $L_w^p(R^n)\}$ holds. Let us show that $(\lambda-\bar{A}_\alpha)R(\lambda, C_\alpha)f=f$ holds for all f in $L_w^p(R^n)$. Suppose that f belongs to $D(R^n)$. By [4. Remark following Theorem 3.7.12] and by the Fubini theorem we can show that

$$\left((B)\int_0^\infty e^{-\lambda t}T_\alpha(t)f\,dt, \phi\right)=\left(\int_0^\infty e^{-\lambda t}T_\alpha(t)f\,dt, \phi\right)$$

for all ϕ in $D(R^n)$. Consequently the Bochner integral of the right hand side of (1.4) is equal to the ordinary Lebesgue integral. We obtain

$$g(x)=R(\lambda, C_\alpha)f(x)=(2\pi)^{-n/2}\int_{R^n}e^{ixy}\frac{2}{2\lambda+|y|^\alpha}\hat{f}(y)dy \tag{1.5}$$

and

$$|g(x)|\leqq\frac{C}{\lambda}m_f(x) \qquad \text{for a constant } C.$$

Let us show that $g\in D(A_\alpha)$. It suffices to show that

$$h(x)=(2\pi)^{-n/2}\int_{R^n}e^{ixy}\frac{|y|^\alpha}{2\lambda+|y|^\alpha}\hat{f}(y)dy$$

belongs to $L_w^p(R^n)$. We see that $h(x)=f(x)-\lambda g(x)$, and hence $h(x)$ belongs to $L_w^p(R^n)$. It is seen from (1.5) that

$$(\lambda-\bar{A}_\alpha)R(\lambda, C_\alpha)f=(\lambda-A_\alpha)g=f \tag{1.6}$$

for f in $D(R^n)$. Since $D(R^n)$ is dense in $L_w^p(R^n)$ and $R(\lambda, C_\alpha)$ is bounded and $\bar{A}_\alpha$ is closed (1.6) holds for all f in $L_w^p(R^n)$. Consequently it is seen that $D(\bar{A}_\alpha) \supset D(C_\alpha)$ and $\bar{A}_\alpha g=C_\alpha g$ for all g in $D(C_\alpha)$. Let us show that $D(A_\alpha)\subset D(C_\alpha)$. When g belongs to $D(A_\alpha)$ let

$$f(x)=(2\pi)^{-n/2}\int_{R^n} e^{ixy}\Big(1+\frac{|y|^\alpha}{2}\Big)\hat{g}(y)dy\,.$$

Since g belongs to $D(A_\alpha)$, by the Fourier inversion formula we see that f belongs to $L_w^p(R^n)$. Recalling (1.5) we can show that $R(1, C_\alpha)f=g$. Thus we obtain $D(A_\alpha)\subset D(C_\alpha)$. Consequently, by the definition of the smallest closed extension of A_α we obtain $D(\bar{A}_\alpha)\subset D(C_\alpha)$. Consequently we obtain that $D(\bar{A}_\alpha)=D(C_\alpha)$ and $\bar{A}_\alpha f=C_\alpha f$ for f in $D(\bar{A}_\alpha)=D(C_\alpha)$. Q.E.D.

§ 2. The infinitesimal generators of the semigroups of the Poisson transforms.

It is known [1] that the Hilbert transform C on $L_w^p(R^n)$ and the Riesz transforms R_j on $L_w^p(R^n)$ to themselves can be defined and they are bounded operators. It is easily seen that the set of linear combinations of functions in $D(R)$ and in $\{(1/x-\xi-i\eta): -\infty<\xi,\ \eta<\infty,\ \eta\neq 0\}$ is dense in the domain of the operator $(d/dx)\cdot C$, $D((d/dx)\cdot C)=\{f\in L_w^p(R): (Cf)(x)$ is absolutely continuous and $d/dx(Cf)(x)\in L_w^p(R)\}$, with the norm $\max\{\|f\|_{p,w}, \|(d/dx)Cf\|_{p,w}\}$. From this fact and from the same arguments as in [3] we obtain

COROLLARY 1. *When* $n=1$, $D(\bar{A}_1)=D((d/dx)\cdot C)$ *and*

$$(\bar{A}_1 f)(x)=-\frac{1}{2}\frac{d}{dx}(Cf)(x)=-\frac{1}{2}\Big(C\frac{d}{dx}f\Big)(x)$$

holds for $f\in D(\bar{A}_1)=D((d/dx)\cdot C)$.

It is seen from [12] that if $f\in L_w^p(R^n)\cap L^2(R^n)$ and $\hat{f}\in L^1(R^n)\cap L^2(R^n)$,

$$(R_j f)(x)=\int_{R^n}(-i)\frac{y_j}{|y|}\hat{f}(y)e^{ixy}dy$$

holds for almost all x with respect to $w(x)dx$. By this equality we see that if $n\geqq 2$

$$(A_1 f)(x)=-\frac{1}{2}\sum_{j=1}^n\frac{\partial}{\partial x_j}(R_j f)(x)=-\frac{1}{2}\sum_{j=1}^n\Big(R_j\frac{\partial}{\partial x_j}f\Big)(x)$$

holds for f in $D(A_1)$. Consequently, by the above theorem we obtain

COROLLARY 2. *The smallest closed extension of the operator*

$$-\frac{1}{2}\sum_{j=1}^{n}\frac{\partial}{\partial x_j}\cdot R_j=-\frac{1}{2}\sum_{j=1}^{n}R_j\cdot\frac{\partial}{\partial x_j}$$

with the domain $D(A_1)$ is the infinitesimal generator of the semigroup of the Poisson transforms on $L_w^p(R^n)$.

References

[1] Coifman, R. R. and Fefferman, C., Weighted norm inequalities for maximal functions and singular integrals, Studia Math., **51** (1974), 245-250.

[2] Dunford, N. and Schwartz, J. T., Linear operators, Part 2, Interscience, New York 1963.

[3] Hille, E., On the generation of semigroups and the theory of conjugate functions, Proc. R. Physiogr. Soc. Lund., (21) **14** (1951), 103-142.

[4] ——— and Phillips, F. S., Functional analysis and semigroups, A. M. S. Colloq. Publ., **31** (1957).

[5] Hunt, R., Muckenhoupt, B. and Wheeden, R., Weighted norm inequalities for the conjugate function and the Hilbert transform, Trans. Amer. Math. Soc., **176** (1973), 227-251.

[6] Ito, K. and Mckean, H. P., Diffusion processes and their sample paths, Second printing, Springer-Verlag, 1974.

[7] Kurt, D. S. and Wheeden, R., Results on weighted norm inequalities for multipliers, Trans. Amer. Math. Soc., **255** (1979), 343-362.

[8] Muckenhoupt, B., Weighted norm inequalities for the Hardy maximal function, Trans. Amer. Math. Soc., **165** (1972), 207-226.

[9] ——— and Wheeden, R., Weighted norm inequalities for singular and fractional integrals, Trans. Amer. Math. Soc., **161** (1971), 249-258.

[10] ———, Norm inequalities for the Littlewood-Paley functions g_λ^*, Trans. Amer. Math. Soc., **191** (1974), 95-111.

[11] Stein, E., Singular integrals and differentiability properties of functions, Princeton Univ. Press, Princeton N. J., 1970.

[12] ——— and Weiss, G., Introduction to Fourier analysis on Euclidean space, Princeton Univ. Press, Princeton N. J., 1971.

[13] Wintner, A., On a class of Fourier transforms, Amer. J. Math., **58** (1936), 45-90.

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