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ON PROJECTIVE COHEN-MACAULAYNESS OF A DEL PEZZO SURFACE EMBEDDED BY A COMPLETE LINEAR SYSTEM

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Let k be an algebraically closed field. We understand by a Del Pezzo surface X over k a non-singular rational surface on which the anti-canonical sheaf $-\omega_X$ is ample. We call the self-intersection number $d=\omega_X^2$ of ω_X the degree of X, then we get that $1\leqq d\leqq 9$. It is well known that X is isomorphic to $\boldsymbol{P}^1\times\boldsymbol{P}^1$, which has degree 8, or an image of $\boldsymbol{P}^2$ under a monoidal transformation with center the union of $r=9-d$ points which satisfies the following conditions:

(a) no three of them lie on a line;

(b) no six of them lie on a conic;

(c) there are no cubics which pass through seven of them and have a double point at the eighth point.

Conversely any surface described above is a Del Pezzo surface of the corresponding degree ([8, III, Theorem 1]). It is also well known that $-\omega_X$ is very ample when $d\geqq 3$ and that ample divisors on X of degree 3, which is a cubic surface, are very ample too. In this paper we will get that *ample divisors on X of degree $d\geqq 3$ are very ample and that ample divisors on X of degree* 2 [*resp.* 1] *other than* $-\omega_X$ [*resp.* $-\omega_X$ *nor* $-2\omega_X$] *are very ample.*

A closed subscheme V in $\boldsymbol{P}^N$ is said to be projectively Cohen-Macaulay if its affine cone is Cohen-Macaulay. It is equivalent to that $H^1(\boldsymbol{P}^N, \mathcal{I}_V(m))=0$ for every $m\in Z$ and $H^i(V, \mathcal{O}_V(m))=0$ for every $m\in Z$ and $0<i<\dim V$. In this paper, we will get that $\phi_{|D|}(X)$ is projectively Cohen-Macaulay for a very ample divisor D on X, where $\phi_{|D|}$ is the morphism from X to $\boldsymbol{P}^{\dim|D|}$ defined by the complete linear system $|D|$ of D. We also study the homogeneous ideal $I(D)=\mathrm{Ker}\left[S\Gamma(D)\longrightarrow\bigoplus_{n\geqq 0}\Gamma(nD)\right]$ defining $\phi_{|D|}(X)$. These results will be stated and proved in §3 and §5. The fourth section will be devoted to a study on $-n\omega_X$ of a Del Pezzo surface X of degree 1 or 2.

In §1 we will compute the dimension $h^i(D)$ of the i-th cohomology group $H^i(X, \mathcal{O}_X(D))$ of the invertible sheaf $\mathcal{O}_X(D)$ corresponding to a divisor D.

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By abuse of terminology we use a divisor D and the corresponding invertible sheaf $\mathcal{O}_X(D)$ interchangeably. In § 2 we have general studies of the equations defining a projective variety. Throughout this paper a curve on a surface will mean a reduced curve.

§ 1. Cohomology groups of a divisor on a Del Pezzo surface.

From now on, a Del Pezzo surface means one which is not $\boldsymbol{P}^1\times\boldsymbol{P}^1$ unless otherwise specified. Let X be a Del Pezzo surface of degree $d\leqq 8$, and $f: X\longrightarrow \boldsymbol{P}^2$ be its representation in the form of monoidal transformation of the plane with center $P_1, \cdots, P_r$. The linearly equivalent class of the exceptional curve $E_i=f^{-1}(P_i)$ is denoted by $e_i\in \mathrm{Pic}(X)$. Put $l=f^*\mathcal{O}_{\boldsymbol{P}}(1)$. Then $(l, e_1, \cdots, e_r)$ is a free basis of $\mathrm{Pic}(X)$ and $\omega_X\sim -3l+\sum_{i=1}^{r} e_i$. We denote by $\mathcal{E}$ the set of all exceptional curves on X, then

$$\mathcal{E}=\{Y \mid \text{an irreducible curve } Y \text{ with } Y^2<0\}.$$

$\mathcal{E}$ is a finite set and it is easy to list up all $E\in\mathcal{E}$, as follows.

a	b_1	b_2	b_3	b_4	b_5	b_6	b_7	b_8
0	0	0	0	0	0	0	0	−1
1	1	1	0	0	0	0	0	0
2	1	1	1	1	1	0	0	0
3	2	1	1	1	1	1	1	0
4	2	2	2	1	1	1	1	1
5	2	2	2	2	2	2	1	1
6	3	2	2	2	2	2	2	2

where $\mathcal{O}_X(E)\sim al-\sum_{i=1}^{r} b_ie_i$ with $b_1\geqq\cdots\geqq b_r$.

We begin with a lemma on X of degree 8, which is isomorphic to a rational ruled surface $\boldsymbol{F}_1$ with invariant one.

LEMMA 1.1. *Let X be a Del Pezzo surface of degree 8 and $D\sim al-b_1e_1$ a divisor on X. Then the following assertions hold:*

(1) *if $a\geqq b_1\geqq -1$ or $a-b_1=-1$, then $h^1(D)=h^2(D)=0$;*

(2) *D is ample $\Longleftrightarrow$ D is very ample $\Longleftrightarrow$ $a>b_1>0$.*

PROOF. We can prove (1) in the manner of [6, § 7]. The statement (2) is found in [1, *V*, Cor. 2.18].

The following remark is available for us.

REMARK 1.2. *Let X be a Del Pezzo surface of degree $d\leqq 7$ and D a divisor with $D.E>0$ for each $E\in\mathcal{E}$. Then there exists a monoidal transformation $f: X \longrightarrow \boldsymbol{P}^2$ such that $D\sim al-\sum_{i=1}^{r} b_i e_i$ with $a\geqq b_1+b_2+b_3$ (in case $r=2$, $a>b_1+b_2$) and $b_1\geqq b_2\geqq\cdots\geqq b_r>0$.*

PROOF. We prove the result by induction on r. The assertion is trivial for $r=2$. In fact for any monoidal transformation $f: X \longrightarrow \boldsymbol{P}^2$ we can assume $D.e_1\geqq D.e_2$. Also we get $D.(l-e_1-e_2)>0$ and $D.e_2=b_2>0$ by the assumption.

For $r\geqq 3$, choose E_r so that $D.E=b_r$ is equal to the minimum value of $D.E$ for any $E\in\mathcal{E}$. Blowing down E_r, we have a monoidal transformation $\pi: X \longrightarrow X'$, where X' is a Del Pezzo surface of degree $d+1$. By the induction hypothesis for a divisor D' on X' such that $\pi^*D'\sim D+b_r e_r$, there exists a monoidal transformation $f': X' \longrightarrow \boldsymbol{P}^2$ satisfying the condition of this remark. Then $f'\circ\pi$ is what we want.

LEMMA 1.3. *Let X be a Del Pezzo surface of degree $d\leqq 7$ and $D\sim al-\sum_{i=1}^{r} b_i e_i$ a divisor on X such that $a\geqq b_1+b_2+b_5$ (if $r=3$ or 4, $a\geqq b_1+b_2+b_r$; if $r=2$, $a>b_1+b_2$) and $b_1\geqq b_2\geqq\cdots\geqq b_r>0$. Then in case $3\leqq d\leqq 7$, D is very ample and in case $d\leqq 2$, D is ample. Moreover in case $d=2$, $|D|$ is free from base points.*

PROOF. If $r=2$, then it is clear that $D\sim -\omega_X+(b_1-1)(l-e_1)+(b_2-1)(l-e_2)+(a-b_1-b_2-1)l$ is very ample. Because $-\omega_X$ is very ample and $|l-e_i|$ and $|l|$ are free from base points. Next assume $r\geqq 5$ [resp. $r=3$ or 4]. We put $D_r=-\omega_X$, $D_0=l$, $D_1=l-e_1$, $D_i=2l-\sum_{j=1}^{i} e_j$, for $2\leqq i\leqq 4$ [resp. $2\leqq i\leqq r-1$], and $D_k=3l-\sum_{j=1}^{k} e_j$, for $5\leqq k\leqq r-1$. Then D is linearly equivalent to $\sum_{i=0}^{r} c_i D_i$, where $c_r=b_r$, $c_i=b_i-b_{i+1}$ for $1\leqq i\leqq r-1$ and $c_0=a-(b_5+b_2+b_1)$ [resp. $c_0=a-(b_r+b_2+b_1)$]. Since $|D_i|$ has no base points and $c_i\geqq 0$ for every i, $0\leqq i\leqq r-1$, D is ample or very ample according as $-\omega_X$ is ample or very ample. Also $|D|$ has no base points if $|-\omega_X|$ has no base points. Since the anti-canonical divisor $3l-\sum_{i=1}^{6} e_i$ is very ample on a cubic surface, it has no unassigned base points. This shows that $|-\omega_X|=|3l-\sum_{i=1}^{7} e_i|$ on X of degree 2 has no base points.

PROPOSITION 1.4.1. *Let $D\sim al-\sum_{i=1}^{r} b_i e_i$ be a divisor on a Del Pezzo surface X of degree $d\leqq 7$. Assume that $|D|$ has an irreducible curve. Then $h^1(D)=h^2(D)=0$ and $h^0(D)=\frac{1}{2}(a+1)(a+2)-\sum_{i=1}^{r}\frac{1}{2}b_i(b_i+1)$.*

PROOF. Let $Y \in |D|$ be an irreducible curve on X and $p_a(Y)$ the arithmetic genus of Y. We consider the following exact sequence

$$0 \longrightarrow \mathcal{O}_X \longrightarrow D \longrightarrow D|_Y \longrightarrow 0.$$

From its long cohomology sequence we have $H^2(D)=0$ and $H^1(D) \cong H^1(Y, D|_Y)$. Since $\deg(D|_Y)-(2p_a(Y)-1)=Y.(-\omega_X)-1$ is not less than zero by the ampleness of $-\omega_X$, we get $h^1(Y, D|_Y)=0$ and $h^1(D)=0$. Finally $h^0(D)=h^0(D|_Y)+1$ is computed by Riemann-Roch theorem.

COROLLARY 1.4.2. *Let* $D \sim al-\sum_{i=1}^{r} b_i e_i$ *be a divisor on a Del Pezzo surface of degree* $d \leqq 7$. *Assume that* $a \geqq b_1+b_2+b_3$ (*in case* $r=2, a \geqq b_1+b_2$) *and* $b_1 \geqq b_2 \geqq \cdots \geqq b_r \geqq 0$, *then* $h^1(D)=h^2(D)=0$.

PROOF. First we consider the case $r=2$. If $a>b_1+b_2$ and $b_2>0$, then D is very ample by Lemma 1.3. So $|D|$ has an irreducible curve, which proves $h^1(D)=h^2(D)=0$ by Proposition 1.4.1. Next if $a>b_1$ and $b_2=0$, then $H^i(D) \cong H^i(\boldsymbol{F}_1, al-b_1e_1)$, which are vanishing by Lemma 1.1 (1) when $i=1$ or 2. In the other case (i.e., $a=b_1+b_2$), we contract $l-e_1-e_2$ to a point on $\boldsymbol{P}^1 \times \boldsymbol{P}^1$. Any divisor on $\boldsymbol{P}^1 \times \boldsymbol{P}^1$ is denoted by a pair of integers $(\ ,\ )$ in $\boldsymbol{Z} \oplus \boldsymbol{Z}$ under the isomorphism

$$\mathrm{Pic}(\boldsymbol{P}^1 \times \boldsymbol{P}^1) \cong p_1^*(\mathrm{Pic}\ \boldsymbol{P}^1) \oplus p_2^*(\mathrm{Pic}\ \boldsymbol{P}^1) = \boldsymbol{Z} \oplus \boldsymbol{Z},$$

where p_1 and p_2 are the projections of $\boldsymbol{P}^1 \times \boldsymbol{P}^1$ onto the two factors. Then we get $H^i(X, D) \cong H^i(\boldsymbol{P}^1 \times \boldsymbol{P}^1, (b_1, b_2))$, since D is $b_1(l-e_1)+b_2(l-e_2)$. By the assumption $b_1 \geqq b_2 \geqq 0$ we get $H^i(\boldsymbol{P}^1 \times \boldsymbol{P}^1, (b_1, b_2))=0$ for $i=1, 2$.

Second we consider the case when $3 \leqq r \leqq 6$. If $b_r>0$, then D is very ample by Lemma 1.3. In that case we can apply Proposition 1.4.1 and conclude $h^i(D)=0$ for $i=1, 2$. If $b_r=0$, then $H^i(D) \cong H^i(X', D')$ for each i, where X' is the contraction of X via E_r and D' is a divisor on X' such that D is its transform. Thus we can prove the corollary by induction on r.

Finally for the case $r=7, 8$, we may assume $b_r>0$ by using the inductive proof above. Let $C \in |-\omega_X|$ be an irreducible curve, which has the arithmetic genus $p_a(C)=1$. Consider the following exact sequence

$$0 \longrightarrow D+t\omega_X \longrightarrow D+(t-1)\omega_X \longrightarrow D+(t-1)\omega_X|_C \longrightarrow 0$$

where $1 \leqq t \leqq b_r$. Since $\deg(D+(t-1)\omega_X)|_C>0$ for $1 \leqq t \leqq b_r$, $H^1(C, D+(t-1)\omega_X)=0$. So we get the surjection $H^1(D+t\omega_X) \longrightarrow H^1(D(t-1)\omega_X)$ and the isomorphism

$$H^2(D+t\omega_X) \cong H^2(D+(t-1)\omega_X).$$

Immediately we get the surjection $H^1(D+b_r\omega_X) \longrightarrow H^1(D)$ and the isomorphism

$H^2(D+b_r\omega_X)\cong H^2(D)$. But $D+b_r\omega_X=(a-3b_r)l-\sum_{i=1}^{r-1}(b_i-b_r)e_i$ satisfies the condition of this corollary too. Hence we obtain that $h^i(D+b_r\omega_X)=0$ for $i=1,2$, and $h^i(D)=0$ for $i=1,2$ as required.

COROLLARY 1.4.3. *Let D be an ample divisor on a Del Pezzo surface X, which may be isomorphic to $\boldsymbol{P}^1\times\boldsymbol{P}^1$. Then*:

(i) $H^i(X, D+\omega_X)=0$ *for* $i=1,2$;

(ii) $H(X, -D)=0$ *for* $i=0,1$.

PROOF. Since (i) and (ii) are equivalent by Serre's duality, it is sufficient to prove (i). In case $X\cong\boldsymbol{P}^2$ or $\boldsymbol{P}^1\times\boldsymbol{P}^1$, the assertion is clear. For the case $X\cong\boldsymbol{F}_1$ see Lemma 1.1. For $r\geqq 2$, by Remark 1.2 we may assume that D is such that $a\geqq b_1+b_2+b_3$ (in case $r=2$, $a>b_1+b_2$) and $b_1\geqq\cdots\geqq b_r>0$, because $D.E>0$ for all $E\in\mathcal{E}$ by Nakai's criterion. It follows that $h^i\left((a-3)l-\sum_{i=1}^{r}(b_i-1)e_i\right)=0$ for $i=1,2$ by Corollary 1.4.2.

This corollary implies that Kodaira's vanishing theorem holds on a Del Pezzo surface in any characteristic. The following lemma is also a vanishing theorem on some divisors which are not ample. This will be used in § 4 and § 5.

LEMMA 1.5. *Let X be a Del Pezzo surface of degree $d\leqq 7$ and E_α's exceptional curves on X. Then*:

(1) $h^0(E_\alpha)=1$ *and* $h^1(E_\alpha)=h^2(E_\alpha)=0$;

(2) $h^i(-E_\alpha)=0$ *for every* i;

(3) $h^2(E_\alpha-E_\beta)=0$;

(4) $h^2(E_\alpha-E_\beta-E_\gamma)=0$ *and* $h^2(-E_\beta-E_\gamma)=0$;

(5) $h^2(E_\alpha-E_\beta-E_\gamma-E_\delta)=0$ *and* $h^2(-E_\beta-E_\gamma-E_\delta)=0$ *unless* $E_\beta.E_\gamma=E_\gamma.E_\delta=E_\delta.E_\beta=1$.

PROOF. In Proposition 1.4.1, (1) is already proved. To prove (2) we assume $X\cong\boldsymbol{F}_1$ and $E_\alpha\sim e_1$. Then (2) is given by Lemma 1.1 (1). For (3) we consider the following exact sequence

$$0\longrightarrow -E_\beta\longrightarrow E_\alpha-E_\beta\longrightarrow\mathcal{O}_{\boldsymbol{P}^1}(-1-E_\alpha.E_\beta)\longrightarrow 0$$

and the resulting cohomology sequence $H^2(-E_\beta)\longrightarrow H^2(E_\alpha-E_\beta)\longrightarrow 0$. Since $H^2(-E_\beta)=0$, we get $H^2(E_\alpha-E_\beta)=0$. Similarly if $H^2(-E_\beta-E_\gamma)=0$ [resp. $H^2(-E_\beta-E_\gamma-E_\delta)=0$], then $H^2(E_\alpha-E_\beta-E_\gamma)=0$ [resp. $H^2(-E_\alpha-E_\beta-E_\gamma-E_\delta)=0$]. To show that $H^2(-E_\beta-E_\gamma)=0$, we consider the following exact sequence

$$0\longrightarrow -E_\beta-E_\gamma\longrightarrow -E_\gamma\longrightarrow\mathcal{O}_{\boldsymbol{P}^1}(-E_\beta.E_\gamma)\longrightarrow 0.$$

Since $-E_\beta.E_\gamma\geqq -1$, $H^1(\mathcal{O}_{\boldsymbol{P}^1}(-E_\beta.E_\gamma))=0$. This means $H^2(-E_\beta-E_\gamma)\cong H^2(-E_\gamma)$, which

is vanishing by (2). Finally assume that $E_\beta.E_\gamma \neq 1$ or $E_\beta.E_\delta \neq 1$. Then in the same manner we get $H^2(-E_\beta-E_\gamma-E_\delta) \cong H^2(-E_\gamma-E_\delta)$ because $H^1(\mathcal{O}_{\boldsymbol{P}^1}(-E_\beta.E_\gamma-E_\beta.E_\delta))=0$. Hence we get that $H^2(-E_\beta-E_\gamma-E_\delta)=0$ unless $E_\beta.E_\gamma=E_\gamma.E_\delta=E_\delta.E_\beta=1$. We have finished the proof.

§2. On the equations defining a projective variety.

In this section let V be a projective variety of dimension $n \geqq 2$ over k.

PROPOSITION 2.1. *Assume $V \subset \boldsymbol{P}^N$. Let H be a hyperplane of $\boldsymbol{P}^N$ such that $V \not\subset H$. Put $V'=V \cap H$. We denote by $\mathcal{I}_V$ the ideal sheaf of V in $\boldsymbol{P}^N$ and by $\mathcal{I}_{V',H}$ the ideal sheaf of V' in H. For a positive integer m, we assume $H^1(\mathcal{I}_V(m))=0$. If $\Gamma(\mathcal{I}_{V',H}(m+1)) \otimes \Gamma(\mathcal{O}_H(1)) \longrightarrow \Gamma(\mathcal{I}_{V',H}(m+2))$ is surjective, then*

$$\Gamma(\mathcal{I}_V(m+1)) \otimes \Gamma(\mathcal{O}_{\boldsymbol{P}}(1)) \longrightarrow \Gamma(\mathcal{I}_V(m+2))$$

is also surjective. Futhermore if $H^1(\mathcal{I}_V(m+1))=0$, then the converse is also true.

PROOF. We note the following exact sequence

$$(*) \qquad 0 \longrightarrow \mathcal{I}_V(m) \xrightarrow{\otimes H} \mathcal{I}_V(m+1) \longrightarrow \mathcal{I}_{V',H}(m+1) \longrightarrow 0,$$

which is obtained from the exact sequence

$$0 \longrightarrow \mathcal{O}_{\boldsymbol{P}^N}(-1) \xrightarrow{\otimes H} \mathcal{O}_{\boldsymbol{P}^N} \longrightarrow \mathcal{O}_H \longrightarrow 0$$

tensored with $\mathcal{I}_V(m+1)$ (cf. [3, p. 101]). Taking cohomology groups of the exact suquences (*) and (*) $\otimes \mathcal{O}_{\boldsymbol{P}^N}(1)$, we have the following commutative diagram.

$$\begin{array}{ccccccc}
 & & H^0(\mathcal{I}_V(m+1)) \otimes H^0(\mathcal{O}_{\boldsymbol{P}}(1)) & \longrightarrow & H^0(\mathcal{I}_{V',H}(m+1)) \otimes H^0(\mathcal{O}_{\boldsymbol{P}}(1)) & \longrightarrow & 0 \\
 & & \searrow^{\alpha} & & \downarrow & & \\
0 & \longrightarrow & H^0(\mathcal{I}_V(m+1)) & \longrightarrow & H^0(\mathcal{I}_V(m+2)) & \longrightarrow & H^0(\mathcal{I}_{V',H}(m+2)) \longrightarrow H^1(\mathcal{I}_V(m+1))
\end{array}$$

If we define the dotted arrow by $t \longmapsto t \otimes H$ where $t \in H^0(\mathcal{I}_V(m+1))$, then the shaded triangle commutes, which proves that the map α is surjective. The rest of the proposition is clear.

COROLLARY 2.2. *Assume $H^1(\mathcal{O}_V)=0$. Let $\mathcal{L}$ be an ample invertible sheaf on V such that $H^1(m\mathcal{L})=0$ for every $m \geqq 1$. Assume that there exists a non-zero section*

$s\in H^0(\mathcal{L})$. Put $V'=(s)_0$ and $\mathcal{L}'=\mathcal{L}\mid_{V'}$. Then $\mathcal{L}$ is normally generated(*) if and only if $\mathcal{L}'$ is normally generated. In this case $I(\mathcal{L})=\mathrm{Ker}\,[S\Gamma(\mathcal{L})\longrightarrow\bigoplus_{m\geqq 0}\Gamma(m\mathcal{L})]$ is generated by its elements of degree $2,3,\cdots$, and ν if and only if $I(\mathcal{L}')$ is generated by its elements of degree $2,3,\cdots$, and ν.

PROOF. First we note that $\mathcal{L}'$ is ample on V'. From the following exact sequence

$$0\longrightarrow(m-1)\mathcal{L}\xrightarrow{\otimes s}m\mathcal{L}\longrightarrow m\mathcal{L}'\longrightarrow 0,\ \text{where } m\geqq 1,$$

we get the following commutative diagram with exact rows

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \Gamma((m-1)\mathcal{L})\otimes\Gamma(\mathcal{L}) & \longrightarrow & \Gamma(m\mathcal{L})\otimes\Gamma(\mathcal{L}) & \longrightarrow & \Gamma(V',m\mathcal{L}')\otimes\Gamma(\mathcal{L}) & \longrightarrow & 0\\
 & & \downarrow & \nearrow & \downarrow\alpha & & \downarrow\beta & & \\
0 & \longrightarrow & \Gamma(m\mathcal{L}) & \longrightarrow & \Gamma((m+1)\mathcal{L}) & \longrightarrow & \Gamma(V',(m+1)\mathcal{L}') & \longrightarrow & 0,
\end{array}$$

where the dotted arrow is defined by $t\longmapsto t\otimes s$ for $t\in\Gamma(m\mathcal{L})$. So α is surjective if and only if β is surjective. This proves that the normal generatedness of $\mathcal{L}$ is equivalent to that of $\mathcal{L}'$. In this case the following diagram commutes

$$\begin{array}{ccc}
\phi_{|\mathcal{L}|}:\ V & \subset\!\longrightarrow & \boldsymbol{P}^{\dim|\mathcal{L}|}\\
\cup & & \cup\\
\phi_{|\mathcal{L}'|}:\ V' & \subset\!\longrightarrow & H\cong\boldsymbol{P}^{\dim|\mathcal{L}'|},
\end{array}$$

where H is a hyperplane section such that $V'=V\cap H$. Since $\mathcal{L}$ is normally generated, $H^1(\mathcal{J}_V(m))=$ for every $m\geqq 0$. Applying Proposition 2.1 to V and V', immediately we get the rest of the corollary.

For curves the following theorem is known.

THEOREM 2.3 ([5], [9] and [10]). *Let C be an irreducible reduced projective curve and D a divisor on C. Then:*

(1) *if* $\deg D\geqq 2p_a(C)+1$, *then D is normally generated and $I(D)$ is generated by its homogeneous parts $I_2(D)$ of degree 2 and $I_3(D)$ of degree 3;*

(2) *if* $\deg D\geqq 2p_a(C)+2$, *then $I(D)$ is generated by $I_2(D)$.*

(*) According to [2], an ample invertible sheaf $\mathcal{L}$ on a projective variety is said to be normally generated if $\Gamma(\mathcal{L})\otimes\Gamma(m\mathcal{L})\longrightarrow\Gamma((m+1)\mathcal{L})$ is surjective for every $m\geqq 1$. By abuse of terminology we say that a divisor D is normally generated if the corresponding invertible sheaf is normally generated. In this case D is very ample.

§ 3. Ample divisors on a Del Pezzo surface of dgree d, where $3 \leqq d \leqq 8$.

Now we enter the main issue of this paper.

PROPOSITION 3.1. *Let X be a Del Pezzo surface of degree $d \leqq 7$. For a divisor D on X the following conditions are equivalent:*

(i) *for every exceptional curve E on X, $E.D > 0$;*

(ii) *D is ample.*

Moreover if $3 \leqq d \leqq 7$, the above conditions are equivalent to the next one.

(iii) *D is very ample.*

PROOF. The implication (ii) $\Longrightarrow$ (i) is clear by Nakai's criterion. Combining Remark 1.2 and Lemma 1.3, we get (i) $\Longrightarrow$ (ii), (iii).

LEMMA 3.2. *Let X be a Del Pezzo surface of degree $d \leqq 8$. For an ample divisor $D \sim al - \sum_{i=1}^{r} b_i e_i$ assume that:*

(a) *$|D|$ has an irreducible curve; and*

(b) $3a - \sum_{i=1}^{r} b_i - 3 \geqq 0$.

Then D is very ample and $\phi_{|D|}(X)$ is projectively Cohen-Macaulay. In this case $I(D)$ is generated by $I_2(D)$ and $I_3(D)$. Moreover if D satisfies the condition

(b′) $3a - \sum_{i=1}^{r} b^i - 3 > 0$,

then $I(D)$ is generated by $I_2(D)$.

PROOF. Let $Y \in |D|$ be an irreducible curve. By the adjunction formula we get $\deg(D|_Y) - (2p_a(Y)+1) = 3a - \sum_{i=1}^{r} b_i - 3$, which is not less than zero. Then $D|_Y$ is normally generated by Theorem 2.3 and so is D by Corollary 2.2, since $H^1(mD) = 0$ for every $m \geqq 0$. Also $H^1(mD)$ vanishes for every $m < 0$ by Corollary 1.4.3 (ii), so we see that $\phi_{|D|}(X)$ is projectively Cohen-Macaulay. The rest follows also Theorem 2.3 and Corollary 2.2.

THEOREM 3.3. *Let X be a Del Pezzo surface of degree $3 \leqq d \leqq 8$ and D be a very ample divisor on X. Then $\phi_{|D|}(X)$ is projectively Cohen-Macaulay. Moreover if D is not linearly equivalent to the anti-canonical divisor on a cubic surface, then $I(D) = \mathrm{Ker}\,[S\Gamma(D) \longrightarrow \bigoplus_{m \geqq 0} \Gamma(mD)]$ is generated by its elements of degree 2.*

PROOF. We have only to apply Lemma 3.2. Since D is very ample, the condition (a) of the lemma is satisfied. If $d = 8$, then D on $\boldsymbol{F}_1$ can be written $al - b_1 e_1$ with $a > b_1 > 0$ by Lemma 1.1. In case $3 \leqq d \leqq 7$, we may assume that $D \sim al - \sum_{i=1}^{r} b_i e_i$ is such as in Remark 1.2. In each case D satisfies the condition (b). The equality

holds if and only if $a=3$ and $b_1=\cdots=b_6=1$. So we get the theorem.

§ 4. Anti-canonical divisors on Del Pezzo surfaces of degree 1 and 2.

A detailed study on anti-canonical divisors on Del Pezzo surfaces is found in [8, IV and V]. Here the author will add a few results on generaters of $I(-n\omega_X)$.

THEOREM 4.1. *Let X be a Del Pezzo surface of degree 2. Then $-n\omega_X$ is very ample if and only if $n\geqq 2$. In this case $\phi_{|-n\omega_X|}(X)$ is projectively Cohen-Macaulay and $I(-n\omega_X)$ is generated by its elements of degree 2.*

PROOF. We prove only that $I(-n\omega_X)$ is generated by its elements of degree 2. For the other assertions are found in [8, V, Theorem 1]. Since $-n\omega_X\sim 3nl-\sum_{i=1}^{7} ne_i$, $3a-\sum_{i=1}^{7} b_i-3=9n-7n-3$ is grater than zero under $n\geqq 2$. Hence $I(-n\omega_X)$ is generated by its elements of degree 2 from Lemma 3.2.

THEOREM 4.2. *Let X be a Del Pezzo surface of degree 1. Then:*

(1) *$Bs|-\omega_X|=\{$one point$\}$ and $Bs|-2\omega_X|=\phi$;*

(2) *$|-n\omega_X|$ has an irreducible member for every $n\geqq 1$;*

(3) *$-n\omega_X$ is very ample if and only if $n\geqq 3$. In this case $\phi_{|-n\omega_X|}(X)$ is projectively Cohen-Macaulay;*

(4) *if $n\geqq 4$, then $I(-n\omega_X)$ is generated by $I_2(-n\omega_X)$;*

(5) *$I(-3\omega_X)$ is generated by its elements of degree 2 and 3 but not generated by only those of degree 2.*

PROOF. The assertions (1), (2) and (3) are found in [8, IV, Proposition 6 and V, Theorem 1]. We will prove (4) and (5), applying Lemma 3.2. The condition (b′) of Lemma 3.2, that is $3a-\sum_{i=1}^{8} b_i-3=9n-8n-3>0$, holds when $n\geqq 4$, hence (4) is proved. When $n=3$, the condition (b) holds, so $I(-3\omega_X)$ is generated by its elements of degree 2 and 3. Let $Y\in|-3\omega_X|$ be a non-singular irreducible curve whose genus is equal to four. To prove (5) we have only to study generaters of $I(-3\omega_X|_Y)$ by Corollary 2.2. By the adjunction formula we get $\omega_Y\sim -2\omega_X|_Y$, this implies $-3\omega_X|_Y\sim\omega_Y+(-\omega_X)|_Y$. We claim that $-\omega_X|_Y$ is an effective divisor of degree 3. In fact considering the following exact sequence

$$0\longrightarrow\Gamma(2\omega_X)\longrightarrow\Gamma(-\omega_X)\longrightarrow\Gamma(Y,-\omega_X|_Y)\longrightarrow H^1(2\omega_X),$$

we get $\Gamma(-\omega_X|_Y)\cong\Gamma(-\omega_X)$, because $H^i(2\omega_X)=0$ for $i=0$ and 1. So $-\omega_X|_Y$ is effective. It is clear that $\deg(-\omega_X|_Y)=3\omega_X^2=3$. An application of the next lemma to $-3\omega_X|_Y$ yields that $I(-3\omega_X|_Y)$ is not generated by only its elements of degree 2. Hence $I(-3\omega_X)$ is not generated by its elements of degree 2 by Corollary 2.2.

LEMMA 4.3. ([4]). *Let C be a non-singular irreducible curve of genus $g \geqq 1$ and D a divisor on C of type $\omega_C + P_1 + P_2 + P_3$, where P_i is a closed point of C. Then D is normally generated and $I(D)$ is generated by its elements of degree 2 and 3 but not generated by only those of degree 2.*

§ 5. Ample divisors on Del Pezzo surfaces of degree 1 and 2.

In this section X is a Del Pezzo surface of degree 1 or 2. We study ample divisors on X from the same point of view as § 3.

THEOREM 5.1. *Let D be an ample divisor other than $-\omega_X$ on X of degree 2. Then D is very ample, $\phi_{|D|}(X)$ is projectively Cohen-Macaulay and $I(D)$ is generated by its elements of degree 2.*

THEOREM 5.2. *Let $D \sim al - \sum_{i=1}^{8} b_i e_i$ be an ample divisor on X of degree 1 such that $a \geqq b_1 + b_2 + b_3$ and $b_1 \geqq \cdots \geqq b_8 > 0$. Assume that D is neither $-\omega_X$ nor $-2\omega_X$. Then the following assertions hold:*

(1) *D is very ample and $\phi_{|D|}(X)$ is projectively Cohen-Macaulay;*

(2) *if D is $4l - 2e_1 - \sum_{j=2}^{8} e_j$ or $6l - 2\sum_{i=1}^{7} e_i - e_8$ or $9l - 3\sum_{i=1}^{8} e_i$, then $I(D)$ is generated by $I_2(D)$ and $I_3(D)$, but not generated by only $I_2(D)$.*

(3) *if D is not any of the three divisors described above, then $I(D)$ is generated by $I_2(D)$.*

Before the proof, we state some lemmas.

LEMMA 5.3. ([2, § 1. Generalized lemma of Castelnuovo]). *Suppose that $\mathcal{M}$ is an invertible sheaf on a variety V such that $\Gamma(\mathcal{M})$ has no base points. Let $\mathcal{F}$ be a coherent sheaf on V such that $H^i(\mathcal{F} \otimes (-i\mathcal{M})) = 0$ for every $i \geqq 1$. Then the map $\Gamma(\mathcal{F} \otimes (i-1)\mathcal{M}) \otimes \Gamma(\mathcal{M}) \longrightarrow \Gamma(\mathcal{F} \otimes i\mathcal{M})$ is surjective for every $i \geqq 1$.*

LEMMA 5.4. *Let D be an ample divisor on X such that $\Gamma(D)$ has no base points. Assume that the map $\beta : \Gamma(D) \otimes \Gamma(D) \longrightarrow \Gamma(2D)$ is surjective. Then D is normally generated.*

PROOF. From Corollary 1.4.2, $H^i((t-i)D) = 0$ for every $i = 1, 2$ and $t \geqq 2$. By Lemma 5.3 we see that $\Gamma(tD) \otimes \Gamma(D) \longrightarrow \Gamma((t+1)D)$ is surjective for each $t \geqq 2$. Under the assumption that β is surjective, this proves that D is normally generated.

LEMMA 5.5.1. *Let D be an ample divisor on X such that $\Gamma(D)$ has no base points. Assume:*

(1) $-D.\omega_X \geqq 3$;

(2) $|D+\omega_X|$ *has no base points*;

(3) $h^2(-D-2\omega_X)=0$.

Then D is normally generated.

PROOF. By Lemma 5.4 we have only to prove the surjectivity of the map β. Let C be an irreducible curve of $|-\omega_X|$ whose arithmetic genus is equal to one. Consider the following commutative diagram.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Gamma(D+\omega_X)\otimes\Gamma(D) & \longrightarrow & \Gamma(D)\otimes\Gamma(D) & \longrightarrow & \Gamma(D|_C)\otimes\Gamma(D) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\gamma} & & \\ 0 & \longrightarrow & \Gamma(2D+\omega_X) & \longrightarrow & \Gamma(2D) & \longrightarrow & \Gamma(2D|_C) & \longrightarrow & 0 \end{array}$$

Since $h^1(mD+\omega_X)=0$ for every $m\geqq 1$, the rows are exact. The assumption (1) implies that $\deg(D|_C)\geqq 2p_a(C)+1$. Hence $D|_C$ is normally generated by Theorem 2.3(1). So γ is surjective. Next we can apply Lemma 5.3 to α by the assumptions (2), (3) and the fact that $h^1(D-(D+\omega_X))=0$. Hence α is surjective, and so is β.

LEMMA 5.5.2. *Let X be a Del Pezzo surface of degree* 1 *and let D be an ample divisor on X such that $\Gamma(D)$ has no base points. Assume*:

(0) $h^1(D+2\omega_X)=0$;

(1) $-D.2\omega_X\geqq 5$;

(2) $|D+2\omega_X|$ *has no base points*;

(3) $h^2(-D-4\omega_X)=0$.

Then D is normally generated.

PROOF. By Theorem 4.2(2) there exists an irreducible curve C of $|-2\omega_X|$ whose arithmetic genus is equal to two. Then we have only to replace ω_X in Lemma 5.5.1 by $2\omega_X$.

Now to prove the theorems we may assume that $D\sim al-\sum_{i=1}^{r} b_ie_i$ is such that $a\geqq b_1+b_2+b_3$ and $b_1\geqq\cdots\geqq b_r>0$ by Remark 1.2. In this case we get $a\geqq 4$ since $D\nsim -\omega_X$. Moreover if a=4, then D is either $4l-\sum_{i=1}^{r} e_i$ (say $D_{(4)}$) or $4l-2e_1-\sum_{j=2}^{r} e_j$ (say $D_{(4')}$).

PROOF OF THEOREM 5.1. We have only to apply Lemma 3.2. For the condition (a) of Lemma 3.2, we will prove that D is very ample, classifying $D's$ as follows.

Case 1. D is either $D_{(4)}$ or $D_{(4')}$.

Case 2. $b_7\geqq 2$.

Case 3. $a>4, a>b_1+b_2+b_5$ and $b_7=1$.

Case 4. $a>4, a=b_1+b_2+b_3, b_1>b_3=b_5$ and $b_7=1$.

Case 5. $a=3b_1, b_1=b_5\geqq 2$ and $b_7=1$.

Since it is a simple calculation to check the condition (b') of Lemma 3.2, it is omitted.

Case 1. We will prove that D is normally generated using Lemma 5.5.1. First we note that $\Gamma(D)$ has no base points by Lemma 1.3. Easily we can check the condition (1) of Lemma 5.5.1. Since $D_{(4)}+\omega_X$ [resp. $D_{(4')}+\omega_X$] is l [resp. $l-e_1$], the condition (2) holds. Finally for (3), since $-D-2\omega_X$ is $2l-\sum_{i=1}^{7}e_i=(2l-\sum_{i=1}^{5}e_i)-e_6-e_7$ [resp. $(2l-\sum_{j=2}^{6}e_j)-e_7$], its second cohomology is vanishing by Lemma 1.5 (4) [resp. Lemma 1.5 (3)]. Thus we can apply Lemma 5.5.1 and get that D is normally generated.

Case 2. We will prove that D is very ample. Put $\nu=[(1/2)b_7]$. then D is linearly equivalent to the sum of the very ample divisor $\nu(-2\omega_X)$ and the divisor $(a-6\nu)l-\sum_{i=1}^{7}(b_i-2\nu)e_i$. Since $|(a-6\nu)l-\sum_{i=1}^{7}(b_i-2\nu)e_i|$ is free from base points by Lemma 1.3, D is very ample.

Case 3. Since $|D-D_{(4)}|$ has no base points, D is very ample.

Case 4. Since $|D-D_{(4')}|$ has no base points, D is very ample.

Case 5. First we note that if $b_1=2$, then D is either $6l-2\sum_{i=1}^{5}e_i-e_6-e_7$ (say $D_{(6)}$) or $6l-2\sum_{i=1}^{6}e_i-e_7$ (say $D_{(6')}$). It is clear that $D_{(6)}\sim D_{(4')}+(2l-\sum_{j=2}^{5}e_j)$ is very ample. For $D_{(6')}$, we can prove its normal generatedness applying Lemma 5.5.1. Indeed it is easy to see that the conditions (1) and (2) of Lemma 5.5.1 are satisfied. Since $h^2(-D_{(6')}-2\omega_X)=h^2(-e_7)=0$ by Lemma 1.5, (3) holds. Finally when $b_1\geqq 3$, we see that $|D-D_{(6)}|$ or $|D-D_{(6')}|$ has no base points. So D is very ample.

Now we will prove Theorem 5.2 on the same lines as above. But in the first place we have to prove the following lemma.

LEMMA 5.6. *Let $D\sim al-\sum_{i=1}^{8}b_ie_i$ be a divisor other than $-\omega_X$ such that $a\geqq b_1+b_2+b_3$ and $b_1\geqq\cdots\geqq b_8=1$ or 2. Then $|D|$ has no base points.*

PROOF. When $b_8=1$, we consider a morphism $\pi: X\longrightarrow X'$ to a Del Pezzo surface X' of degree 2 such that $\pi(E_8)$ is a point. By abuse of notation we also denote by $(l, e_1, \cdots, e_7)$ the basis of $\mathrm{Pic}(X')$ such that $e_i\sim\pi^*e_i$ for $1\leqq i\leqq 7$. Then the divisor $al-\sum_{i=1}^{7}b_ie_i$ on X' is ample, hence very ample by Theorem 5.1. So it has no unassigned base points, which shows that $|al-\sum_{i=1}^{7}b_ie_i-e_8|$ has no base points on X. Next if $b_8=2$, then we may assume $D\nsim -2\omega_X$ since we have already known

that $|-2\omega_X|$ has no base points. Then $|D+2\omega_X|$ is free from base points, and so is $|D|$.

PROOF OF THEOREM 5.2. We classify ample divisors D other than $-\omega_X$ nor $-2\omega_X$ into the following six cases.

Case 1. D is either $D_{(4)}$ or $D_{(4')}$.

Case 2. $b_8 \geqq 3$.

Case 3. $a>4, a>b_1+b_2+b_5$ and $b_8=1,2$.

Case 4. $a>4, a=b_1+b_2+b_3, b_1>b_3=b_5$ and $b_8=1,2$.

Case 5. $b_8=1, a=3b_1$ and $b_1=b_5 \geqq 2$.

Case 6. $b_8=2, a=3b_1$ and $b_1=b_5 \geqq 3$.

Case 1. By Lemma 5.6 $|D|$ is free from base points, so we get that D is normally generated from Lemma 5.5.1.

Case 2. Put $\nu=[(1/3)b_8]$, then D is linearly equivalent to the sum of the two divisors $\nu(-3\omega_X)$ and $(a-9\nu)l-\sum_{i=1}^{8}(b_i-3\nu)e_i$. If the latter is $-\omega_X$, then $D\sim-(3\nu+1)\omega_X$ with $\nu \geqq 1$, which is very ample by Theorem 4.2. In the other case, $|(a-9\nu)l-\sum_{i=1}^{8}(b_i-3\nu)e_i|$ is free from base points by Lemma 5.6, hence D is very ample.

Case 3. When $b_8=1$, $|D-D_{(4)}|$ has no base points, so D is very ample. When $b_8=2$, we replace $D_{(4)}$ by $4l-\sum_{i=1}^{7}e_i-2e_8$.

Case 4. Unless $D\sim 7l-3e_1-2\sum_{j=2}^{8}e_j$, D is very ample. Because $|D-D_{(4')}|$ is free from base points. When $D\sim 7l-3e_1-2\sum_{j=2}^{8}e_j$, we get that it is normally generated from Lemma 5.5.1.

Case 5. Similarly to Case 5 of the previous proof, first we note that if $b_1=2$, then D is one of the $D_{(6,k)}$'s, where $D_{(6,k)}\sim 6l-2\sum_{i=1}^{5+k}e_i-\sum_{j=6+k}^{8}e_j$, $k=0,1,2$. It is clear that $D_{(6,0)}$, which is the sum of $4l-\sum_{i\neq 5}^{8}e_i-2e_5$ and $2l-\sum_{i=1}^{4}e_i$, is very ample. Applying Lemma 5.5.1 we can prove that $D_{(6,1)}$ and $D_{(6,2)}$ are normally generated. When $b_1 \geqq 3$, $|D-D_{(6,k)}|$ has no base points, for some k. Hence D is very ample.

Case 6. In the same manner as above, we have only to prove that D with $b_1=3$ is very ample. Such D is one of the $D_{(9,k)}$'s, where $D_{(9,k)}\sim 9l-3\sum_{i=1}^{5+k}e_i-\sum_{i=6+k}^{8}e_j$, $k=0,1,2$. For $k=0$ or 1, since $D_{(9,k)}\sim D_{(6,k)}+(3l-\sum_{i=1}^{5+k}e_i-e_8)$, it is very ample. Next applying Lemma 5.5.2 to $D_{(9,2)}$, we conclude that $D_{(9,2)}$ is normally generated.

Finally we will examine the condition (b) of Lemma 3.2. By a simple calculation we see that $3a-\sum_{i=1}^{8}b_i-3$ is zero when D is either $D_{(4')}, D_{(6,2)}$ or $-3\omega_X$, and that it is grater than zero for the other cases. Hence we get the assertions (1) and (3) of the theorem.

Now to complete the proof we will show that $I(D_{(4')})$ and $I(D_{(6,2)})$ are not generated by their elements of degree 2. Since the homogeneous part of $I(D_{(4')})$ of degree 2 is the kernel of the surjection $S^2\Gamma(D_{(4')}) \longrightarrow \Gamma(2D_{(4')})$, its dimension is equal to ${}_5H_2-h^0(2D_{(4')})=(1/2)6\times5-(1/2)(9\times10-4\times5-7\times2\times3)=1$. This implies that $I(D_{(4')})$ cannot be generated by its elements of degree 2. Next when $D\sim D_{(6,2)}$, the proof is similar to that of Theorem 4.2(5) for $-3\omega_X$. Let Y be an irreducible curve of $|D|$, then $D|_Y\sim\omega_Y+(-\omega_X)|_Y$. Looking at the following exact sequence

$$0 \longrightarrow \Gamma(-D-\omega_X) \longrightarrow \Gamma(-\omega_X) \longrightarrow \Gamma(-\omega_X|_Y) \longrightarrow H^1(-D-\omega_X),$$

since $h^i(-D-\omega_X)=h^{2-i}(D+2\omega_X)=h^{2-i}(e_8)=0$ for $i=0,1$, we get that

$$\Gamma(-\omega_X)\cong\Gamma(-\omega_X|_Y)\neq0.$$

Thus $-\omega_X|_Y$ is an effective divisor of degree 3. Applying Theorem 2.3 and Corollary 2.2, we conclude that $I(D)$ is not generated by $I_2(D)$. We have done.

References

[1] Hartshorne, R., Algebraic Geometry, Graduate Text in Math. 52, Springer-Verlag, Berlin-Heidelberg-New York, (1977).

[2] Mumford, D., Varieties defined by quadratic equations, Questions on Algebraic Varieties, C.I.M.E., Cremonese, Rome, (1969), 29-100.

[3] ———, Lectures on Curves on an Algebraic Surface, Annals of Math. Studies 59, Princeton Univ. Press, Princeton (1966).

[4] Homma, M., Theorem of Enriquess-Petri type for a very ample invertible sheaf on a curve of genue three, preprint.

[5] ———On the equations defining a projective curve embedded by a non-special divisor, Tsukuba J. Math. **3** (1979), 31-39.

[6] Homma, Y., Projective normality and the defining equations of ample invertible sheaves on elliptic ruled surfaces with $e\geqq0$, Natural Sci. Rep. of the Ochanomizu Univ. **31** (1980).

[7] Manin, Yu. I., Cubic Forms, Algebra, Geometry, Arithmetic, North-Holland, Amsterdam, (1974).

[8] Demazure, M., Surfaces de Del Pezzo II, III, IV, V, Séminaire sur les Singularités des Surfaces, Lecture Note in Math. 777, Springer-Verlag, Berlin-Heidelberg-New York, (1980), 23-69.

[9] Fujita, T., Defining equations for certain types of polarized varieties, Complex analysis and algebraic geometry, Iwanami-shoten and Cambridge Univ. Press, Tokyo-Cambridge, (1977), 165-173.

[10] Saint-Donat, B., Sur les équations définissant une courbe algebrique, C. R. Acad. Sci. Paris **274**, (1972), 324-327 and 487-489.

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