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# ON A RELATION BETWEEN THE TOTAL CURVATURE AND THE MEASURE OF RAYS

Dedicated to Professor I. Mogi on his 60th birthday

By

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## § 0. Introduction.

Let $X$ be a 2-dimensional manifold, then we say that $X$ is finitely connected if the fundamental group $\pi_1(X)$ is finitely generated. If $X$ is noncompact and finitely connected, then it is homeomorphic to a compact surface with a finite number of points removed. Let $M$ be a 2-dimensional finitely connected complete noncompact Riemannian manifold without boundary. The Euler characteristic of $M, \chi(M)$, equals the Euler characteristic of the associated compact surface minus the number of points removed. A geodesic $\gamma:[0, \infty) \rightarrow M$ is called a ray when any subarc of $\gamma$ is the shortest connection between its end points. And all geodesics are assumed to be parametrized by arc length. Let $T_pM$ be the tangent space of $M$ at $p$ and $S_pM$ be the unit circle of $T_pM$ centered at the origin. $S_pM$ may be regarded as a standard unit circle $S^1$ from the Euclidean metric on $T_pM$. Hence we can consider the Riemannian measure on $S_pM$. Let $A(p)$ be the subset of $S_pM$ consisting of vectors $v$ in $S_pM$ such that the geodesic $\gamma_v:[0, \infty) \rightarrow M, \gamma_v(t)=\exp_p tv$, is a ray, where $\exp_p$ is the exponential map of $M$.

Recently, Maeda has proved in [4] the following theorem with interest in a problem whether less curvedness of a Riemannian manifold in some sense implies the existence of rays on it in large quantities or not when the manifold is nonnegatively curved;

THEOREM ([4]). Let $M$ be a 2-dimensional complete Riemannian manifold with nonnegative Gaussian curvature $G \geqq 0$ diffeomorphic to a Euclidean plane. If $\int_M G\,dv < 2\pi$, then for any point $p$ in $M$ such that $\# A(p) \geqq 2$, we have

$$\text{measure } A(p) \geqq 2\pi - \int_M G\,dv.$$

Here the total curvature $\int_M G\,dv$ of a noncompact Riemannian manifold $M$ is by

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definition the limit of a sequence $\left\{\int_{V_i} G\,dv\right\}_{j\in N}$ which does not depend on the choice of a sequence of compact domains $\{V_j\}_{j\in N}$ such that $V_j\subset V_{j+1}$ and $\bigcup_{j=1}^{\infty}V_j=M$. And we admit $+\infty$ and $-\infty$ to be the value of a total curvature. Hence the total curvature always exists if the Gaussian curvature is nonpositive or nonnegative. Moreover, we know that if there exists the total curvature of a complete finitely connected surface $M$, the following well know inequality of Cohn-Vossen holds ([3]);

$$\int_M G\,dv \leqq 2\pi\chi(M).$$

The aim of this note is to give a relation between the total curvature and the measure of rays, the abundance of rays, on a 2-dimensional complete finitely connected Riemannian manifold $M$. We shall prove the following theorem;

THEOREM 1. Let $M$ be a 2-dimensional finitely connected complete noncompact Riemannian manifold with nonpositive Gaussian curvature $G$. If $\int_M G\,dv>2\pi(\chi(M)-1)$, then we have

$$\text{measure } A(p) \leqq 2\pi\chi(M)-\int_M G\,dv \qquad \text{for any point } p\in M.$$

And from the proof we can get the following theorem which includes Maeda's result;

THEOREM 2. Let $M$ be a 2-dimensional complete Riemannian manifold homeomorphic to a Euclidean plane. If $\int_M G^+\,dv < 2\pi$, then we have

$$\text{measure } A(p) \geqq 2\pi-\int_M G^+\,dv \qquad \text{for any point } p\in M,$$

where $G^+=(|G|+G)/2$.

We remark that the right quantity of the inequality in Theorem 1 is not guaranteed to be bounded above by $2\pi$. The assumption, $\int_M G\,dv> 2\pi(\chi(M)-1)$, is put for the inequality to have geometric meaning. The assumption, $\int_M G^+\,dv < 2\pi$, in Theorem 2 is put by the same reason.

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## §1. Preliminaries.

In this section, we shall introduce the various terminologies which follow [2], [3] and modifications of Shiohama [5]. Hereafter $M$ always denotes a 2-dimensional finitely connected complete noncompact Riemannian manifold without boundary unless otherwise mentioned. Now let $M$ be homeomorphic to $M_0/\{p_1, p_1, \cdots, p_n\}$ under a homeomorphism $f$, where $M_0$ is a compact surface and $p_1, p_2, \cdots, p_n$ are points of $M$.

**Definition 1.** An open set $U$ in $M$ is called an open tube if $U$ is homeomorphic to $S^1\times(0, \infty)$ and the boundary of $U(:=\partial U)$ is homeomorphic to $S^1$. And a closed set of $M$ is called a tube or an $R_0$-tube if it is homeomorphic to $S^1\times[0, \infty)$ and its boundary is a noncontractible simply closed geodesic polygon $R_0$. It is written as $U(R_0)$.

Now, for each point $p_j, j=1, 2, \cdots, n$, we can choose mutually disjoint open neighbourhood $\tilde{U}_j$ of $p_j$ in $M_0$ such that $U_j := f^{-1}(\tilde{U}_j\setminus\{p_j\})$ is a tube.

Let $U(R_0)$ be a given tube of $M$ and let $\rho_{U(R_0)}$ be the distance function on $U(R_0)$, that is, for any points $p, q \in U(R_0)$, $\rho_{U(R_0)}(p, q)$ is defined to be the infimum of the lengths of all piecewise smooth curves joining $p$ and $q$ in $U(R_0)$. Then the function $X_{U(R_0)} : [0, \infty) \to \boldsymbol{R}$ is defined as follows; $X_{U(R_0)}(t)$ is the infimum of the lengths of all piecewise smooth noncontractible closed curves $R$ in $U(R_0)$ which satisfies $\rho_{U(R_0)}(R, R_0) \leqq t$. It is easily seen that the function $X_{U(R_0)}$ is Lipschitz continuous. We shall classify tubes by making use of $X_{U(R_0)}$ in accordance with [2]. The following three cases may occur for $R_0$-tubes;

Case 1. $X_{U(R_0)}$ does not attain $\inf\{X_{U(R_0)}(s) : s\geqq 0\}$,

Case 2. $X_{U(R)}$ attains $\inf\{X_{U(R)}(s) : s\geqq 0\}$ for any subtube $U(R)$ in $U(R_0)$,

Case 3. $X_{U(R_0)}$ attains $\inf\{X_{U(R_0)}(s) : s\geqq 0\}$ but $X_{U(R)}$ does not attain $\inf\{X_{U(R)}(s) : s\geqq 0\}$ for some subtube $U(R)$ in $U(R_0)$.

**Definition 2.** An $R_0$-tube $U(R_0)$ is said to be contracting, expanding or bulging if the function $X_{(R_0)}$ satisfies Case 1, Case 2 or Case 3, respectively.

According to this definition, a bulging tube is essentially a contracting tube. Hence we have only to consider the contracting or expanding tubes. And note that subtubes of a contracting (expanding) tubes are also contracting (expanding).

**Definition 3.** Let $U(R_0)$ be a given tube and $R$ be a noncontractible simply closed geodesic polygon in $U(R_0)$. If all vertical angles of $R$ which are measured in $U(R)$ are less (more) than $\pi$, then the geodesic polygon $R$ is said to be convex (concave).

**Definition 4.** Let an $R_0$-tube $U(R_0)$ and a nonnegative number $t$ be arbitrarily given. If a noncontractible closed curve $R(t)$ in $U(R_0)$ satisfies following two conditions, then $R(t)$ is called the solution of Minimal Problem (or simply M.P.) for

$U(R_0)$ and $t$;

$$L(R(t))=X_{U(R_0)}(t) \quad \text{and} \quad \rho_{U(R_0)}(R_0, R(t))\leqq t.$$

**Definition 5.** Let the following objects be arbitrarily given; a nonnegative number $t$, a tube $U(R_0)$ and a ray $\gamma:[0,\infty)\to M$ such that $\gamma([a,\infty))\subset U(R_0)$ and $\gamma(a)\in R_0$ $(a>0)$. If a noncontractible closed curve $R(t)$ in $U(R_0)$ which passes through $\gamma(a+t)$ satisfies $L(R(t))=Y_{U(R_0)}(t)$, then $R(t)$ is called the solution of Minimal Problem along $\gamma$ (or simply $\gamma$-M.P.) for $U(R_0)$ and $t$. Here the function $Y_{U(R_0)}:[0,\infty)\to R$ is defined as follows; $Y_{U(R_0)}(t)$ is the infimum of the lengths of piecewise smooth noncontractible closed curves $R$ in $U(R_0)$ which pass through $\gamma(a+t)$.

As is seen in [2] and [3], two kinds of solutions surely exist and they satisfy the following facts;

**Fact 1.** Let $U(R_0)$ be a contracting tube. Then the solution of M.P. $R(t)$ for $U(R_0)$ and $t\geqq 0$ is either a closed geodesic or a convex geodesic loop. Hence the distance between $R(t)$ and $R_0$ is equal to the distance between the vertex of $R(t)$ and $R_0$ if $R(t)$ is a convex geodesic loop. The solution of $\gamma$-M.P. for $U(R_0)$ and $t\geqq 0$ is either a closed geodesic or a geodesic loop.

**Fact 2.** Let $U(R_0)$ be an expanding tube. Then the solution of M.P. $R(t)$ for $U(R_0)$ and $t\geqq 0$ is either a closed geodesic or a concave geodesic polygon. And for some $t_0\geqq 0$, $R(t_0)$ is the shortest noncontractible closed curve in $U(R(t_0))$. The solution of $\gamma$-M.P. for $U(R_0)$ and $t$ is either a closed geodesic or a geodesic polygon whose vertical angles except for the vertical angle at $\gamma\cap R_0$ measured in $U(R_0)$ are more than $\pi$.

For the solution of $\gamma$-M.P. we can not get the general information about the vertical angle which is on $\gamma$. See Cohn-Vossen ([3]), Busemann ([2]) and Bleecker ([1]) for more details of the properties on the solution of M.P.

## §2. Construction of an expanding filtration.

Throughout this section, let $p$ be an arbitrarily fixed point of $M$. And let $\mathbf{N}$ denote the set of natural numbers. It is our purpose in this section to construct a family of compact domains $\{V_j\}_{j\in N}$ with properties (1), (2) and (3);

(1) $V_1\ni p$,

(2) $V_j\subset V_{j+1}$ and $\bigcup_{j=1}^{\infty} V_j=M$,

(3) $\partial V_j$ is a closed geodesic or a geodesic polygon which intersects any ray emanating from $p$ at most once.

LEMMA 1. If $U(R_0)$ is a contracting tube which does not contain the point $p$, then there exist noncontractible closed curves $R_j, j \in \boldsymbol{N}$, in $U(R_0)$ such that

(1) $R_j$ is either a closed geodesic or a convex geodesic loop whose vertex lies on a fixed ray,

(2) $\lim_{j\to\infty} \rho_{U(R_0)}(R_0, R_j)=\infty$,

(3) $R_j$ intersects any ray with at most one point.

PROOF. Let $C_0$ be the length of $R_0$ and let $\gamma$ be a ray emanating from $p$ and diverging in $U(R_0)$. Set $X(t):=X_{U(R_0)}(t)$ and $Y(t):=Y_{U(R_0)}(t)$. Then we know the existence of a number $t_j \in (C_0+j, \infty)$ with $Y(t_j)<Y(0) \leqq C_0$. In fact, the contracting condition implies the existence of a number $s_j \in (C_0+j, \infty)$ with $X(s_j)<X(0) \leqq C_0$, $X(s_j)=L(\bar{R}(s_j))$ and $\rho_{U(R_0)}(R_0, \bar{R}(s_j))=s_j$, where $\bar{R}(s_j)$ is the solution of M.P. for $U(R_0)$ and $s_j$. Let $t_j$ be the number with $\gamma(a+t_j):=\bar{R}(s_j) \cap \gamma$. Then we can get the following relations; $t_j>C_0+j$ and $Y(t_j) \leqq X(s_j)<X(0) \leqq Y(0) \leqq C_0$ Hence $t_j$ is a required number.

Now let $R_j:=R(t_j)$ be the solution of $\gamma$-M.P. for $U(R_0)$ and $t_j$, then $R_j$ satisfies $\rho_{U(R_0)}(R_0, R_j)>j$. This implies $R_j \cap R_0=\phi$. Hence $R_j$ is either a closed geodesic or a geodesic loop. Let $s_j' \in (t_j, \infty)$ be the number such that $X(s_j')<X(t_j)$. Such a number surely exists from the contracting condition. And putting $\gamma(a+t_j'):=\bar{R}(s_j') \cap \gamma$, we have $Y(t_j') \leqq X(s_j')<X(t_j) \leqq Y(t_j)$. Therefore there exists a number $u_j \in (t_j, t_j')$ such that $Y$ is decreasing at $u_j$. $R(u_j)$ must not be a concave geodesic loop. Set newly $R_j:=R(u_j)$, then $R_j$ satisfies (1) and (2). Moreover it can be easily proved that any ray which is divergent in $U(R_0)$ never intersects $R_j$ twice because of their minimality.

LEMMA 2. If $U(R_0)$ is an expanding tube which does not contain the point $p$, then there exist noncontractible closed curves $R_j, j \in \boldsymbol{N}$, in $U(R_0)$ such that

(1) $R_j$ is either a closed geodesic or a concave geodesic polygon,

(2) $\lim_{j\to\infty} \rho_{U(R_0)}(R_0, R_j)=\infty$,

(3) $R_j$ intersects any ray with at most one point.

PROOF. From Fact 2, we know the existence of the shortest noncontractible closed curve $R_1$ in $U(R_1)$ which is either a closed geodesic or a convave geodesic polygon in $U(R_0)$. Let $\sigma$ be any ray emanating from $p$ and diverging in $U(R_0)$. Then $\sigma$ does not meet $R_1$ at more than one point. In fact if $R_1$ is a closed geodesic, then our assertion is trivial because of the minimality of $R_1$ and $\sigma$. Hence we may assume that $R_1$ is a concave geodesic polygon. Let $q_1:=\sigma(t_1)$ and $q_2:=\sigma(t_2), t_1<t_2$, be the first point of intersection and the second point of intersection of $\sigma$ and $R_1$, respectively. Then $\sigma([t_1, t_2])$ is contained in $U(R_1)$ because of the concavity of $R_1$. Let $R_1'$ be a new noncontractible geodesic polygon which is gotten by exchanging

the subarc of $R_1$ between $q_1$ and $q_1$ for $\sigma|[t_1, t_2]$. The $R_1'$ is contained in $U(R_1)$ and has the same length as that of $R_1$ because of the minimality of $\sigma$ and $R_1$. Since $R_1'$ has a vertex at $q_1$, we can get a shorter noncontractible curve in $U(R_1)$ by exchanging a subarc of $R_1'$ for a minimal geodesic in a neighbourhood of $q_1$. This contradicts the shortestness of $R_1$ in $U(R_1)$. Consequently, $\sigma$ does not meet $R_1$ at more than one point. For $j\geqq 2$, let $R_j'$ be a noncontractible geodesic polygon such that $\rho_{U(R_0)}(R_1, R_j')>j$ and let $R_j$ be the shortest noncontractible closed curve in $U(R_j')$. Then we can see that $R_j$ satisfies (1), (2) and (3).

Since $M$ is finitely connected, $M\setminus K$ can be represented to a union of $n$ tubes $U_\alpha$, $\alpha=1, 2, \cdots, n$, for a large compact set $K$ whose boundary consists of $n$ geodesic polygons each of which may be considered such as an $R_0$ in the preceeding Lemmas. Thus Lemma 1 and Lemma 2 imply the existence of noncontractible closed curves $R_{j\alpha}$, $j\in \boldsymbol{N}$, in each $U_\alpha$. Let $V_j$ be the compact domains in $M$ bounded by $\bigcup_\alpha R_{j\alpha}$. Then we have

(1) $V_1\ni p$, $V_j\subset V_{j+1}$ and $\bigcup_{j=1}^\infty V_j=M$,

(2) $\partial V_j\cap U_\alpha(=R_{j\alpha})$ is a noncontractible geodesic polygon in $U_\alpha$ which does not intersect any ray at more than one point.

The proof of our theorem is achieved by constructing such a special family of compact domains that are chosen by taking into account of the position of rays emanating from $p$. $F(p)$ is by definition the set of all points on rays emanating from $p$. And set $D(p):=M\setminus F(p)$. $F(p)$ is a closed set which is homeomorphic to a closed set of $T_pM$ under $\exp_p$. Hence $F(p)$ contains no handles on it. To compute the total curvature of $M$, we must compute that of $V_j$. And it is a sum of those of $F(p)\cap V_j$ and $\mathrm{cl}\,D(p)\cap V_j$, where $\mathrm{cl}\,D(p)$ denotes the closure of $D(p)$. It is difficult to compute the total curvature of $\mathrm{cl}\,D(p)\cap V_j$ because of the existence of handles. However, we can get an information about the total curvature of $\mathrm{cl}\,D(p)$. Hence we must take into account of the position of rays emanating from $p$ to relate the total curvature of $F(p)\cap V_j$ and that of $D(p)\setminus\mathrm{int}\,V_j$. Namely we need the following lemma.

LEMMA 3. There exists a family of compact domains $\{V_j\}_{j\in\boldsymbol{N}}$ in $M$ which satisfies the above properties (1), (2) and the following properties; For each $\alpha$

(a) if $U_\alpha$ is expanding, then there is no vertices of $\partial V_j\cap U_\alpha$ on the rays which are boundaries of $D(p)$,

(b) if $U_\alpha$ is contracting and if $\mathrm{int}\,(F(p)\cap U_\alpha)$ is not empty, then there is no vertices of $\partial V_j\cap U_\alpha$ on the rays which are boundaries of $D(p)$,

(c) if $U_\alpha$ is contracting and if $\mathrm{int}\,(F(p)\cap U_\alpha)$ is empty, then the vertex of $\partial V_j\cap U_\alpha$ lies on a ray which is a boundary of $D(p)$ if the vertex exists.

PROOF. (a) In the case of $U_\alpha$ being expanding, the construction follows from Lemma 2. Take $R_0$ and $R'_j$ in the proof of Lemma 2 so that their vertices do not lie on the rays which are boundaries of $D(p)$. This is possible because the rays which are boundaries of $D(p)$ are measure zero. Since $R_j$ is a solution of M.P. for $U(R'_j)$, $R_j$ is either a closed geodesic or a concave geodesic polygon whose vertices are on those of $R'_j$. In this way, we can get a family $\{R_j\}_{j\in N}$ of closed geodesics or concave geodesic polygons without their vertices on the rays which are boundaries of $D(p)$.

(b) and (c). In the case of $U_\alpha$ being contracting, the construction follows from Lemma 1. Take a ray $\gamma$ which passes through the interior of $F(p)\cap U_\alpha$ if it is not empty and take a ray which is a boundary of $D(p)\cap U_\alpha$ if $\mathrm{int}(F(p)\cap U_\alpha)$ is empty. And applying Lemma 1, we get a family $\{R_j\}_{j\in N}$ of closed geodesics or convex geodesic loops which has the desired properties.

## § 3. Proof of Theorems.

Let $\{V_j\}_{j\in N}$ be the family of compact domains obtained in Lemma 3. Let $\bar{D}$ be one of the connected components of $D(p)\setminus V_1$. And let $\sigma$ and $\tau$ be the rays which are boundaries of $\bar{D}$. Let $\underline{D}$ be one of the connected components of $D(p)\cap V_1$ and set $F:=F'\cup\{p\}$ and $\underline{F}:=F\cap V_1$, where $F'$ is one of the connected components $F(p)\setminus\{p\}$. Let $\Psi^+$ and $\Psi^-$ be the vertical angles of $\mathrm{cl}\,\bar{D}$ at $\partial V_1\cap\sigma$ and $\partial V_1\cap\tau$, respectively. And let $\theta^\beta$ be a vertical angle of $\partial V_1\cap\mathrm{cl}\,\bar{D}$ measured in $\mathrm{cl}\,\bar{D}$.

Under these notations, we can prove the following Lemma by following Maeda [4].

LEMMA 4. The following inequality holds good;

$$\int_{\underline{D}} G\, dv \geqq \Psi^+ + \Psi^- - \pi - \sum_{\partial V_1\cap\mathrm{cl}\bar{D}} (\pi-\theta^\beta),$$

where the summation is taken over all vertices of $\partial V_1\cap\mathrm{cl}\,\bar{D}$. And the equality holds if the Gaussian curvature $G$ is nonpositive.

PROOF. Let $E_j:=\partial V_j\cap\mathrm{cl}\,D$. $A^+_{j\theta}$ and $A^-_{j\theta}$ are by definition the set of all initial tangent vectors $\dot\gamma(0)$ of the shortest geodesic $\gamma$ connecting between $p$ and $q$ of $E_j$ which satisfy $\measuredangle(\dot\gamma(0),\dot\sigma(0))\leqq\theta$ and $\measuredangle(\dot\gamma(0),\dot\tau(0))\leqq\theta$, respectively. And let $A^+_\theta$ and $A^-_\theta$ be the set of all unit vectors $v$ which satisfy $\measuredangle(v,\dot\sigma(0))\leqq\theta$ and $\measuredangle(v,\dot\tau(0))\leqq\theta$, respectively. Moreover define the number $\theta(j)$ for each natural number $j$ by

$$\theta(j):=\inf\{\theta\in\boldsymbol{R}\,;\ \text{there exists a geodesic } \gamma \text{ in } G(p,q) \text{ such that } \dot\gamma(0)\in A^+_{j\theta}\cup A^-_{j\theta} \text{ for any } q\in E_j\}.$$

Here $G(p, q)$ denotes the set of all the shortest geodesic connections from $p$ to $q$. We assert that $\theta(j)$ tends to zero as $j$ goes to infinity. In fact, if $\theta(j)$ does not tend to zero, then there is a constant $C_0>0$ and a subsequence $\{j_i\}\subset\{j\}$ such that $\theta(j_i)\geqq C_0$ for any $j_i\in\{j_i\}$. Hence for any $j_i$, there is a point $q_{j_i}$ in $E_{j_i}$ and $\gamma_{j_i}\in G(p, q_{j_i})$ such that $\dot\gamma_{j_i}(0)$ does not belong to $A^+_{j_i,(1/2)C_0}\cup A^-_{j_i,(1/2)C_0}$. From the sequence $\{\dot\gamma_{j_i}(0)\}$, we can choose a convergent subsequence $\{\dot\gamma_{j_{i_k}}(0)\}$. Let $v_0\in S_pM$ be the limit vector of $\{\dot\gamma_{j_{i_k}}(0)\}$, then from the construction $v_0$ is not contained in $A^+_{(1/3)C_0}\cup A^-_{(1/3)C_0}$ and the geodesic $\gamma_0:[0,\infty)\to M$ defined by $\gamma_0(t):=\exp_p tv_0$ is a ray. This contradicts the fact that $\gamma_0$ belongs to the domain which no ray passes through. Let the set $E^+_j$ and $E^-_j$ be defined as follows;

$$E^+_j:=\{q\in E_j;\ \text{there exists a geodesic } \gamma\in G(p,q) \text{ such that } \dot\gamma(0)\in A^+_{j,\theta(j)}\},$$
$$E^-_j:=\{q\in E_j;\ \text{there exists a geodesic } \gamma\in G(p,q) \text{ such that } \dot\gamma(0)\in A^-_{j,\theta(j)}\}.$$

Then it is easily seen that $E_j=E^+_j\cup E^-_j$ and $E^+_j$ and $E^-_j$ are nonempty closed sets in $E_j$ from the connectivity of the cut point. The connectivity of $E_j$ implies the existence of a point $q_j\in E^+_j\cap E^-_j$ such that the initial vectors $\dot\gamma^+_j(0)$ and $\dot\gamma^-_j(0)$ of minimal geodesics between $p$ and $q_j$ which belong to $A^+_{\theta(j)}$ and $A^-_{\theta(j)}$, respectively. Therefore $\gamma^+_j$ tends to $\sigma$ and $\gamma^-_j$ tends to $\tau$ as $j$ goes to infinity. Let $\bar D_j$ be the subset of $\bar D$ bounded by $\gamma^+_j, \gamma^-_j$ and $\partial V_1$. Then we can get the following inequality from Theorem of Gauss-Bonnet,

$$\begin{aligned}\int_{\mathrm{cl}\bar D} G\,dv &= \lim_{j\to\infty}\int_{\mathrm{cl}\bar D} G\,dv\\ &= \lim_{j\to\infty}\Big[2\pi\chi(\bar D_j)-(\pi-\Psi^+_j)-(\pi-\Psi^-_j)-(\pi-\varphi_j)-\sum_{\partial V_1\cap\mathrm{cl}\bar D_j}(\pi-\theta^{\beta(j)})\Big)\\ &\geqq \Psi^+ + \Psi^- -\pi-\sum_{\partial V_1\cap\mathrm{cl}\bar D}(\pi-\theta^\beta),\end{aligned}$$

where $\Psi^+_j$ and $\Psi^-_j$ are the vertical angles of $\mathrm{cl}\,\bar D_j$ at $\partial V_1\cap\gamma^+_j$ and $\partial V_1\cap\gamma^-_j$, respectively. And $\theta_\beta{}^{(i)}$ is a vertical angle of $\partial V_1\cap\mathrm{cl}\,\bar D_j$ measured in $\mathrm{cl}\,\bar D_j$ and $\varphi_j=\measuredangle(\dot\gamma^+_j(t_j), \dot\gamma^+_j(t_j))$, where $t_j$ is the distance from $p$ to $q_j$. Thus the inequality is verified.

Next, consider the case that the Gaussian curvature of $M$ is nonpositive. Let $p_j:=\partial V_1\cap\gamma^+_j$ and $r_j:=\partial V_1\cap\gamma^-_j$. And let $(p_j, q_j, r_j)$ be the geodesic triangle determined by the three shortest geodesic segments. Let $c:[0,1]\to M$ be the shortest geodesic segment with $c(0)=p_j$ and $c(1)=r_j$. Since $(p_j, q_j, r_j)$ is contractible, we can consider the homotopy $H:[0,1]\times[0,1]\to M$ such that for any $s\in[0,1]$, $H(0,s)=c(s)$, $H(1,s)=q_j$ and $H([0,1],s)=$ the shortest geodesic segment between $c(s)$ and $q_j$. Let $(\tilde p_j, \tilde q_j, \tilde r_j)$ be a lift of $(p_j, q_j, r_j)$ in the universal Riemannian covering space $\tilde M$ of $M$ which is gotten by making use of the homotopy $H$ and let $\tilde\varphi_j$ be the vertical angle of $(\tilde p_j, \tilde q_j, \tilde r_j)$ at $\tilde q_j$. Then from the construction we have $\varphi_j=\tilde\varphi_j$ and it is seen that $\rho_{\tilde M}(\tilde p_j, \tilde q_j)\to\infty$, $\rho_{\tilde M}(\tilde r_j, \tilde q_j)\to\infty$ and $\rho_{\tilde M}(\tilde p_j, \tilde r_j)<C$ as $j\to\infty$, where $C$ is a constant.

Hence making use of the law of cosines, we can see that $\tilde{\varphi}_j \to 0$ as $j \to \infty$. Therefore the equality holds when Gaussian curvature of $M$ is nonpositive.

Hereafter let $\bar{D}^\lambda$ and $\underline{D}^\mu$ be connected components of $D(p)\setminus V_1$ and $D(p)\cap V_1$, respectively. And let $F^{\lambda *}$ be a connected component of $(F(p)\cap V_1)\setminus\{p\}$ and set $F^\lambda := F^{\lambda *}\cup\{p\}$. Then we can get the following Proposition which implies our theorem.

PROPOSITION 5. The following inequality holds good;

$$\text{measure } A(p) \geqq 2\pi\chi(M) - \int_{\mathrm{cl}\,D(p)} G\,dv$$

at any point $p$ of $M$. And the equality holds when Gaussian curvature of $M$ is nonpositive.

PROOF. Let $\underline{F}^\lambda$ be the one such that $\operatorname{int}\underline{F}^\lambda \neq \emptyset$. Since $\underline{F}^\lambda$ is diffeomorphic to a polygon in $T_pM$, we have

$$\int_{\underline{F}^\lambda} G\,dv = a^\lambda + (\Psi^\lambda)^+ + (\Psi^\lambda)^- - \pi + \sum_{\partial V_1 \cap \underline{F}^\lambda} (\pi - \theta^\beta) \qquad \cdots (*)$$

where $a^\lambda$ is a vertical of $\underline{F}^\lambda$ at $p$, $(\Psi^\lambda)^+$ and $(\Psi^\lambda)^-$ are the vertical angles of $\underline{F}^\lambda$ formed with $\partial V_1$ and the rays which are the boundaries of $\underline{F}^\lambda$ and $\theta^\beta$ is a vertical angle of $\partial V_1 \cap \underline{F}^\lambda$ measured in $M\setminus V_1$. From our construction, there is no vertex of $\partial V_1$ on the rays which are the boundaries of $F^\lambda$'s. Hence we can get the following inequality by using (*), Lemma 4 and the fact that vertically opposite angles are identical;

$$\begin{aligned}\int_{V_1} G\,dv &= \sum_{\mathrm{all}\,F^\lambda} \int_{\underline{F}^\lambda} G\,dv + \sum_{\mathrm{all}\,\underline{D}^\mu} \int_{\mathrm{cl}\,d^\mu} G\,dv \\ &= \sum_{\mathrm{all}\,\underline{F}^\lambda} \Big[ a^\lambda + (\Psi^\lambda)^+ + (\Psi^\lambda)^- - \pi + \sum_{\partial V_1 \cap \underline{F}^\lambda} (\pi - \theta^\beta) \Big] + \sum_{\mathrm{all}\,\underline{D}^\mu} \int_{\mathrm{cl}\,\underline{D}^\mu} G\,dv \\ &\leqq \text{measure } A(p) + \sum_{\partial V_1} (\pi - \theta^\beta) + \int_{\mathrm{cl}\,D(p)} G\,dv.\end{aligned}$$

On the other hand, we have

$$\int_{V_1} G\,dv = 2\pi\chi(M) + \sum_{\partial V_1} (\pi - \theta^\beta).$$

Hence we get the desired inequality and the equality holds when Gaussian curvature of $M$ is nonpositive.

Now Theorem 1 and 2 are the direct consequences of Proposition 5.

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