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# EQUIVALENCE PROBLEM AND AUTOMORPHISM GROUPS OF CERTAIN COMPACT RIEMANN SURFACES

By

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## Introduction.

Let $V$ be the compact Riemann surface defined by the equation:

$$y^n = f(x),$$

where $n$ is a positive integer and $f$ is a rational function of $x$. For such $V$, there is a conjecture:

CONJECTURE: *The moduli of such $V$ can be determined by the branch locus of the map $(x, y) \in V \to x \in \mathbb{P}^1$.*

Here, $\mathbb{P}^1$ is the complex projective line. The purpose of this paper is to give affirmative answers to the conjecture under various conditions. It is separated into 3 parts.

In Part 1, we assume that $n=p$ is a prime number and obtain a result. Recently, Kato [2] has improved this result extensively.

In Part 2, we assume that $f(x)$ is a polynomial of degree $p$ with $p$ a prime number, and obtain a result.

In Part 3, we assume:

$$f(x) = (x-\alpha_1)\cdots(x-\alpha_n),$$

where $\alpha_1, \cdots, \alpha_n$ are mutually distinct complex numbers, and obtain an affirmative answer to the conjecture. In this case, the result can be extended to the case of non-singular hypersurfaces in $\mathbb{P}^{r+1}$, the $(r+1)$-dimensional complex projective space.

Corresponding to each case, we naturally obtain information on the automorphism group $\mathrm{Aut}(V)$ of $V$.

We note such compact Riemann surfaces were treated by Picard [8], Lefschetz [4], Shimura [9] and Kuribayashi [3] in connection with the study of Jacobian varieties and the concrete construction of some modular fuctions of several variables.

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**Part 1.**

**1.1. Reduction of the Problem.**

Let $p$ be a prime number. Let $V$ be the compact Rieman surface defined by the equation:

$$V: y^p = f(x).$$

The precise meaning of this is that $V$ is a non-singular model of the closure in $\mathbb{P}^1 \times \mathbb{P}^1$ of the affine curve: $y^p = f(x)$. We write the rational function $f(x)$ as follows:

$$f(x) = c(x-\alpha_1)^{k_1} \cdots (x-\alpha_m)^{k_m}, \tag{1}$$

where $c$ is a non-zero constant, $\alpha_1, \cdots, \alpha_m$ are mutually distinct complex numbers and $k_1, \cdots, k_m$ are integers. We may assume that $c=1$ and

$$k_1 \not\equiv 0, \cdots, k_m \not\equiv 0 \pmod{p}.$$

In fact, if $k_1 = k_1' p$, then $V$ and

$$V_1: y^p = (x-\alpha_2)^{k_2} \cdots (x-\alpha_m)^{k_m}$$

are conformally equivalent (i.e., biholomorphic) under the birational transformation:

$$(x, y) \in V \longrightarrow (x, y/(c^{1/p}(x-\alpha_1)^{k_1'})) \in V_1.$$

Moreover, it is easy to see that if

$$k_1 \equiv j_1, \cdots, k_m \equiv j_m \pmod{p},$$

then the compact Riemann surfaces $V$ and

$$V': y^p = (x-\alpha_1)^{j_1} \cdots (x-\alpha_m)^{j_m}$$

are conformally equivalent. Now, in (1), one of the following cases occurs:

(i) $k_1 + \cdots + k_m \equiv 0 \pmod{p}$,

(ii) $k_1 + \cdots + k_m \not\equiv 0 \pmod{p}$.

In the case (i), the point infinity, $\infty$, is *not* a branch point of the meromorphic function

$$x: (x, y) \in V \longrightarrow x \in \mathbb{P}^1.$$

Hence, by the Riemann-Hurwitz formula (see, e.g., Griffiths-Harris [1, p. 219]), the genus of $V$ is $(p-1)(m-2)/2$. In the case (ii), $\infty$ is a branch point of $x$. The genus of $V$ in this case is $(p-1)(m-1)/2$.

In the case (i), we associate $V$ with the divisor

$$D=k_1(\alpha_1)+\cdots+k_m(\alpha_m)$$

on $\mathbb{P}^1$ and write $V=V(D)$. Here, we denote by $(\alpha)$ the point divisor on $\mathbb{P}^1$ corresponding to the complex number $\alpha$.

In the case (ii), we associate $V$ with the divisor

$$D=k_1(\alpha_1)+\cdots+k_m(\alpha_m)+k_{m+1}(\infty)$$

on $\mathbb{P}^1$ and write $V=V(D)$, where $k_{m+1}$ satisfies

$$k_1+\cdots+k_m+k_{m+1}\equiv 0 \pmod p .$$

($k_{m+1}$ is determined up to modulo $p$.)

Now, we fix a positive integer $m$ ($\geqq 2$) and consider the set

$$\Omega=\{D=k_1(\alpha_1)+\cdots+k_m(\alpha_m)\mid \alpha_\nu\in\mathbb{P}^1 \text{ are mutually distinct, } k_\nu\not\equiv 0 \pmod p \text{ for any } \nu \text{ and } \Sigma_{\nu=1}^m k_\nu\equiv 0 \pmod p\} .$$

We introduce an equivalence relation $\sim$ in $\Omega$ by

$$k_1(\alpha_1)+\cdots+k_m(\alpha_m)\sim j_1(\alpha_1)+\cdots+j_m(\alpha_m)$$

if $k_1\equiv j_1, \cdots, k_m\equiv j_m \pmod p$. Put

$$\Gamma=\Omega/\sim ,$$
$$\pi:\Omega\longrightarrow\Gamma, \text{ the canonical projection.}$$

If $D\sim D'$, the compact Riemann surfaces $V(D)$ and $V(D')$ are conformally equivalent as noted above. Hence we may define $V(\pi(D))$. By abuse of notation, we idendify $\pi(D)$ with $D$ and write $V(D)$ instead of $V(\pi(D))$.

The multiplicative group $(\mathbb{Z}/p\mathbb{Z})^*=\mathbb{Z}/p\mathbb{Z}-\{0\}$ acts on $\Gamma$ as follows:

$$(r, D)\in(\mathbb{Z}/p\mathbb{Z})^*\times\Gamma\longrightarrow rD\in\Gamma .$$

Note that $V(D)$ and $V(rD)$ are conformally equivalent. In fact, the following transformation is birational:

$$(x, y)\in V(D)\longrightarrow(x, y^r)\in V(rD) .$$

The automorphism group $\mathrm{Aut}(\mathbb{P}^1)$ of $\mathbb{P}^1$ also acts on $\Gamma$ as follows:

$$(B, D)=(B, k_1(\alpha_1)+\cdots+k_m(\alpha_m))\in\mathrm{Aut}(\mathbb{P}^1)\times\Gamma$$
$$\longrightarrow B(D)=k_1B(\alpha_1)+\cdots+k_mB(\alpha_m)\in\Gamma .$$

Note that $V(B(D))$ is conformally equivalent to $V(D)$. In fact, if neither $D$ nor $B(D)$ contain $\infty$, then the transformation

$$\phi_B:(x, y)\in V(D)\longrightarrow(B(x), cy/(x-\gamma)^{k'})\in V(B(D))$$

is birational, where $c$ is a suitable non-zero constant, $\gamma=B^{-1}(\infty)$ and $k'=(\Sigma k_\nu)/p$.

When $D$ (or $B(D)$) contains $\infty$, a birational transformation of $V(D)$ onto $V(B(D))$ can be constructed in a similar way.

Let $G$ be the group of transformations on the set $\Gamma$ generated by $(\mathbb{Z}/p\mathbb{Z})^*$ and $\mathrm{Aut}(\mathbb{P}^1)$. Note that

$$r(B(D))=B(rD) \qquad \text{for} \quad r\in(\mathbb{Z}/p\mathbb{Z})^*, B\in\mathrm{Aut}(\mathbb{P}^1) \quad \text{and} \quad D\in\Gamma .$$

Hence $G$ can be written as

$$G=(\mathbb{Z}/p\mathbb{Z})^*\cdot\mathrm{Aut}(\mathbb{P}^1) .$$

We write $D\sim D'$ (mod $G$) if there is $rB\in G$ such that $rB(D)=D'$.

### 1.2. Theorems.

We have shown that if $D\sim D'$ (mod $G$), then $V(D)$ and $V(D')$ are conformally equivalent.

THEOREM 1.1. *Under the notations above, assume* $m\geqq 2p+1$. *Then* $V(D)$ *and* $V(D')$ *are conformally equivalent if and only if* $D\sim D'$ (mod $G$).

PROOF. It is enough to show the "only if" part. Put

$$D=k_1(\alpha_1)+\cdots+k_m(\alpha_m) ,$$
$$D'=j_1(\beta_1)+\cdots+j_m(\beta_m) .$$

We may assume that neither $D$ nor $D'$ contain $\infty$. Then $V(D)$ and $V(D')$ are defined by the equations:

$$y^p=(x-\alpha_1)^{k_1}\cdots(x-\alpha_m)^{k_m} , \qquad \text{for} \quad \alpha_\nu\in\mathbb{C} ,$$
$$y^p=(x-\beta_1)^{j_1}\cdots(x-\beta_m)^{j_m} , \qquad \text{for} \quad \beta_\nu\in\mathbb{C} ,$$

respectively. Assume that there is a birational transformation

$$\phi : V(D) \longrightarrow V(D') .$$

$V(D)$ has the genus $g=(p-1)(m-2)/2$. The assumption that $m\geqq 2p+1$ implies $(p-1)^2\leqq g-1$. Hence, by Namba [7, Corollary 2.4.5], the linear pencil determined by the meromorphic function $x$ is the *unique* linear pencil of degree $p$ on $V(D)$. (Here, we use the assumption that $p$ is prime.) Thus, there is $B\in\mathrm{Aut}(\mathbb{P}^1)$ such that the diagram

$$\begin{array}{ccc} V(D) & \xrightarrow{\ \phi\ } & V(D') \\ x\downarrow & & \downarrow x \\ \mathbb{P}^1 & \xrightarrow{\ B\ } & \mathbb{P}^1 \end{array}$$

is commutative. Let $\phi=\phi_{B^{-1}}$ be the birational transformation defined in 1.1. Then, the diagram

$$\begin{tikzcd} V(D) \arrow[rr, "\psi\phi"] \arrow[dr, "x"'] & & V(B^{-1}(D')) \arrow[dl, "x"] \\ & \mathbb{P}^1 & \end{tikzcd} \tag{2}$$

is commutative. We may write

$$B^{-1}(D')=j_1(\alpha_1)+\cdots+j_m(\alpha_m)\,.$$

Thus $V(B^{-1}(D'))$ is defined by

$$z^p=(x-\alpha_1)^{j_1}\cdots(x-\alpha_m)^{j_m}\,.$$

By the diagram (2), the meromorphic function $w=z\cdot\psi\cdot\phi$ on $V(D)$ also satisfies

$$w^p=(x-\alpha_1)^{j_1}\cdots(x-\alpha_m)^{j_m}\,.$$

Consider the meromorphic function $v=w^{k_1}/y^{j_1}$ on $V(D)$. It satisfies

$$v^p=(x-\alpha_2)^{k_1j_2-j_1k_2}\cdots(x-\alpha_m)^{k_1j_m-j_1k_m}\,. \tag{3}$$

We show that $v$ is a rational function of $x$. In fact, otherwise, the subfield $\mathbb{C}(x,v)$ of the field $\mathbb{C}(V(D))$ of all meromorphic functions on $V(D)$ satisfies $[\mathbb{C}(x,v):\mathbb{C}(x)]\geqq 2$. Since

$$p=[\mathbb{C}(V(D)):\mathbb{C}(x)]=[\mathbb{C}(V(D)):\mathbb{C}(x,v)]\cdot[\mathbb{C}(x,v):\mathbb{C}(x)]\,,$$

we have $\mathbb{C}(x,v)=\mathbb{C}(V(D))$. But the genus of the compact Riemann surface defined by (3) in $(x,v)$-plane is less than $(p-1)(m-2)/2$, a contradiction. Thus $v$ is a rational function of $x$. We write $v=h(x)$. Then (3) is written as

$$h(x)^p=(x-\alpha_2)^{k_1j_2-j_1k_2}\cdots(x-\alpha_m)^{k_1j_m-j_1k_m}\,.$$

This implies that

$$\begin{aligned} &k_1j_2-j_1k_2\equiv 0 \pmod p\,,\\ &\cdots\cdots\cdots\cdots\\ &k_1j_m-j_1k_m\equiv 0 \pmod p\,, \end{aligned}$$

i.e.,

$$j_1/k_1=j_2/k_2=\cdots=j_m/k_m=r\in(\mathbb{Z}/p\mathbb{Z})^*\,.$$

Hence $B^{-1}(D')=rD$, i.e., $D'=rB(D)$. This proves the theorem.

*Q.E.D.*

THEOREM 1.2. *Let $p$ be a prime number and $m\geqq 2p+1$. Let $K_p$ be the subgroup*

*of the automorphism group* $\mathrm{Aut}(V(D))$ *of* $V(D)$ *defined by*

$$K_p=\{\omega^j:(x,y)\longrightarrow(x,\omega^j y)\,|\,\omega=\exp 2\pi\sqrt{-1}/p\,,\quad 0\leqq j\leqq p-1\}\,.$$

*Let* $L_D$ *be the subgroup of* $\mathrm{Aut}(\mathbb{P}^1)$ *defined by*

$$L_D=\{B\in\mathrm{Aut}(\mathbb{P}^1)\,|\,\textit{there is } r\in(\mathbb{Z}/p\mathbb{Z})^* \textit{ such that } B(D)=rD\}\,.$$

*Then there is the following exact sequence*:

$$0\longrightarrow K_p\longrightarrow \mathrm{Aut}(V(D))\longrightarrow L_D\longrightarrow 0\,.$$

PROOF. For any $\phi\in\mathrm{Aut}(V(D))$, there is a unique $B\in\mathrm{Aut}(\mathbb{P}^1)$ such that the diagram:

$$\begin{tikzcd} V(D) \arrow[r, "\phi"] \arrow[d, "x"'] & V(D) \arrow[d, "x"] \\ \mathbb{P}^1 \arrow[r, "B"] & \mathbb{P}^1 \end{tikzcd}$$

is commutative. Then

$$\phi\in\mathrm{Aut}(V(D))\longrightarrow B\in\mathrm{Aut}(\mathbb{P}^1)$$

is a homomorphism. The kernel is clearly $K_p$. The proof of Theorem 1 shows that the image is $L_D$.

*Q.E.D.*

### 1.3. Examples.

The following examples show that the conjecture in Introduction is affirmative for $n=2$ and 3.

EXAMPLE 1.1. If $p=2$ and $s\geqq 5$ ($s$: odd), then

$$V: y^2=(x-\alpha_1)\cdots(x-\alpha_s)$$

is nothing but a hyperelliptic Riemann surface. In this case, Theorems 1.1 and 1.2 are well known.

If $p=2$ and $s=3$, then

$$V: y^2=(x-\alpha_1)(x-\alpha_2)(x-\alpha_3)$$

is an elliptic Riemann surface. In this case, it is also well known that the conclusion of Theorem 1.1 still holds, while the conclusion of Theorem 1.2 is clearly false.

EXAMPLE 1.2. Put $p=3$ and $m=6$. Then the condition $m\geqq 2p+1$ in the theorems

is not satisfied. In this case, the genus of $V(D)$ is 4. Every element of $\Gamma$ is equivalent modulo $G$ to one of the divisors of the following forms:

(i) $D=(\alpha_1)+\cdots+(\alpha_5)+(\infty)$,

(ii) $D=(\alpha_1)+(\alpha_2)+(\alpha_3)-(\beta_1)-(\beta_2)-(\beta_3)$.

In general, the canonical curve in $\mathbb{P}^3$ of a non-hyperelliptic compact Rieman surface $V$ of genus 4 is the complete intersection of a cubic surface and a quadric surface. (1) If the quadric surface is a cone, then $V$ has a unique linear system $g_3^1$. (2) If it is non-singular, then $V$ has two $g_3^1$'s, (see, e.g., Mumford [6, p. 55]). The cases (i) and (ii) correspond to (1) and (2), respectively.

In the case (i), the conclusions of Theorems 1.1 and 1.2 hold (by the uniqueness of $g_3^1$).

In the case (ii), the conclusion of Theorem 1.2 does not hold. For example,

$$V: y^3=\frac{x^3-1}{x^3+1}$$

has the following automorphism:

$$(x, y) \longrightarrow (-y, -x).$$

(The order of $\mathrm{Aut}(V)$ is $2\cdot 3\cdot 12=72$.) But, direct calculations snow that the conclusion of Theorem 1.1 *still holds* in this case.

EXAMPLE 1.3. Put $p=3$ and $m=5$. Then every element of $\Gamma$ is equivalent modulo $G$ to the divisor of the following form:

$$D=(0)+(1)+(\alpha_1)+(\alpha_2)+2(\infty).$$

$V=V(D)$ has the genus 3 and is defined by:

$$y^3=x(x-1)(x-\alpha_1)(x-\alpha_2).$$

This equation defines a non-singular curve of degree 4 in $\mathbb{P}^2$. We identify $V$ and the curve.

In general, a *non-singular* curve of degree $d$ ($d\geqq 4$) in $\mathbb{P}^2$ has a unique linear system $g_d^2$ of degree $d$ and dimeuion 2, which is nothing but the linear system of the line sections of the curve (see, e.g., Namba [7, Theorem 5.1.5]). Thus $V$ and

$$V': y^3=x(x-1)(x-\beta_1)(x-\beta_2)$$

are conformally equivalent if and only if there is $\tau\in\mathrm{Aut}(\mathbb{P}^2)$ such that $\tau(V)=V'$. Direct calculations show that this happens if and only if there is $B\in\mathrm{Aut}(\mathbb{P}^1)$ such that

$$\{B(0), B(1), B(\alpha_1), B(\alpha_2)\}=\{0, 1, \beta_1, \beta_2\} \quad \text{and} \quad B(\infty)=\infty,$$

i.e., if and only if the divisor $(\beta_1)+(\beta_2)$ is equal to one of the following:

$$(\alpha_1)+(\alpha_2)\,,\quad (1-\alpha_1)+(1-\alpha_2)\,,$$

$$(1/\alpha_1)+(\alpha_2/\alpha_1)\,,\quad (1/\alpha_2)+(\alpha_1/\alpha_2)\,,$$

$$((\alpha_1-1)/\alpha_1)+((\alpha_1-\alpha_2)/\alpha_1)\,,\quad ((\alpha_2-1)/\alpha_2)+((\alpha_2-\alpha_1)/\alpha_2)\,,$$

$$(1/(1-\alpha_1))+((1-\alpha_2)/(1-\alpha_1))\,,\quad (1/(1-\alpha_2))+((1-\alpha_1)/(1-\alpha_2))\,,$$

$$(\alpha_1/(\alpha_1-1))+((\alpha_1-\alpha_2)/(\alpha_1-1))\,,\quad (\alpha_2/(\alpha_2-1))+((\alpha_2-\alpha_1)/(\alpha_2-1))\,,$$

$$(\alpha_1/(\alpha_1-\alpha_2))+((\alpha_1-1)/(\alpha_1-\alpha_2))\,,\quad (\alpha_2/(\alpha_2-\alpha_1))+((\alpha_2-1)/(\alpha_2-\alpha_1))\,.$$

Thus the conclusion of Theorem 1.1 *still holds* in this case.

## Part 2.

### 2.1. Theorems.

The purpose of Part 2 is to prove the following theorems:

THEOREMS 2.1. *Let $p$ be a prime number and $n$ an integer such that $n\geqq 2p+1$ and $n\not\equiv 0 \pmod p$. Let $V$ and $V'$ be the compact Riemann surfaces defined by the equations:*

$$V: y^n=(x-\alpha_1)\cdots(x-\alpha_p)\,,$$

$$V': y^n=(x-\beta_1)\cdots(x-\beta_p)\,,$$

*where $\alpha_1,\cdots,\alpha_p$ (resp. $\beta_1,\cdots,\beta_p$) are mutually distinct complex numbers. Then $V$ and $V'$ are conformally equivalent if and only if there is $B\in\mathrm{Aut}(\mathbb{C})$ (the automorphism group of the complex plane $\mathbb{C}$) such that $\{B(\alpha_1),\cdots,B(\alpha_p)\}=\{\beta_1,\cdots,\beta_p\}$.*

COROLLARY. *Let $n$ be an integer such that $n\geqq 2$ and $n\not\equiv 0 \pmod 3$. Let $\lambda$ and $\mu$ be complex numbers different from 0 and 1, and $V_\lambda$ and $V_\mu$ be the compact Riemann surfaces defined by the equations:*

$$V_\lambda: y^n=x(x-1)(x-\lambda)\,,$$

$$V_\mu: y^n=x(x-1)(x-\mu)\,.$$

*Then $V_\lambda$ and $V_\mu$ are conformally equivalent if and only if $\mu$ is equal to one of the following:*

$$\lambda,\ 1/\lambda,\ 1-\lambda,\ 1/(1-\lambda),\ (\lambda-1)/\lambda,\ \lambda/(\lambda-1)\,.$$

THEOREM 2.2. *Let $V$ be as in Theorem 2.1. Then there is the following exact sequence:*

$$0 \longrightarrow K \longrightarrow \mathrm{Aut}\,(V) \longrightarrow L \longrightarrow 0\,,$$

*where*

$$K=\{\sigma^j \in \mathrm{Aut}\,(V) \mid \sigma:(x,y) \longrightarrow (x,\zeta y)\,,\quad \zeta=\exp 2\pi\sqrt{-1}/n\,,\quad 0 \leqq j \leqq n-1\}\,,$$
$$L=\{B \in \mathrm{Aut}\,(\mathbb{C}) \mid \{B(\alpha_1),\cdots,B(\alpha_p)\}=\{\alpha_1,\cdots,\alpha_q\}\}\,.$$

### 2.2. Proof of the theorems.

Let $p, n, V$ and $V'$ be as in Theorem 2.1. The point infinity: $\infty=(\infty,\infty)$ has a unique place on $V$, which we denote by $\infty$ again. The genus of $V$ (and $V'$) is $g=(1/2)(p-1)(n-1)$. As a basis of the space of all holomorphic differentials on $V$, we may take:

$$(x^j/y^k)dx\,,\quad 0 \leqq j<p, 0<k<n, kp-jn \geqq n+1\,,$$

(see, e.g., Shimura [9]). From this, we can easily show that the space of all holomorphic differentials $\omega$ on $V$ such that $(\omega) \geqq n(\infty)$ ($(\omega)$ is the zero divisor of $\omega$) is spanned by

$$(x^j/y^k)dx\,,\quad 0 \leqq j<p, 0<k<n, kp-jn \geqq 2n+1\,.$$

Hence the index of speciality $i(n(\infty))$ of the divisor $n(\infty)$ is equal to $g-n+1+[n/p]$, where [ ] is Gauss' notation. Hence, by Riemann-Roch theorem,

$$\dim L(n(\infty))=[n/p]+2\,,$$

where $L(n(\infty))$ is the vector space of all meromorphic functions on $V$ whose polar divisors are contained in $n(\infty)$, and 0-function. Hence we may take $\{1, y, \cdots, y^{[n/p]}, x\}$ as a basis of $L(n(\infty))$.

Now, the 'if' part of Theorem 2.1 is clear. We show the 'only if' part. Assume that there is a biholomorphic map $A: V \longrightarrow V'$. Then the compositions $z=x\cdot A$ and $w=y\cdot A$ are meromorphic functions on $V$ and satisfy the equation:

$$w^n=(z-\beta_1)\cdots(z-\beta_p)\,. \tag{1}$$

The order of the function $w$ is $p$. Since $n \geqq 2p+1$, we have $(p-1)^2 \leqq g-1$. Hence, by Namba [7, Corollary 2.4.5], there are complex numbers $c, c', d$ and $d'$ with $cd'-c'd \neq 0$ such that $w=(cy+d)/(c'y+d')$. Put $Q=A^{-1}(\infty)$. We first assume:

CASE 1: $Q=\infty$. In this case, the polar divisor of the function $z$ is $n(\infty)$, so that we can write

$$z=a_0+a_1y+\cdots+a_sy^s+bx\,, \tag{2}$$

where $s=[n/p]$ and $a_\nu$ and $b$ are complex numbers. Note that $b \neq 0$, since $sp<n$.

On the other hand, since

$$\infty = w(\infty) = (cy(\infty)+d)/(c'y(\infty)+d') = (c\infty+d)/(c'\infty+d'),$$

we have $c'=0$ (and $d'=1$), i.e.,

$$w = cy+d \quad (c \neq o). \tag{3}$$

From (2) and (3), (1) becomes

$$(cy+d)^n = (a_0+a_1y+\cdots+a_sy^s+bx-\beta_1)\cdots(a_0+a_1y+\cdots+a_sy^s+bx-\beta_p). \tag{4}$$

Since $sp<n$, this is an equation of degree $n$ with respect to $y$, so that this must coincide with the original equation: $y^n=(x-\alpha_1)\cdots(x-\alpha_p)$, up to constant. Arranging the terms in (4) in the descending degree with respect to $x$, the coefficient of $x^{p-1}$ is

$$b^{p-1}\{p(a_0+a_1y+\cdots+a_sy^s)-(\beta_1+\cdots+\beta_p)\},$$

which must be a constant (i.e., $(-1)c^n(\alpha_1+\cdots+\alpha_p)$). Thus

$$a_1=\cdots=a_s=0,$$

so that (4) becomes

$$(cy+d)^n=(a_0+bx-\beta_1)\cdots(a_0+bx-\beta_p).$$

Hence we get $d=0$, $c^n=b^p$ and

$$\{\alpha_1,\cdots,\alpha_p\}=\{(\beta_1-a_0)/b,\cdots,(\beta_p-a_0)/b\}.$$

Let $B\in Aut(\mathbb{C})$ be defined by $B(t)=bt+a_0$ for $t\in\mathbb{C}$. Then

$$\{B(\alpha_1),\cdots,B(\alpha_p)\}=\{\beta_1,\cdots,\beta_p\}.$$

This proves Theorem 2.1 in this case. Note that, from (2), the following diagram is commutative:

$$\begin{tikzcd} V \arrow[r, "A"] \arrow[d, "x"'] & V' \arrow[d, "x"'] \\ \mathbb{P}^1 \arrow[r, "B"] & \mathbb{P}^1 \end{tikzcd} \tag{5}$$

CASE 2: $Q\neq\infty$. We show that this case dose not occur. In fact, put $Q=(x_o, y_o)\in\mathbb{C}^2$. Since $w=(cy+d)/(c'y+d')$, we have

$$p\cdot Q=D_\infty(w)=y^{-1}(-d'/c').$$

($D_\infty(w)$ is the polar divisor of the function $w$.) Hence $y_o=-d'/c'$ and the equation for $x$:

$$(x-\alpha_1)\cdots(x-\alpha_p)-y_o^n=0$$

has $x_o$ as the $p$-ple root, i.e.,

$$(x-\alpha_1)\cdots(x-\alpha_p)-y_o^n=(x-x_o)^p.$$

Thus $V$ is defined by

$$V: y^n=(x-x_o)^p+y_o^n.$$

Note that $y_o\neq 0$, for, $\alpha_1,\cdots,\alpha_p$ are mutually distinct.

In a similar way, there is $(x_1, y_1)\in\mathbb{C}^2$ $(y_1\neq 0)$ such that $V'$ is defined by

$$V': y^n=(x-x_1)^p+y_1^n.$$

Let $V_o$ be the compact Riemann surface defined by the equation:

$$V_o: y^p=x^n-1.$$

Then the maps

$$A_o: V\ni(x, y)\longrightarrow(y/y_o, (x-x_o)/y_o')\in V_o,$$

$$A_o': V'\ni(x, y)\longrightarrow(y/y_1, (x-x_1)/y_1')\in V_o,$$

are biholomorphic, where $y_o'^p=y_o^n$ and $y_1'^p=y_1^n$. Then, $A_oA^{-1}A_o'^{-1}$ is an automorphism of $V_0$, mapping $\infty$ to $(1, 0)$. But, by Theorem 1.2 in Part 1, every automorphism of $V_o$ can be written as

$$V_0\in(x, y)\longrightarrow(\zeta^j x, \rho^k y)\in V_0,$$

where $\zeta=\exp 2\pi\sqrt{-1}/n$, $0\leqq j\leqq n-1$, $\rho=\exp 2\pi\sqrt{-1}/p$, $0\leqq k\leqq p-1$. In particular, every automorphism of $V_o$ fixes $\infty$, a contradiction. Thus, this does not occur. This completes the proof of Theorem 2.1.

Theorem 2.2 follows from the above proof of Theorem 2.1. In fact, if we put $V'=V$ and associate $B$ to $A$ in (5), then this correspondence gives a homomorphism of $\mathrm{Aut}(V)$ into $\mathrm{Aut}(\mathbb{C})$. Its image and kernel are clearly $L$ and $K$ in the theorem, respectively.

Finally, we show the corollary to Theorem 2.1.

If $n\geqq 7$, then the corollary is a direct consequence of Theorem 2.1 putting $p=3$.

If $n-5$, then the genus of $V_\lambda$ is 4. We may take

$$\{(1/y^4)dx, (x/y^4)dx, (1/y^3)dx, (1/y^2)dx\}$$

as a basis of the space of all holomorphic differentials on $V_\lambda$. Thus the canonical map is

$$V_\lambda\ni(x, y)\longrightarrow(Z_0: Z_1: Z_2: Z_3)=(1: x: y: y^2)\in\mathbb{P}^3,$$

where $(Z_0:Z_1:Z_2:Z_3)$ is a homogeneous coordinate system on $\mathbb{P}^3$. Hence its image (the canonical curve) is contained in the quadric cone $S: Z_2^2=Z_0Z_3$. In general, a non-hyperelliptic canonical curve of genus 4 is the complete intersection of a quadric surface and a cubic surface in $\mathbb{P}^3$. In our case, the quadric surface is the cone $S$, so that the curve has a *unique* linear pencil of degree 3. (It is, in fact, the projection with the center a line on $S$.) Thus a similar argument to the proof of Theorem 2.1 shows that the conclusion of the corollary holds for $n=5$.

If $n=4$, then the equation: $y^4=x(x-1)(x-\lambda)$ defines a non-singular curve of degree 4 in $\mathbb{P}^2$. It is the canonical curve of the non-hyperelliptic $V_\lambda$ of genus 3. We may identify $V_\lambda$ with the curve. Since the linear system of all line sections is the canonical linear system, it is the unique linear system of degree 4 and dimension 2 on $V_\lambda$. Thus $V_\lambda$ and $V_\mu$ are biholomorphic if and only if there is $\tau\in\mathrm{Aut}(\mathbb{P}^2)$ such that $\tau(V_\lambda)=V_\mu$. But direct calculations show that this happens if and only if there is $B\in\mathrm{Aut}(\mathbb{P}^1)$ such that $\{B(0), B(1), B(\lambda), B(\infty)\}=\{0, 1, \mu, \infty\}$, (see also Theorem 3.1 below). This shows that the conclusion of the corollary holds for $n=4$.

If $n=2$, then $V_\lambda$ is an elliptic curve, so that the conclusion of the corollary in this case is classically well known.

This completes the proof of the corollary.

## Part 3.

### 3.1. Theorems.

Let $\mathbb{P}^{r+1}$ be the $(r+1)$-dimensional complex projective space and $(X_0:\cdots:X_{r+1})$ be a homogeneous coordinate system on it. The purpose of Part 3 is to prove the following theorems:

THEOREM 3.1. *Let $V$ and $W$ be non-singular hypersurfaces of degree $n$ in $\mathbb{P}^{r+1}$ defined by the equations:*

$$V: X_{r+1}^n=F(X_0, \cdots, X_r),$$

$$W: X_{r+1}^n=G(X_0, \cdots, X_r).$$

*Suppose $(n, r)\neq(4, 2)$. Then $V$ and $W$ are biholomorphic if and only if there is a $B\in\mathrm{Aut}(\mathbb{P}^r)$ mapping the non-singular hypersurfaces $\{F=0\}$ in $\mathbb{P}^r$ onto $\{G=0\}$.*

COROLLARY. *Let $V$ and $W$ be the compact Riemann surfaces defined by the equations:*

$$V: y^n=(x-\alpha_1)\cdots(x-\alpha_n),$$

$$W: y^n=(x-\beta_1)\cdots(x-\beta_n),$$

*where* $\alpha_1, \cdots, \alpha_n$ *(resp.,* $\beta_1, \cdots, \beta_n$*) are mutually distinct complex numbers. Then* $V$ *and* $W$ *are biholomorphic if and ouly if there is a* $B \in \mathrm{Aut}(\mathbb{P}^1)$ *such that* $\{B(\alpha_1), \cdots, B(\alpha_n)\} = \{\beta_1, \cdots, \beta_n\}$.

REMARK. We do not know if the conclusion of Theorem 3.1 still holds for $(n, r)=(4, 2)$, i.e., for non-singular quartic surfaces in $\mathbb{P}^3$.

Let $V$ be a non-singular hypersurface of degree $n$ in $\mathbb{P}^{r+1}$ defined by the equation:

$$V: X_{r+1}^n + \cdots + X_{k+1}^n = F_1(X_0, \cdots, X_k), \quad 1 \leqq k \leqq r.$$

Assume that $V$ is *not* biholomorphic to any hypersurface defined by the equation:

$$X_{r+1}^n + \cdots + X_k{}^n = G_1(X_0, \cdots, X_{k-1}),$$

(i.e., $k$ is the least integer such that $V$ is expressed as above). Let $\sigma$ and $\sigma_{ij}$, $k+1 \leqq i, j \leqq r+1$, be the automorphisms of $V$ defined by

$$\sigma: (X_0 : \cdots : X_r : X_{r+1}) \longrightarrow (X_0 : \cdots : X_r : \zeta X_{r+1}), \quad \zeta = \exp 2\pi\sqrt{-1}/n,$$

$$\sigma_{ij}: (X_0 : \cdots : X_i : \cdots : X_j : \cdots : X_{r+1}) \longrightarrow (X_0 : \cdots : X_j : \cdots : X_i : \cdots : X_{r+1}).$$

Let $K$ be the subgroup of $\mathrm{Aut}(V)$, the automorphism group of $V$, generated by $\sigma$ and $\sigma_{ij}$, $k+1 \leqq i, j \leqq r+1$. Its order is $(r+1-k)!n^{r+1-k}$. Let $L$ be the subgroup of $\mathrm{Aut}(\mathbb{P}^k)$ defined by

$$L = \{B \in \mathrm{Aut}(\mathbb{P}^k) \mid B(\{F_1 = 0\}) = \{F_1 = 0\}\}.$$

THEOREM 3.2. *Let* $V$ *be as above. Assume that* (i) $n \geqq 4$ *if* $r=1$, (ii) $n \geqq 3$ *if* $r \geqq 2$ *and* (iii) $(n, r) \neq (4, 2)$. *Then there is the following exact sequence*:

$$0 \longrightarrow K \longrightarrow \mathrm{Aut}(V) \longrightarrow L \longrightarrow 0.$$

Let $V$ be a compact Riemann surface as in the corollary to Theorem 3.1. Let $K$ be the subgroup of $\mathrm{Aut}(V)$ generated by

$$\sigma: (x, y) \longrightarrow (x, \zeta y).$$

Let $L$ be the subgroup of $\mathrm{Aut}(\mathbb{P}^1)$ defined by

$$L = \{B \in \mathrm{Aut}(\mathbb{P}^1) \mid \{B(\alpha_1), \cdots, B(\alpha_n)\} = \{\alpha_1, \cdots, \alpha_n\}\}.$$

COROLLARY TO THEOREM 3.2. *Let* $V$ *be as in the corollary to Theorem* 3.1 *Assume that* (i) $n \geqq 4$ *and* (ii) $V$ *is not biholomorphic to the Fermat curve*: $x^n + y^n = 1$. *Then there is the following exact sequence*:

$$0 \longrightarrow K \longrightarrow \mathrm{Aut}(V) \longrightarrow L \longrightarrow 0.$$

### 3.2. Proof of Theorem 3.1.

The 'if' part of Theorem 3.1 is trivial. We show the 'only if' part. If $n=1$ or 2, or $(n,r)=(3,1)$, then Theorem 3.1 is trivial. (Any non-singular curve: $y^3=(x-\alpha_1)(x-\alpha_2)(x-\alpha_3)$ is biholomorphic to the elliptic curve: $y^3=x(x-1)$.) Hence we may assume that

(i) $n\geqq 4$, if $r=1$,

(ii) $n\geqq 3$, if $r\geqq 2$, and

(iii) $(n,r)\neq(4,2)$.

Then any biholomorphic map of $V$ onto $W$ can be uniquely extended to $\sigma\in\mathrm{Aut}(\mathbb{P}^{r+1})$ such that $\sigma(V)=W$, (see Namba [7, Theorem 5.1.5] for $r=1$ and Matsumula-Monsky [5] for $r\geqq 2$).

In order to aboid confusion, we prepare another $\mathbb{P}^{r+1}$ (which is denoted by $\mathbb{P}_1{}^{r+1}$) with a homogeneous coordinate system $(X_0':\cdots:X'_{r+1})$ and regard $W$ as a hypersurface in $\mathbb{P}_1{}^{r+1}$ defined by the equation:

$$X'^n_{r+1}=G(X_0',\cdots,X_r').$$

Now our proof is based on the following two trivial lemmas, whose proofs are omitted.

Put

$$P_\infty=(0:\cdots:0:1)\in\mathbb{P}^{r+1} \quad \text{and}$$

$$H=\text{the hyperplane } \{X_{r+1}=0\} \text{ in } \mathbb{P}^{r+1}.$$

Let $s$ and $t$ be complex numbers.

LEMMA 1. *Let* $\sigma:\mathbb{P}^{r+1}\ni(X_0:\cdots:X_{r+1})\longrightarrow(X_0':\cdots:X'_{r+1})\in\mathbb{P}_1{}^{r+1}$ *be a linear isomorphism, mapping* $P_\infty$ *and* $H$ *to* $(0:\cdots:s:1)$ *and* $\{X'_{r+1}=0\}$, *respectively. Then* $\sigma(V)$ *is defined by*

$$\sigma(V):X'^n_{r+1}=F'(X_0',\cdots,X'_{r-1},X_r'-sX'_{r+1}),$$

*where* $F'$ *is a homogeneous polynomial of degree* $n$.

LEMMA 2. *Let* $\sigma:\mathbb{P}^{r+1}\longrightarrow\mathbb{P}_1{}^{r+1}$ *be a linear isomorphism, mapping* $P_\infty$ *and* $H$ *to* $(s:0:\cdots:0:1:t)$ *and* $\{X_r'=0\}$, *respectively. Then* $\sigma(V)$ *is defined by*

$$\sigma(V):X_r'^n=F'(sX_r'-X_0',X_1',\cdots,X'_{r-1},tX_r'-X'_{r+1}),$$

*where* $F'$ *is a homogeneous polynomial of degree* $n$.

Now let $V\subset\mathbb{P}^{r+1}$ and $W\subset\mathbb{P}_1^{r+1}$ be as above and

$$\sigma:\mathbb{P}^{r+1}\ni(X_0:\cdots:X_{r+1})\longrightarrow(X_0':\cdots X_{r+1}')\in\mathbb{P}_1^{r+1}$$

be a linear isomorphism mapping $V$ onto $W$. This means that, if $\sigma$ is given by a non-singular matrix $(a_{jk})$, i.e.,

$$X_j'=\Sigma_{k=0}^{r+1}a_{jk}X_k\,,$$

then there is a non-zero constant $c$ such that

$$(1)\qquad (\Sigma_{k=0}^{r+1}a_{(r+1)k}X_k)^n-G((\Sigma_{k=0}^{r+1}a_{0k}X_k),\cdots,(\Sigma_{k=0}^{r+1}a_{rk}X_k))=cX_{r+1}^n-cF(X_0,\cdots,X_r)\,.$$

Let $P_\infty$ and $H$ be as above. Put

$$P_\infty'=(0:\cdots:0:1)\in\mathbb{P}_1^{r+1}\quad\text{and}$$

$$H'=\{X_{r+1}'=0\}\,.$$

We first consider the case:

*Case* 1: $\sigma(H)=H'$. This means that $X_{r+1}=0$ if and only if $X_{r+1}'=0$. Hence, by (1),

$$G((\Sigma_{k=0}^{r}a_{0k}X_k),\cdots,(\Sigma_{k=0}^{r}a_{rk}X_k))=cF(X_0,\cdots,X_r)\,,\quad\text{on}\quad H\,.$$

Thus the restriction $\sigma|H:H\longrightarrow H'$ maps $\{F=0\}$ onto $\{G=0\}$.

Next, we consider the case:

*Case* 2: $\sigma(H)\neq H'$. This case is divided further into the following 2 cases, depending on the positions of $\sigma(P_\infty), P_\infty', \sigma(H)$ and $H'$.

*Case* 2-i: *There is a line* $\mathfrak{l}$ *passing through* $P_\infty'$ *and* $\sigma(P_\infty)$ *but not* $H'\cap\sigma(H)$. Put $R=\mathfrak{l}\cap H'$ and $S=\mathfrak{l}\cap\sigma(H)$. Note that $P_\infty'\notin H'$ and $\sigma(P_\infty)\notin\sigma(H)$. We prepare another $\mathbb{P}^{r+1}$, which is denoted by $\mathbb{P}_2^{r+1}$ and a homogeneous coordinate system $(X_0'':\cdots:X_{r+1}'')$ on it. Let $\tau$ be a liuear isomorphism of $\mathbb{P}_1^{r+1}$ onto $\mathbb{P}_2^{r+1}$, mapping $P_\infty', \sigma(P_\infty), R, S, H'$ and $\sigma(H)$ to $(0:\cdots:0:s:1)$, $(0:\cdots:0:1:t)$, $(0:\cdots:0:1:0)$, $(0:\cdots:0:0:1)$, $\{X_{r+1}''=0\}$ and $\{X_r''=0\}$, respectively. Then, by Lemma 1, $\tau(W)$ is defined by

$$(2)\qquad \tau(W):X_{r+1}''^n=F''(X_0'',\cdots,X_{r-1}'',X_r''-sX_{r+1}'')\,,$$

where $F''$ is a homogeneous polynomial of degree $n$. On the other hand, $\tau(W)=\tau\sigma(V)$ and $\tau\sigma$ satisfies the condition of Lemma 2, so that $\tau\sigma(V)$ is defined by

$$(3)\qquad \tau\sigma(V):X_r''^n=G''(X_0'',\cdots,X_{r-1}'',X_{r+1}''-tX_r'')\,,$$

where $G''$ is a homogeneous polynomial of degree $n$.

By (2) and (3), there is a *non-zero* constant $c$ such that

$$F''(X_0'', \cdots, X_{r-1}'', X_r''-sX_{r+1}'')-X_{r+1}''^n \tag{4}$$
$$=cG''(X_0'', \cdots, X_{r-1}'', X_{r+1}''-tX_r'')-cX_r''^n$$

In (4), we put $X_0''=\cdots=X_{r-1}''=0$, $X_r''=X$ and $X_{r+1}''=Y$. Then we get

$$a(X-sY)^n-Y^n=cb(Y-tX)^n-cX^n,$$

for some constants $a$ and $b$. Expanding and comparing terms, we get

$$a=cb(-1)^n t^n-c, \tag{5-1}$$

$$(-1)^n as^n-1=cb, \tag{5-2}$$

$$-as=cb(-1)^{n-1}t^{n-1}, \tag{5-3}$$

$$as^2=cb(-1)^{n-2}t^{n-2}, \tag{5-4}$$

$$\cdots\cdots\cdots\cdots\cdots$$

Now, assume $s\neq 0$. Then $b\neq 0$. In fact, if $b=0$, then, by (5-3), $a=0$. Hence, by (5-1), $c=0$, a contradiction. Hence $a\neq 0$, so that $b\neq 0$. Now, by (5-3), (5-4), $\cdots$,

$$a/(cb(-1)^n)=t^{n-1}/s=t^{n-2}/s^2=\cdots. \tag{6}$$

Note that $t\neq 0$, for $a\neq 0$. Since we have assumed $n\geqq 3$, we get $st=1$ by (6). Thus, by (6) again, $cb(-1)^n t^n=a$. Hence, by (5-1), $a=a-c$, so that $c=0$, a contradiction. Consequently, $s$ must be zero. By (5-2), $b\neq 0$. Hence, by (5-3), $t=0$.

Thus we conclude that $P_\infty'=S$, $\sigma(P_\infty)=R$ and (4) can be written as

$$F''(X_0'', \cdots, X_{r-1}'', X_r'')-X_{r+1}''^n=cG''(X_0'', \cdots, X_{r-1}'', X_{r+1}'')-cX_r''^n.$$

Hence there is a homogeneous polynomial $K''$ of $X_0'', \cdots, X_{r-1}''$ of degree $n$ such that

$$F''(X_0'', \cdots, X_{r-1}'', X_r'')=K''(X_0'', \cdots, X_{r-1}'')-cX_r''^n, \tag{7-1}$$

$$G''(X_0'', \cdots, X_{r-1}'', X_{r+1}'')=(1/c)K''(X_0'', \cdots, X_{r-1}'')-(1/c)X_{r+1}''^n \tag{7-2}$$

Now we define an automorphism $\eta$ of $\mathbb{P}_2^{r+1}$ by

$$\eta:(X_0'':\cdots:X_{r-1}'':X_r'':X_{r+1}'')\longrightarrow(X_0'':\cdots:X_{r-1}'':c_1X_{r+1}'':X_r''),$$

where $c_1^n=1/c$. Then $\eta$ maps $\{G''(X_0'', \cdots, X_{r-1}'', X_{r+1}'')=0\}$ onto $\{F''(X_0'', \cdots, X_{r-1}'', X_r'')=0\}$.

Note that, by (2), $\tau$ maps $\{G=0\}$ onto $\{F''=0\}$. Also, by (3), $\tau\sigma$ maps $\{F=0\}$ onto $\{G''=0\}$.

Thus, $\lambda=\tau^{-1}\eta\tau\sigma$ is a linear isomorphism of $\mathbb{P}^{r+1}$ onto $\mathbb{P}_1^{r+1}$ mapping $H$ to $H'$, and $\{F=0\}$ onto $\{G=0\}$.

*Case* 2-ii: *$P_\infty'\neq\sigma(P_\infty)$ and the line $\mathfrak{l}$ connecting them intersects with $H'\cap\sigma(H)$.*

We show that this case does *not* occur. Put $R=\mathfrak{l}\cap H'\cap\sigma(H)$. Note that neither $P'_\infty$ nor $\sigma(P_\infty)$ belongs to $H'\cup\sigma(H)$. Let $\mathbb{P}_2^{r+1}$ and $(X_0'':\cdots:X''_{r+1})$ be as above. Let $\tau$ be a linear isomorphism of $\mathbb{P}_1^{r+1}$ onto $\mathbb{P}_2^{r+1}$, mapping $P'_\infty, \sigma(P_\infty), R, H'$ and $\sigma(H)$ to $(0:0:\cdots:0:s:1)$, $(1:0:\cdots:0:s:1)$, $(1:0:\cdots:0)$, $\{X''_{r+1}=0\}$ and $\{X_r''=0\}$, respectively. Note that $s\neq 0$.

By Lemma 1, $\tau(W)$ is defined by

$$\tau(W): X''^n_{r+1}=F''(X_0'',\cdots,X''_{r-1},X_r''-sX''_{r+1}), \tag{8}$$

where $F''$ is a homogeneous polynomial of degree $n$. On the other hand, $\tau(W)=\tau\sigma(V)$ and $\tau\sigma$ satisfies the condition of Lemma 2, so that $\tau\sigma(V)$ is defiued by

$$\tau\sigma(V): X_r''^n=G''(X_r''-sX_0'',X_1''\cdots,X''_{r-1},X_r''-sX''_{r+1}), \tag{9}$$

where $G''$ is a homogeneous polynomial of degree $n$. By (8) and (9), there is a *non-zero* constant $c$ such that

$$\begin{aligned} &F''(X_0'',\cdots,X''_{r-1},X_r''-sX''_{r+1})-X''^n_{r+1}\\ &\quad=cG''(X_r''-sX_0'',X_1'',\cdots,X''_{r-1},X_r''-sX''_{r+1})-cX_r''^n. \end{aligned} \tag{10}$$

Put $X_1''=\cdots=X''_{r-1}=0$, $X_0''=X$, $X_r''-sX_0''=Y$ and $X_r''-sX''_{r+1}=Z$. Then (10) can be written as

$$\begin{aligned} &a_0X^n+a_1X^{n-1}Z+\cdots+a_nZ^n-(1/s^n)(Y-Z+sX)^n\\ &\quad=cb_0Y^n+cb_1Y^{n-1}Z+\cdots+cb_nZ^n-c(sX+Y)^n, \end{aligned} \tag{11}$$

where $a_0,\cdots,a_n,b_0,\cdots,b_n$ are complex numbers. In (11), put $X=0$. Then

$$cb_0Y^n+cb_1Y^{n-1}Z+\cdots+cb_nZ^n=a_nZ^n+cY^n-(1/s^n)(Y-Z)^n.$$

Substituting this into (11), we get

$$\begin{aligned} &a_0X^n+a_1X^{n-1}Z+\cdots+a_{n-1}XZ^{n-1}\\ &\quad=cY^n+(1/s^n)(Y-Z+sX)^n-(1/s^n)(Y-Z)^n-c(sX+Y)^n. \end{aligned} \tag{12}$$

In (12), put $Y=0$. Then

$$\begin{aligned} &a_0X^n+a_1X^{n-1}Z+\cdots+a_{n-1}XZ^{n-1}\\ &\quad=(1/s^n)(sX-Z)^n-(-1/s)^nZ^n-cs^nX^n. \end{aligned}$$

Substituting this into (12), we get

$$\begin{aligned} &(1/s^n)(sX-Z)^n-(-1/s)^nZ^n-cs^nX^n\\ &\quad=cY^n+(1/s^n)(Y-Z+sX)^n-(1/s^n)(Y-Z)^n-c(sX+Y)^n. \end{aligned} \tag{13}$$

In (13), put $Z=0$. Then we get

$$(1-cs^n)X^n+((1/s^n)-c)Y^n=((1/s^n)-c)(sX+Y)^n.$$

The left hand side does not have the $(X^{n-1}Y)$-term. Hence $c=1/s^n$. Thus, by (13), we get

$$(14)\qquad (Y-Z+sX)^n-(Y-Z)^n=(sX-Z)^n+(sX+Y)^n-(-1)^nZ^n-Y^n-s^nX^n\,.$$

Since $n\geqq 3$, the first term in the left hand side of (14) is expanded as

$$nsXY^{n-1}-n(n-1)sXY^{n-2}Z+\cdots+s^nX^n\,,$$

while the right hand side of (14) does not have the $(XY^{n-2}Z)$-term, a contradiction. Hence this case does not occur.

This completes the proof of Theorem 3.1.

### 3.3. Proof of Theorem 3.2

In the following lemmas, suppose that $V$ is a non-singular hypersurface of of degree $n$ in $\mathbb{P}^{r+1}$.

LEMMA 3. *Let $P_\infty$ and $H$ be a point in $\mathbb{P}^{r+1}-V$ and a hyperplane in $\mathbb{P}^{r+1}$, respectively, with the following conditions*: (i) $P_\infty\notin H$, (ii) *$H\cap V$ is a non-singular hypersurface in $H$, and* (iii) *the line connecting $P_\infty$ and every point $Q$ of $H\cap V$ meets $V$ at the unique point $Q$. Then $V$ is defined by the equation as in Theorem* 3.1. *Conversely, if $V$ is defined as in Theorem* 3.1, *then $P_\infty=(0:\cdots:0:1)$ and $H=\{X_{r+1}=0\}$ satisfy the above conditions* (i)-(iii).

PROOF. Take a homogeneous coordinate system $(X_0:\cdots:X_{r+1})$ such that $P_\infty=(0:\cdots:0:1)$ and $H=\{X_{r+1}=0\}$. Let

$$\tilde{F}(X_0,\cdots,X_{r+1})=A_0(X)X_{r+1}^n+A_1(X)X_{r+1}^{n-1}+\cdots+A_n(X)\,,$$

$(X=(X_0,\cdots,X_r))$, be the irreducible homogeneous polynomial of degree $n$ defining $V$, i.e., $V=\{\tilde{F}=0\}$. Every $A_j(X)$ is then a homogeneous polynomial of degree $j$. In particular, $A_0(X)=A_0$ is a constant, which is non-zero, because $P_\infty\notin V$. Now, by (iii), for any point $(X^o:0)=(X_0^o:\cdots:X_r^o:0)$ of $V\cap H$, the equation for $X_{r+1}$:

$$A_0X_{r+1}^n+A_1(X^o)X_{r+1}^{n-1}+\cdots+A_n(X^o)=0$$

has $X_{r+1}=0$ as the $n$-ple root. Hence

$$A_1(X^o)=\cdots=A_n(X^o)=0\,.$$

Since the degree of the non-singular hypersurface $V\cap H$ in $H$ is $n$, we have $A_1=\cdots=A_{n-1}=0$. Thus

$$\tilde{F}(X_0,\cdots,X_{r+1})=A_0X_{r+1}^n+A_n(X)\,,\quad A_0\neq 0\,.$$

The converse is trivial.

*Q.E.D.*

LEMMA 4. *Suppose* $n \geqq 2$. *If the pair* $(P'_\infty, H)$ *satisfies same the conditions as the pair* $(P_\infty, H)$ *in Lemma* 3, *then* $P'_\infty = P_\infty$, *i.e.,* $P_\infty$ *is unique with respect to* $H$.

PROOF. Suppose $P'_\infty \neq P_\infty$. Let $\mathfrak{l}$ be the line connecting them. Put $Q_0 = \mathfrak{l} \cap H$. Take any point $Q \in V \cap H$. By the condition (iii) in Lemma 3, the tangent space $T_Q V$ to $V$ at $Q$ passes both $P_\infty$ and $P'_\infty$ so that it contains $\mathfrak{l}$ (and hence $Q_0$). Note that the tangent space $T_Q(V \cap H)$ to $V \cap H$ at $Q$ is $(T_Q V) \cap H$, so that it contains the fixed point $Q_0$. This is impossible, unless $V \cap H$ is of degree 1.

*Q.E.D.*

The following lemma can be shown by a similar argument to Case 2-i in 3.2.

LEMMA 5. *Let* $P_\infty$ *and* $H$ *be as in Lemma* 3. *Suppose* $n \geqq 3$. *Then* $H$ *is unique with respect to* $P_\infty$.

Finally we need

LEMMA 6. *Suppose* $n \geqq 3$. *Let* $(P_{r+1}, H_{r+1}), \cdots, (P_{k+1}, H_{k+1})$ *be mutually distinct pairs satisfying the same conditions as* $(P_\infty, H)$ *in Lemma* 3. *Then there is a homogeneous coordinate system* $(X_0 : \cdots : X_{r+1})$ *on* $\mathbb{P}^{r+1}$ *such that*

(i) $P_j = (0 : \cdots : \overset{(j)}{1} : \cdots : 0)$, $k+1 \leqq j \leqq r+1$,

(ii) $H_j = \{X_j = 0\}$, $k+1 \leqq j \leqq r+1$, and

(iii) $V : X_{r+1}^n + \cdots + X_{k+1}^n = F_1(X_0, \cdots, X_k)$.

PROOF. A similar argument to Case 2-i in 3.2 shows that $P_j \in H_j$, if $i \neq j$. This implies that $H_{k+1}, \cdots, H_{r+1}$ are in general position. In fact, since $P_{k+1}$, say, is contained in $H_{k+2} \cap \cdots \cap H_{r+1}$, this linear subspace is not contained in $H_{k+1}$. Thus we may take a homogeneous coordinate system $(X_0 : \cdots : X_{r+1})$ satisfying (i) and (ii). Using (2) and (7-1) (or (3) and (7-2)) in 3.2, an inductive argument shows that $V$ is defined as in (iii).

*Q.E.D.*

Now we are ready to prove Theorem 3.2. Let $V$ be as in Theorem 3.2. Put

$$P_j = (0 : \cdots : \overset{(j)}{1} : \cdots : 0),$$

$$H_j = \{X_j = 0\},$$

for $k+1 \leqq j \leqq r+1$. Then every pair $(P_j, H_j)$ satisfies the same conditions as $(P_\infty, H)$ in Lemma 3. By the assumption on the integer $k$, and by Lemma 6, they are all

the pairs satisfying the same conditions as $(P_\infty, H)$ in Lemma 3.

Note that any $\sigma \in \mathrm{Aut}(V)$ can be *uniquely* extended to $\sigma \in \mathrm{Aut}(\mathbb{P}^{r+1})$ such that $\sigma(V)=V$, as was noted in 3.2. It is clear that $\sigma$ maps $\{P_{k+1}, \cdots, P_{r+1}\}$ and $\{H_{k+1}, \cdots, H_{r+1}\}$ onto themselves. Hence it maps $S=H_{k+1}\cap\cdots\cap H_{r+1}$ onto itself. The restriction $\sigma|S$ then maps $\{F_1=0\}$ onto itself.

Now the correspondence

$$\mathrm{Aut}(V)\ni\sigma \longrightarrow \sigma|S\in\mathrm{Aut}(\mathbb{P}^k)$$

is a homomorphism, whose image and kernel are clearly $L$ and $K$ in Theorem 3.2, respectively.

This completes the proof of Theorem 3.2.

### 3.4. A remark.

In Theorem 3.2, we assumed $k\geqq 1$. If $k=0$, then we have the Fermat variety:

$$F(n,r): X_0{}^n+\cdots+X_{r+1}^n=0\,.$$

A similar argument to the proof of Theorem 3.2 shows that, if $(n,r)\neq(4,2)$, then $\mathrm{Aut}(F(n,s))$ is generated by $\sigma$ and $\sigma_{ij}$, $0\leqq i,j\leqq r+1$. Since $(X_0:\cdots:X_{r+1})\longrightarrow(\zeta X_0:\cdots:\zeta X_{r+1})$ is the identity map, the order of $\mathrm{Aut}(F(n,r))$ is $(r+2)!n^{r+1}$, a well known result.

## References

[1] Griffiths, P. and Harris, J., Principles of algebraic geometry. A Wiley-Interscience Publ., 1978.

[2] Kato, T., Conformal equivalence of compact Riemann surfaces, to appear.

[3] Kuribayashi, A., On analytic families of compact Riemann surfaces with non-trivial automorphism. Nagoya Math. J., **28** (1966), 119-164.

[4] Lefschetz, S., On certain numberical invariants of algebraic varieties with applications to Abelian varieties. Transac. Amer. Math. Soc., **22** (1921), 327-482.

[5] Matsumura, H. and Monsky, P, On the automorphisms of hypersurfaces. J. Math. Kyoto Univ., **3-3** (1964), 347-361

[6] Mumford, D., Curves and their Jacobians. Univ. Michigan Press, 1975.

[7] Namba, M., Families of meromorphic functions on compact Riemann surfaces. Lec. Notes in Math., **767** (1979), Springer-Verlag.

[8] Picard, E., Sur les fonctions de deux variables independentes analogues aux fonctions modulaires. Acta Math., **2** (1883), 114-135.

[9] Shimura, G., On purely transcendental fields of automorphic functions of several variables. Osaka J. Math., **1** (1964), 1-14.

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