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ZETA FUNCTIONS OF INTEGRAL GROUP RINGS OF METACYCLIC GROUPS

By

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Recently, Solomon has introduced a zeta function which counts sublattices of a given lattice over an order ([5]). Let us recall the definition of this zeta function. Let Σ be a (finite dimensional) semisimple algebra over the rational field $\boldsymbol{Q}$ or over the p-adic field $\boldsymbol{Q}_p$, and let Λ be an order in Σ. Λ is a $\boldsymbol{Z}$-order when Σ is a $\boldsymbol{Q}$-algebra, while Λ is a $\boldsymbol{Z}_p$-order when Σ is a $\boldsymbol{Q}_p$-algebra, where $\boldsymbol{Z}_p$ is the ring of p-adic integers. Throughout this paper, p stands for a rational prime and the subscript p indicates the p-adic completion.

Let V be a finitely generated left Σ-module, and let L be a full Λ-lattice in V. Solomon's zeta function is defined as

$$\zeta_\Lambda(L;s)=\sum_N (L:N)^{-s},$$

where the sum $\sum_N$ extends over all full Λ-sublattices N in L, $(L:N)$ denotes the index of N in L and s is a complex variable. We shall omit the subscript Λ and write $\zeta(L;s)$, unless there is danger of confusion. When Σ is a field K and L is the ring of integers in K, $\zeta_K(L;s)$ is the classical Dedekind zeta function, and we shall denote this by $\zeta_L(s)$.

We denote by C_n the cyclic group of order n. The explicite form of $\zeta(\boldsymbol{Z}G;s)$ has been given for each of the cases $G=C_p$ and C_{p^2} ([4], [5]).

Let q be a prime and let n be a square-free integer coprime to q. Let $C_n\cdot C_q$ be the semidirect product of C_n by C_q in which C_q acts faithfully on the subgroup C_p of C_n for every $p|n$. The aim of this paper is to give an explicit form of $\zeta(\boldsymbol{Z}(C_n\cdot C_q);s)$. We shall use the method introduced in [1].

§1. Let Λ be a $\boldsymbol{Z}$-order in a semisimple $\boldsymbol{Q}$-algebra Σ, and let $\mathfrak{M}$ be a maximal $\boldsymbol{Z}$-order containing Λ. Denote by S the set of primes p for which $\Lambda_p\neq\mathfrak{M}_p$. Since the zeta function satisfies the Euler product identity ([5]), we have

$$\zeta_\Lambda(\Lambda;s)=\zeta_{\mathfrak{M}}(\mathfrak{M};s)\times\prod_{p\in S}\frac{\zeta_{\Lambda_p}(\Lambda_p;s)}{\zeta_{\mathfrak{M}_p}(\mathfrak{M}_p;s)}. \tag{1.1}$$

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Let $\mathfrak{P}$ be a set of representatives of the isomorphism classes of full Λ_p-lattices in Σ_p for each $p\in S$. Then

$$\zeta_{\Lambda_p}(\Lambda_p;s)=\sum_{L\in\mathfrak{P}} Z_{\Lambda_p}(\Lambda_p,L;s) \quad\text{and}\quad Z_{\Lambda_p}(\Lambda_p,L;s)=\sum_{M}(\Lambda_p:M)^{-s},$$

where the sum extends over all full Λ_p-sublattices M in Λ_p isomorphic to L.

The following notation will be often used in this paper.

For a ring R, $R^*=$ the unit group of R.

For a $\boldsymbol{Z}_p$-order Λ in a semisimple $\boldsymbol{Q}_p$-algebra Σ, and for full lattices L, M in Σ,

$$(L:M)=(L:L\cap M)/(M:L\cap M),$$

where the right hand side is defined by the usual index.

$$\{L:M\}=\{x\in\Sigma \mid Lx\subseteq M\}.$$

$$\|x\|_\Sigma=(Lx:L)\quad\text{for } x\in\Sigma^*,$$

this norm is independent of the choice of a full Λ-lattice L.

For a $\boldsymbol{Q}_p$-algebra Σ, $d^*x=$ the Haar measure on Σ^* such that the measure $\mu(\mathfrak{M}^*)=1$ for a maximal $\boldsymbol{Z}_p$-order $\mathfrak{M}$ in Σ. A Haar measure is decomposed canonically according to a decomposition of Σ as $\boldsymbol{Q}_p$-algebras.

Then it is known that

$$Z_\Lambda(L,M;s)=\mu(\mathrm{Aut}_\Lambda(M))^{-1}(M:L)^s\int_{\{M:L\}\cap\Sigma^*}\|x\|^s d^*x \quad ([1,(11)]). \tag{1.2}$$

§2. Let ε_d be a primitive d-th root of unity for every integer $d\geqq 1$, and let $\varphi(\ )$ be the Euler function. The next result has been given in [5]:

$$\zeta(\boldsymbol{Z}C_p;s)=\zeta_{\boldsymbol{Z}}(s)\zeta_{\boldsymbol{Z}[\varepsilon_p]}(s)(1-p^{-s}+p^{1-2s}). \tag{2.1}$$

(2.1) is also proved in [1]. Using the method there, we have immediately the following generalization.

PROPOSITION 2.2. *Let G be the cyclic group of square-free order n. Then*

$$\zeta(\boldsymbol{Z}G;s)=\prod_{m\mid n}\zeta_{\boldsymbol{Z}[\varepsilon_m]}(s)\prod_{p\mid n}\prod_{d\mid n/p}(1-p^{-f_d s}+p^{f_d(1-2s)})^{g_d},$$

where for each prime $p\mid n$ and $d\mid n/p$, g_d is the number of distint prime ideals over (p) in $\boldsymbol{Z}[\varepsilon_d]$ and $f_d=\varphi(d)/g_d$.

For each $p\mid n$, there is a decomposition as $\boldsymbol{Z}_p$-orders

$$\boldsymbol{Z}_pG=\bigoplus_{d\mid n/p}(\boldsymbol{Z}_p[\varepsilon_d]C_p)^{g_d}.$$

Since $\bigoplus_{m|n} \boldsymbol{Z}[\varepsilon_m]$ is a maximal order of $\boldsymbol{Q}G$ containing $\boldsymbol{Z}G$, we have, by virtue of (1.1),

$$\zeta(\boldsymbol{Z}G\,;s)=\prod_{m|n}\zeta_{\boldsymbol{Z}[\varepsilon_m]}(s)\prod_{p|n}\prod_{d|n/p}\left(\frac{\zeta(\boldsymbol{Z}_p[\varepsilon_d]C_p\,;s)}{\zeta_{\boldsymbol{Z}_p[\varepsilon_d]}(s)\zeta_{\boldsymbol{Z}_p[\varepsilon_{dp}]}(s)}\right)^{g_d}$$
$$=\prod_{m|n}\zeta_{\boldsymbol{Z}[\varepsilon_m]}(s)\prod_{p|n}\prod_{d|n/p}\left(\frac{\zeta(\boldsymbol{Z}_p[\varepsilon_d]C_p\,;s)}{(1-p^{-f_d s})^2}\right)^{g_d}.$$

Hence (2.2) follows from the next lemma.

LEMMA 2.3. *Let K be a finite unramified extension of $\boldsymbol{Q}_p$ of degree f, and let R be the ring of integers in K. Then*

$$\zeta(RC_p\,;s)=\frac{1-p^{-fs}+p^{f(1-2s)}}{(1-p^{-fs})^2}.$$

PROOF. There are two isomorphism classes of full RC_p-lattices in KC_p, which are represented by RC_p and $R\oplus R[\varepsilon_p]$. Along the same way as in [1, §3.4], we have

$$Z(RC_p, RC_p\,;s)=1+(p^f-1)\left(\frac{1}{p^{fs}(1-p^{-fs})}\right)^2 \quad \text{and}$$
$$Z(RC_p, R\oplus R[\varepsilon_p]\,;s)=p^{fs}\left(\frac{1}{p^{fs}(1-p^{-fs})}\right)^2.$$

Thus it follows that

$$\zeta(RC_p\,;s)=\frac{1-p^{-fs}+p^{f(1-2s)}}{(1-p^{-fs})^2}.$$

§3. Let q be a prime and let n be a square-free integer coprime to q. Denote by G_n the semidirect product $C_n\cdot C_q$ of C_n by C_q in which $H=C_q$ acts faithfully on the subgroup C_p of C_n for each $p|n$. Write

$$G_n=\langle\sigma,\tau\,|\,\sigma^n=\tau^q=1,\tau\sigma=\sigma^r\tau\rangle,$$

where r is a primitive q-th root of unity modulo p for every $p|n$. Let ε_d be a primitive d-th root of unity. For each $d|n$, $d\neq 1$, H acts on $\boldsymbol{Q}(\varepsilon_d)$ by $\tau\cdot\varepsilon_d=\varepsilon_d{}^r$. Denote by K_d the invariant subfield $\boldsymbol{Q}(\varepsilon_d)^H$ and by R_d the ring of integers in K_d. We will calculate $\zeta(\boldsymbol{Z}G_n\,;s)$.

In this section, assume that $n=p$ is a prime. Let us denote $G=G_p$, $K=K_p$ and $R=R_p$. Then $\mathfrak{M}=\boldsymbol{Z}\oplus\boldsymbol{Z}[\varepsilon_q]\oplus M_q(R)$ is a maximal $\boldsymbol{Z}$-order in $\boldsymbol{Q}G$. Denote by $\hat{K}$ (resp. $\hat{R}$) the p-adic completion of K (resp. R). To begin with, $\zeta(\boldsymbol{Z}G\,;s)$ is reduced as follows.

LEMMA 3.1.

$$\zeta(\boldsymbol{Z}G\,;s)=\zeta_{\boldsymbol{Z}}(s)\zeta_{\boldsymbol{Z}[\varepsilon_q]}(s)\prod_{i=0}^{q-1}\zeta_R(qs-i)\times(1-q^{-s}+q^{1-2s})\frac{\zeta(\boldsymbol{Z}_pG\,;s)}{\zeta(\mathfrak{M}_p\,;s)}\quad\text{and}$$

$$\zeta(\mathfrak{M}_p\,;s)^{-1}=(1-p^{-s})^q\prod_{i=0}^{p-1}(1-p^{i-qs})\,.$$

PROOF. $\zeta(\mathfrak{M}\,;s)=\zeta_{\boldsymbol{Z}}(s)\zeta_{\boldsymbol{Z}[\varepsilon_q]}(s)\zeta(M_q(R)\,;s)$, and by Hey's formula [2, C.7 §8],

$$\zeta(M_q(R)\,;s)=\prod_{i=0}^{q-1}\zeta_R(qs-i)\,.$$

Since $q|p-1$, we have

$$\zeta(\mathfrak{M}_p\,;s)=\zeta_{\boldsymbol{Z}_p}(s)^q\zeta(M_q(\hat{R})\,;s)=(1-p^{-s})^{-q}\prod_{i=0}^{q-1}(1-p^{i-qs})^{-1}\,.$$

Only primes l for which $\boldsymbol{Z}_lG\neq\mathfrak{M}_l$ are p and q. Since $\boldsymbol{Z}_qG$ is decomposed as $\boldsymbol{Z}_qH\oplus(\boldsymbol{Z}_q\underset{\boldsymbol{Z}}{\otimes}\boldsymbol{Z}[\varepsilon_p]\cdot H)$ and the latter is a maximal order, we have

$$\frac{\zeta(\boldsymbol{Z}_qG\,;s)}{\zeta(\mathfrak{M}_q\,;s)}=\frac{\zeta(\boldsymbol{Z}_qH;s)}{\zeta_{\boldsymbol{Z}_q}(s)\zeta_{\boldsymbol{Z}_q[\varepsilon_q]}(s)}=1-q^{-s}+q^{1-2s}\,,\quad\text{by (2.1)}$$

Then the result follows from the formula (1.1).

By (3.1), it suffices to calculate $\zeta(\boldsymbol{Z}_pG\,;s)$. Hereafter we denote $\Lambda=\boldsymbol{Z}_pG$. Since $q|p-1$, there is a primitive q-th root ω of unity in $\boldsymbol{Z}_p$, and $\boldsymbol{Z}_pH$ is decomposed as $\boldsymbol{Z}_pe_1\oplus\cdots\oplus\boldsymbol{Z}_pe_q$ where e_i $(1\leqq i\leqq q)$ is the idempotent for which $\tau e_i=\omega^{i-1}e_i$. Then we have

$$\begin{aligned}\Lambda&=\Lambda e_1\oplus\cdots\oplus\Lambda e_q\\&=\boldsymbol{Z}_pC_pe_1\oplus\cdots\oplus\boldsymbol{Z}_pC_pe_q\quad\text{as }\Lambda\text{-lattices.}\end{aligned}$$

Let $N_0e_i=\boldsymbol{Z}_pe_i\oplus\boldsymbol{Z}_p[\varepsilon_p]e_i$ and $N_1e_i=\boldsymbol{Z}_pC_pe_1$, these are Λ-lattices in a natural way.

There are 2^q isomorphism classes of full Λ-lattices in $\boldsymbol{Q}_p\Lambda$, which are represented by

$$L_{(\delta_1,\dots,\delta_q)}=N_{\delta_1}e_1\oplus\cdots\oplus N_{\delta_q}e_q\,,\quad\text{where }\delta_i=0\text{ or }1.$$

There is a relation: $\Lambda=L_{(1,\dots,1)}\subseteqq L_{(\delta_1,\dots,\delta_q)}\subseteqq L_{(0,\dots,0)}=\mathfrak{H}$ (say). We have $\mathfrak{H}=\boldsymbol{Z}_pH\oplus\boldsymbol{Z}_p[\varepsilon_p]\circ H$. Since $\mathfrak{A}=\boldsymbol{Z}_p[\varepsilon_p]\circ H$ is a hereditary order in $M_q(\hat{K})$,

$$\mathfrak{A}=\{(x_{ij})\in M_q(\hat{R})|x_{ij}\in\pi\hat{R}\quad\text{if}\quad i<j\}\,,$$

where π is a prime element of $\hat{R}$. Further, by force of the pullback diagram

$$\begin{array}{ccc}\Gamma & \longrightarrow & \mathfrak{A}=\begin{pmatrix}\hat{R} & & \pi\hat{R}\\ & \ddots & \\ \hat{R} & & \hat{R}\end{pmatrix}\\ \big\downarrow & & \big\downarrow\\ \underbrace{\boldsymbol{Z}_p\oplus\cdots\oplus\boldsymbol{Z}_p}_{q}=\boldsymbol{Z}_pH & \longrightarrow & \boldsymbol{F}_pH=\underbrace{\boldsymbol{F}_p\oplus\cdots\oplus\boldsymbol{F}_p}_{q}\end{array}\,,$$

we may identify

$$\Lambda=\left\{(x_1,\cdots,x_q;(y_{ij}))\in \boldsymbol{Z}_p\oplus\cdots\oplus Z_p\oplus\mathfrak{A}\;\middle|\;\begin{matrix}x_i\equiv y_{ii} \bmod \pi\hat{R}\\ 1\leqq i\leqq q\end{matrix}\right\},$$

under some rearrangement of e_i if necessary.

LEMMA 3.2. *Let* $L=L_{(\delta_1,\cdots,\delta_q)}$ *and let* $r=\sum_{i=0}^{q}\delta_i$. *Then*

i) $(L:\Lambda)=p^{q-r}$.

ii) $\mu(\mathrm{Aut}_\Lambda(L))^{-1}=\prod_{i=1}^{q}\left(\frac{p^i-1}{p-1}\right)\times(p-1)^r$.

PROOF. i) $(L:\Lambda)=(\boldsymbol{Z}_p\oplus\boldsymbol{Z}_p[\varepsilon_p]:\boldsymbol{Z}_pC_p)^{q-r}=p^{q-r}$.

ii) For every $i,j,1\leqq i,j\leqq q$, it is clear that

$$\mathrm{Hom}_\Lambda(\boldsymbol{Z}_pe_i,\boldsymbol{Z}_p[\varepsilon_p]e_j)=\mathrm{Hom}_\Lambda(\boldsymbol{Z}_p[\varepsilon_p]e_i,\boldsymbol{Z}_pe_j)=0.$$

Let $i\neq j$. Then we have $\mathrm{Hom}_\Lambda(\boldsymbol{Z}_pe_i,\boldsymbol{Z}_pe_j)=0$. Further, for every f in $\mathrm{Hom}(\boldsymbol{Z}_p[\varepsilon_p]e_i,\boldsymbol{Z}_p[\varepsilon_p]e_j)$, we see that $f(e_i)\in(\varepsilon_p-1)\boldsymbol{Z}_p[\varepsilon_p]e_j$. On the other hand, $f\longmapsto f(e_i)$ induces $\mathrm{End}_\Lambda(\boldsymbol{Z}_pe_i)\cong\boldsymbol{Z}_p$, $\mathrm{End}_\Lambda(\boldsymbol{Z}_p[\varepsilon_p]e_i)\cong(\boldsymbol{Z}_p[\varepsilon_p])^H$ and $\mathrm{End}_\Lambda(\boldsymbol{Z}_pC_pe_i)\cong(\boldsymbol{Z}_pC_p)^H$.

Thus, each $f\in\mathrm{End}\,\Lambda(L)$ is given uniquely by

$$(a_1,\cdots,a_q;(b_{ij}))\in\boldsymbol{Z}_p\oplus\cdots\oplus\boldsymbol{Z}_p\oplus M_q(\boldsymbol{Z}_p[\varepsilon_p]),$$

where $b_{ii}\in\hat{R}$, $b_{ij}\in(\varepsilon_p-1)\boldsymbol{Z}_p[\varepsilon_p]$ if $i\neq j$, and $a_i\equiv b_{ii} \bmod \pi\hat{R}$ if $\delta_i=1$. It can be shown that $f\in\mathrm{Aut}_\Lambda(L)$ if and only if $a_i\in\boldsymbol{Z}_p^*$ and $b_{ii}\in\hat{R}^*$ for every i, $1\leqq i\leqq q$. Therefore we see that

$$(\mathrm{Aut}_\Lambda(\mathfrak{H}):\mathrm{Aut}_\Lambda(L))=(p-1)^r,$$

and so we have

$$\mu(\mathrm{Aut}_\Lambda(L))=\mu(\mathrm{Aut}_\Lambda(\mathfrak{H}))\times(p-1)^{-r}.$$

By the way

$$\mu(\mathrm{Aut}_\Lambda(\mathfrak{H}))=\mu(\mathfrak{H}^*)=\mu(\mathfrak{A}^*)=(GL_q(\hat{R}):\mathfrak{A}^*)^{-1}=\prod_{i=1}^{q}\left(\frac{p-1}{p^i-1}\right).$$

Thus we have

$$\mu(\mathrm{Aut}_\Lambda(L))^{-1}=\prod_{i=1}^{q}\left(\frac{p^i-1}{p-1}\right)\times(p-1)^r.$$

Let $F=\boldsymbol{Z}_p/p\boldsymbol{Z}_p\cong R/\pi R$. For $a_i\in F$, $1\leqq i\leqq q$, let us denote

$$\tilde{\Delta}(a_1, \cdots, a_q)=\left\{(x_1, \cdots, x_q; (y_{ij}))\in\Lambda \,\middle|\, \begin{matrix} x_i \bmod p\boldsymbol{Z}_p = y_{ii} \bmod \pi\hat{R}=a_i\,, \\ 1\leqq i\leqq q \end{matrix}\right\}$$

and

$$\Delta(a_1, \cdots, a_q)=\{(y_{ij})\in\mathfrak{A}\,|\,y_{ii} \bmod \pi\hat{R}=a_i\,,\quad 1\leqq i\leqq q\}\,.$$

LEMMA 3.3. i) *Let* $L=L_{(\delta_1,\cdots,\delta_q)}$ *and let* $r=\sum_{i=1}^{q}\delta_i$. *Then*

$$\{L:\Lambda\}=\dot{\bigcup_{\substack{a_i\in F \text{ if } \delta_i=1\\ a_i=0 \text{ if } \delta_i=0\\ 1\leqq i\leqq q}}}\tilde{\Delta}(a_1, \cdots, a_q) \quad (\textit{disjoint union}).$$

Let $a_i\in F$, $1\leqq i\leqq q$, *and let k be the number of i such that* $a_i\neq 0$. *Then*

ii) $$\int_{\Delta(a_1,\cdots,a_q)\cap GL_q(\hat{K})}\|x\|_{M_q(\hat{K})}^s d^*x=\int_{\Delta(\underbrace{1,\cdots,1}_{k},0,\cdots,0)\cap GL_q(\hat{K})}\|x\|_{M_p(\hat{K})}^s d^*x$$

iii) $$\int_{\tilde{\Delta}(a_1,\cdots,a_q)\cap Q_{q}\Lambda^*}\|x\|_{Q_p\Lambda}^s d^*x=\frac{1}{(p^s-1)^{q-k}(p-1)^k}\int_{\Delta(\underbrace{1,\cdots,1}_{k},0,\cdots,0)\cap GL_q(\hat{K})}\|x\|_{M_q(\hat{K})}^s d^*x\,.$$

PROOF. i) $\{L:\Lambda\}=\{x\in\Lambda\,|\,Lx\subseteqq\Lambda\}$, since $1\in L$,

$$=\left\{x\in\Lambda\,\middle|\,e_ix\in\Lambda \quad\text{for every } i\,,\quad \frac{\Phi_p(\sigma)}{p}e_jx\in\Lambda \quad\text{if}\quad \delta_j=0\right\}$$

$$=\left\{x\in\Lambda\,\middle|\,x\frac{\Phi_p(\sigma)}{p}e_j\in p\boldsymbol{Z}_pe_j\,,\quad x\left(1-\frac{\Phi_p(\sigma)}{p}\right)e_j\in(1-\varepsilon_p)\boldsymbol{Z}_p[\varepsilon_p]e_j \text{ if } \delta_i=0\right\}$$

$$=\dot{\bigcup_{\substack{a_i\in F \text{ if } \delta_i=1\\ a_i=0 \text{ if } \delta_i=1\\ 1\leqq i\leqq q}}}\tilde{\Delta}(a_1, \cdots, a_q)\,,$$

where $\Phi_p(\sigma)$ is the p-th cyclotomic polynomial.

ii) Since there exist $A, B\in GL_q(\hat{R})$ such that

$$A\Delta(a_1, \cdots, a_q)B=\Delta(\underbrace{1, \cdots, 1}_{k}, 0, \cdots, 0)\,,$$

the integral over $\Delta(a_1, \cdots, a_q)$ is equal to that over $\Delta(\underbrace{1, \cdots, 1}_{k}, 0, \cdots, 0)$.

iii) Let $Z(a_i)=\{z\in\boldsymbol{Z}_p\,|\,z \bmod p\boldsymbol{Z}_p=a_i\}$, $1\leqq i\leqq q$. Then

$$\tilde{\Delta}(a_1, \cdots a_q)=\bigoplus_{i=1}^{q}Z(a_i)\oplus\Delta(a_1, \cdots, a_q)\,,$$

and we see that

$$\int_{Z(a_i)\cup Q_p^*}\|x\|_{Q_p}^s d^*x=\begin{cases}\dfrac{1}{p-1} & \text{if} \quad a_i\neq 0\\[2mm] \dfrac{1}{p^s-1} & \text{if} \quad a_i=0.\end{cases}$$

Thus we have, by force of ii),

$$\int_{\mathfrak{Z}(a_1,\cdots,a_q)\cap Q_{qA^*}}\|x\|_{Q_qA}^s d^*x=\frac{\int_{\Delta(\overbrace{1,\cdots,1}^{k},0,\cdots,0)\cap GL_q(\hat{K})}\|x\|_{M_q(\hat{K})}^s d^*x}{(p^s-1)^{q-k}(p-1)^k}.$$

We shall use the following notation:
for an integer $n\geqq 1$,

$$\Sigma_n=M_n(\hat{K});$$

$$\Gamma_n=\{(x_{ij})\in M_n(\hat{R})\,|\,x_{ij}\in\pi\hat{R} \quad \text{for } 1\leqq i<j\leqq n\};$$

$$E_n=\Gamma_n{}^*=\{(x_{ij})\in\Gamma_n\,|\,x_{ii}\in\hat{R}^* \quad \text{for } 1\leqq i\leqq n\};$$

$$d_n=\mu(E_n)=\prod_{i=1}^{n}\left(\frac{p-1}{p^i-1}\right);$$

and for an integer k, $0\leqq k\leqq n$,

$$\Delta_n(k)=\left\{(x_{ij})\in\Gamma_n\,\middle|\,\begin{matrix}x_{ii}\in\hat{R}^* & \text{for } 1\leqq i\leqq k\\ x_{ii}\in\pi\hat{R} & \text{for } k+1\leqq i\leqq n\end{matrix}\right\}.$$

We shall omit the subscript n, unless there is danger of confusion.

E_n acts on $\Gamma_n\cap\Sigma_n{}^*$ by left multiplication. As a full set of representatives of $E_n\backslash\Gamma_n\cap\Sigma_n{}^*$, we can take the set $T_n=\dot{\bigcup}_{\sigma\in S_n}T_{n,\sigma}$, where S_n is the symmetric group on n symbols, and each $T_{n,\sigma}$ is the set of matrices $(x_{ij})\in\Sigma_n{}^*$ such that

i) for $1\leqq j\leqq n$, $x_{\sigma(j),j}=\pi^{m_j}$, where $m_j\geqq 0$ if $\sigma(j)\geqq j$ and $m_j\geqq 1$ if $\sigma(j)<j$,

ii) for $j+1\leqq i\leqq n$, $x_{\sigma(i),j}=0$

iii) for $1\leqq i\leqq j-1$, $x_{\sigma(i),j}$ ranges over all representatives of

$$\begin{cases}\pi\hat{R}/\pi^{m_j+1}\hat{R} & \text{if} \quad \sigma(i)<j \quad \text{and} \quad \sigma(i)<\sigma(j)\\ \hat{R}/\pi^{m_j+1}\hat{R} & \text{if} \quad j\leqq\sigma(i)<\sigma(j)\\ \pi\hat{R}/\pi^{m_j}\hat{R} & \text{if} \quad \sigma(j)<\sigma(i)<j\\ \hat{R}/\pi^{m_j}\hat{R} & \text{if} \quad \sigma(i)\geqq j \quad \text{and} \quad \sigma(i)>\sigma(j),\end{cases}$$

where m_j, $1\leqq j\leqq n$, are as in i).

We note here that, for the matrix (x_{ij}) as above, $det(x_{ij})=\pm p^{\sum_{i=1}^{n}m_j}$.

LEMMA 3.4. *Let $n\geqq 1$ be an integer. Then*

i) *There exists a polinomial* $G_n(x)$ *over* $\boldsymbol{Z}$ *with* $p^{n(n-1)/2}X^n$ *as the highest term and* X *as the lowest term such that*

$$\int_{\Delta_n(0)\cap\Sigma_n^*}\|x\|_{\Sigma_n}^s d^*x=\frac{d_nG_n(p^{-ns})}{\prod_{i=0}^{n-1}(1-p^{i-ns})}.$$

ii) *For every integer* k, $0\leqq k\leqq n$,

$$\int_{\Delta_n(k)\cap\Sigma_n^*}\|x\|_{\Sigma_n}^s d^*x=\frac{d_nG_{n-k}(p^{k-ns})}{\prod_{i=k}^{n-1}(1-p^{i-ns})}.$$

PROOF. i) We see that E_n acts on $\Delta_n(0)\cap\Sigma_n^*$, and that $T_n\cap\Delta_n(0)$ form a full set of representatives of $E_n\backslash\Delta_n(0)\cap\Sigma_n^*$. Thus we have

$$\int_{\Delta_n(0)\cap\Sigma_n^*}\|x\|_{\Sigma_n}^s d^*x=\mu(E_n)\sum_{\sigma\in S_n}\sum_{M\in T_{n,\sigma}\cap\Delta_n(0)}\|detM\|_{\hat{K}}^{-ns}.$$

Let $\sigma\in S_n$. For each j, $1\leqq j\leqq n$, let $m_j\geqq 0$ if $\sigma(j)>j$ and let $m_j\geqq 1$ if $\sigma(j)\leqq j$. Further, let $t_j=\#\{i|1\leqq i\leqq j-1$ and $j<\sigma(i)<\sigma(j)\}$ and let $v_j=\#\{i|1\leqq i\leqq j-1$ and $\sigma(j)<\sigma(i)\leqq j\}$. Then $0\leqq t_j,\ v_j\leqq j-1$ and $t_jv_j=0$. There are $p^{(j-1)m_j}p^{t_j-v_j}$ ways of the choice of the j-th column among $\{(x_{ij})\in T_{n,\sigma}\cap\Delta_n(0)|x_{\sigma(j),j}=\pi^{m_j}$ for $1\leqq j\leqq n\}$. Thus we have

$$\begin{aligned}\sum_{M\in T_{n,\sigma}\cap\Delta_n(0)}\|detM\|_{\hat{K}}^{-ns}&=\sum_{\substack{m_j\geqq 0 \text{ if } \sigma(j)>j\\ m_j\geqq 1 \text{ if } \sigma(j)\leqq j\\ 1\leqq j\leqq n}}\left(\prod_{j=1}^{n}p^{t_j-v_j}p^{(j-1)m_j}p^{-nm_js}\right)\\&=\frac{1}{\prod_{i=0}^{n-1}(1-p^{i-ns})}\left(\prod_{\substack{1\leqq j\leqq n\\ \sigma(j)>j}}p^{t_j}\prod_{\substack{1\leqq j\leqq n\\ \sigma(j)\leqq j}}(p^{j-1-v_j}p^{-ns})\right)\\&=\frac{p^{c_\sigma}(p^{-ns})^{e_\sigma}}{\prod_{i=0}^{n-1}(1-p^{i-ns})},\end{aligned}$$

where $e_\sigma=\#\{j|1\leqq j\leqq n$ and $\sigma(j)\leqq j\}$ and $c_\sigma=\sum_{\substack{1\leqq j\leqq n\\ \sigma(j)>j}}t_j+\sum_{\substack{1\leqq j\leqq n\\ \sigma(j)\leqq j}}(j-1-v_j)$. If $\sigma=id$, then $e_\sigma=n$ and $t_j=v_j=0$ for $1\leqq j\leqq n$, and hence $c_\sigma=n(n-1)/2$. If $\sigma=(12\cdots n)$, then $e_\sigma=1$, $t_j=0$ for $1\leqq j\leqq n$, $v_j=0$ for $1\leqq j\leqq n-1$, and $v_n=n-1$, and hence $c_\sigma=0$. It is easy to see that $2\leqq e_\sigma\leqq n-1$ if $\sigma\neq id$, $\sigma\neq(12\cdots n)$. Let $G_n(X)=\sum_{\sigma\in S_n}p^{c_\sigma}X^{e_\sigma}$, then the highest term $p^{n(n-1)/2}X^n$ comes from $\sigma=id$ and the lowest term X comes from $\sigma=(12\cdots n)$. Finally we have

$$\int_{\Delta_n(0)\cap\Sigma_n^*}\|x\|_{\Sigma_n}^s d^*x=\frac{d_nG_n(p^{-ns})}{\prod_{i=0}^{n-1}(1-p^{i-ns})}.$$

ii) We see that E_n acts on $\Delta_n(k)\cap\Sigma_n^*$, and that $T_n\cap\Delta_n(k)$ form a full set of representatives of $E_n\backslash\Delta_n(k)\cap\Sigma_n{}^*$. They are of the form

$$\overbrace{}^{k}\;\overbrace{}^{n-k}$$
$$\left(\begin{array}{c|c} \begin{matrix}1 & & 0\\ & \ddots & \\ & & 1\end{matrix} & B \\ \hline 0 & A \end{array}\right), \quad \text{where } A\in T_{n-k}\cap\Delta_{n-k}(0),$$

and for each A of $detA=\pm p^m$, there are p^{km} ways of the choice of B. Let $l=n-k$ and let a_m be the cardinary of the set

$$\{A\in T_l\cap\Delta_l(0)|detA=\pm p^m\}.$$

Then we have

$$\int_{\Delta_n(k)\cap\Sigma_n{}^*}||x||_{\Sigma_n}^s d^*x=d_n\Big(\sum_{M\in T_n\cap\Delta_n(k)}||detM||_{\hat{R}}^{-ns}\Big)$$
$$=d_n\sum_{m\geq 0}p^{km}a_m(p^{-ns})^m=d_n\sum_{m\geq 0}a_m(p^{k-ns})^m.$$

Since $\sum_{m\geq 0}a_m(p^{-ls})^m=\dfrac{G_l(p^{-ls})}{\prod_{i=0}^{l-1}(1-p^{i-ls})}$, we have

$$\sum_{m\geq 0}a_m(p^{k-ns})^m=\frac{G_l(p^{k-ns})}{\prod_{i=0}^{l-1}(1-p^{i+k-ns})}.$$

Therefore we have

$$\int_{\Delta_n(k)\cap\Sigma_n{}^*}||x||_{\Sigma_n}^s d^*x=\frac{d_nG_{n-k}(p^{k-ns})}{\prod_{i=k}^{n-1}(1-p^{i-ns})}.$$

EXAMPLE 3.5. If n is given, then $G_n(X)$ can be written explicitly. It is easy to see that $G_1(X)=X$ and $G_2(X)=pX^2+X$. For the case that $n=3$, we have the following table

σ	$(T)_{3,\sigma}$	e_σ	c_σ
id	$\left\{\begin{pmatrix}\pi^l & a & b\\ 0 & \pi^m & c\\ 0 & 0 & \pi^n\end{pmatrix}\middle\vert \begin{array}{l} l,m,n\geq 1\\ a\in\pi\hat{R}/\pi^{m+1}\hat{R},\quad b,c\in\pi\hat{R}/\pi^{n+1}\hat{R}\end{array}\right\}$	3	3
(23)	$\left\{\begin{pmatrix}\pi^l & a & b\\ 0 & 0 & \pi^n\\ 0 & \pi^m & c\end{pmatrix}\middle\vert \begin{array}{l} l,n\geq 1,\quad m\geq 0\\ a\in\pi\hat{R}/\pi^{m+1}\hat{R},\quad b\in\pi\hat{R}/\pi^{n+1}\hat{R},\quad c\in\pi\hat{R}/\pi^{n}\hat{R}\end{array}\right\}$	2	1

(12)	$\left\{\begin{pmatrix} 0 & \pi^m & b \\ \pi^l & a & c \\ 0 & 0 & \pi^n \end{pmatrix} \middle\vert \begin{array}{l} l\geqq 0, \quad m,n\geqq 1 \\ a\in\pi\hat{R}/\pi^m\hat{R}, \quad b,c\in\pi\hat{R}/\pi^{n+1}\hat{R} \end{array}\right\}$	2	2
(123)	$\left\{\begin{pmatrix} 0 & 0 & \pi^n \\ \pi^l & a & b \\ 0 & \pi^m & c \end{pmatrix} \middle\vert \begin{array}{l} l,m\geqq 0, \quad n\geqq 1 \\ a\in\pi\hat{R}/\pi^{m+1}\hat{R}, \quad b,c\in\pi\hat{R}/\pi^n\hat{R} \end{array}\right\}$	1	0
(132)	$\left\{\begin{pmatrix} 0 & \pi^m & b \\ 0 & 0 & \pi^n \\ \pi^l & a & c \end{pmatrix} \middle\vert \begin{array}{l} l\geqq 0, \quad m,n\geqq 1 \\ a\in\hat{R}/\pi^m\hat{R}, \quad b\in\pi\hat{R}/\pi^{n+1}\hat{R}, \quad c\in\pi\hat{R}/\pi^n\hat{R} \end{array}\right\}$	2	2
(13)	$\left\{\begin{pmatrix} 0 & 0 & \pi^n \\ 0 & \pi^m & b \\ \pi^l & a & c \end{pmatrix} \middle\vert \begin{array}{l} l\geqq 0, \quad m,n\geqq 1 \\ a\in\hat{R}/\pi^m\hat{R}, \quad b,c\in\pi\hat{R}/\pi^n\hat{R} \end{array}\right\}$	2	1

Thus we see that

$$G_3(X)=p^3X^3+2(p^2+p)X^2+X.$$

Now we have prepared to show

PROPOSITION 3.6.

$$\zeta(\boldsymbol{Z}_pG\,;\,s)=\sum_{k=0}^{q}\frac{{}_qC_k(1+(p-1)p^{-s})^{q-k}G_{q-k}(p^{k-qs})}{\prod_{i=k}^{q-1}((1-p^{-s})(1-p^{i-qs}))},$$

where ${}_qC_k$ is the binomial coefficient, and we define $G_0(X)=1$ and $\prod_{i=q}^{q-1}((1-p^{-s})(1-p^{i-qs}))=1$.

PROOF. Let $L=L_{(\delta_1,\dots,\delta_q)}$ and let $r=\sum_{i=1}^{q}\delta_i$. Then, by force of (1.2), (3.2) and (3.3),

$$Z(\Lambda,L\,;\,s)=d_q^{-1}(p-1)^rp^{(q-r)s}\left[\sum_{k=0}^{r}(p-1)^k{}_rC_k\int_{\tilde{\Delta}(\underbrace{1,\dots,1}_{k},0,\dots,0)\cap Q_p\Lambda^*}\|x\|_{Q_p\Lambda}^{s}d^*x\right]$$

$$=d_q^{-1}(p-1)^rp^{(q-r)s}\left[\sum_{k=0}^{r}\frac{{}_rC_k}{(p^s-1)^{q-k}}\int_{\Delta(\underbrace{1,\dots,1}_{k},0,\dots,0)\cap\Sigma^*}\|x\|_{\Sigma}^{s}d^*x\right]$$

$$=d_q^{-1}(p-1)^rp^{(q-r)s}\left[\sum_{k=0}^{r}\frac{{}_rC_k}{(p-1)^k(p^s-1)^{q-k}}\int_{\Delta(k)\cap\Sigma^*}\|x\|_{\Sigma}^{s}d^*x\right]$$

$$=\sum_{k=0}^{r}\left[\frac{{}_rC_k((p-1)p^{-s})^{r-k}}{(1-p^{-s})^{q-k}}\cdot\frac{G_{q-k}(p^{k-qs})}{\prod_{i=k}^{q-1}(1-p^{i-qs})}\right],\quad\text{by (3.4 ii).}$$

Thus we have

$$\begin{aligned}\zeta(\Lambda;s)&=\sum_{\substack{L=L(\delta_1,\cdots,\delta_q)\\ \delta_i=0,1}} Z(\Lambda,L;s)\\ &=\sum_{r=0}^{q}{}_qC_r Z(\Lambda,L_{(\underbrace{1\ldots1}_{r},0,\cdots,0)};s)\\ &=\sum_{r=0}^{q}\sum_{k=0}^{r}\frac{{}_qC_r\cdot{}_rC_k((p-1)p^{-s})^{r-k}G_{q-k}(p^{k-qs})}{(1-p^{-s})^{q-k}\prod_{i=k}^{q-1}(1-p^{i-qs})}.\end{aligned}$$

Since ${}_qC_r\cdot{}_rC_k={}_qC_k\cdot{}_{q-k}C_{r-k}$, we have

$$\begin{aligned}\zeta(\Lambda;s)&=\sum_{k=0}^{q}\sum_{r=k}^{q}\frac{{}_qC_k\cdot{}_{q-k}C_{r-k}\cdot((p-1)p^{-s})^{r-k}G_{q-k}(p^{k-qs})}{(1-p^{-s})^{q-k}\prod_{i=k}^{q-1}(1-p^{i-qs})}\\ &=\sum_{k=0}^{q}\frac{{}_qC_k(1+(p-1)p^{-s})^{q-k}G_{q-k}(p^{k-qs})}{\prod_{i=k}^{q-1}((1-p^{-s})(1-p^{i-qs}))}.\end{aligned}$$

Combining (3.1) with (3.6), we have

THEOREM 3.7.

$$\begin{aligned}\zeta(\boldsymbol{Z}G;s)=&\zeta_{\boldsymbol{Z}}(s)\zeta_{\boldsymbol{Z}[\varepsilon_q]}(s)\prod_{i=0}^{q-1}\zeta_R(qs-i)\times(1-q^{-s}+q^{1-2s})\\ &\times\left[\sum_{k=0}^{q}\left\{{}_qC_k(1+(p-1)p^{-s})^{q-k}G_{q-k}(p^{k-qs})\prod_{i=0}^{k-1}((1-p^{-s})(1-p^{i-qs}))\right\}\right].\end{aligned}$$

EXAMPLE 3.8. We note here for the case that $q=2$ (dihedral group) and $q=3$.

$$\begin{aligned}\zeta(\boldsymbol{Z}D_p;s)=&\zeta_{\boldsymbol{Z}}(s)^2\zeta_R(2s)\zeta_R(2s-1)\times(1-2^{-s}+2^{1-2s})\\ &\times(1-2p^{-s}+(p+1)p^{-2s}+2p^{2-3s}-(p^2+p)p^{-4s}+p^{3-6s}),\end{aligned}$$

where $R=\boldsymbol{Z}[\varepsilon_p+\varepsilon_p^{-1}]$, and

$$\begin{aligned}\zeta(\boldsymbol{Z}(C_p\cdot C_3);s)=&\zeta_{\boldsymbol{Z}}(s)\zeta_{\boldsymbol{Z}[\varepsilon_3]}(s)\zeta_R(3s)\zeta_R(3s-1)\zeta_R(3s-2)\times(1-3^{-s}+3^{1-2s})\\ &\times\begin{pmatrix}(1-y)^4(1-y^3)(1-py^3)(1-p^2y^3)\\ +3(1-y)^2(1-y^3)(1-py^3)(1+(p-1)y)p^2y^3\\ +3(1-y)(1-y^3)(1+(p-1)y)^2(py^3+p^3y^6)\\ +(1+(p-1)y)^3(y^3+2(p^2+p)y^6+p^3y^9)\end{pmatrix},\end{aligned}$$

where $y=p^{-s}$ and R is the ring of integers in $\boldsymbol{Q}(\varepsilon_p)^{C_3}$.

§4. Let G_n and H be the groups defined at the beginning of §3:

$$G_n=\langle\sigma,\tau\mid\sigma^n=\tau^q=1,\tau\sigma=\sigma^r\tau\rangle \quad\text{and}\quad H=\langle\tau\rangle.$$

Then we have $\boldsymbol{Q}G_n=\boldsymbol{Q}H\oplus\bigoplus_{\substack{d|n/p\\ d\neq 1}} M_q(K_d)$ as algebras. For each $p|n$, there is a decomposition as $\boldsymbol{Z}_p$-orders

$$\boldsymbol{Z}_pG_n=\boldsymbol{Z}_pG_p\oplus\bigoplus_{\substack{d|n/p\\ d\neq 1}}(\boldsymbol{Z}_p[\xi_d]\circ G_p)^{g_d}.$$

Here g_d is the number of distinct prime ideals over (p) in R_d, and $\boldsymbol{Z}_p[\xi_d]=\boldsymbol{Z}_p[X]/(\Psi_d(X))$, where $\Psi_d(X)$ is the minimal monic polynomial over $\boldsymbol{Z}_p$ such that $\Psi_d(\varepsilon_d{}^{r^i})=0$, $0\leqq i\leqq q-1$. On the other hand, there is a decomposition as $\boldsymbol{Z}_q$-orders

$$\boldsymbol{Z}_qG_n=\boldsymbol{Z}_qH\oplus(\bigoplus_{\substack{d|n\\ d\neq 1}}\boldsymbol{Z}_q\otimes_{\boldsymbol{Z}}\boldsymbol{Z}[\varepsilon_d]\circ H),$$

where the latter factor is a maximal $\boldsymbol{Z}_q$-order.

Let $\mathfrak{M}=\boldsymbol{Z}\oplus\boldsymbol{Z}[\varepsilon_q]\oplus\bigoplus_{\substack{d|n\\ d\neq 1}}M_q(R_d)$. Then $\mathfrak{M}$ is a maximal $\boldsymbol{Z}$-order in $\boldsymbol{Q}G_n$. Then, by virtue of (1.1) and Hey's formula, we have

LEMMA 4.1.

$$\zeta(\boldsymbol{Z}G_n\,;s)=\zeta(\mathfrak{M}\,;s)\times(1-q^{-s}+q^{1-2s})\prod_{p|n}\frac{\zeta(\boldsymbol{Z}_pG_p\,;s)\prod_{d|n/p,d\neq 1}(\zeta(\boldsymbol{Z}_p[\xi_d]\circ G_p\,;s))^{g_d}}{\zeta(\mathfrak{M}_p\,;s)},$$

$$\zeta(\mathfrak{M}\,;s)=\zeta_{\boldsymbol{Z}[\varepsilon_q]}(s)\prod_{\substack{d|n\\ d\neq 1}}\prod_{i=0}^{q-1}\zeta_{R_d}(qs-i),\quad \textit{and for each } p|n,$$

$$\zeta(\mathfrak{M}_p\,;s)^{-1}=(1-p^{-s})^q\prod_{i=0}^{q-1}(1-p^{i-qs})\prod_{\substack{d|n/p\\ d\neq 1}}\prod_{i=0}^{q-1}(1-p_d{}^{i-qs})^{2g_d},$$

where $p_d=p^{\varphi(d)/qg_d}$.

Let $\Lambda=\boldsymbol{Z}_p[\xi_d]\circ G_p$, where $d|n/p$ and $d\neq 1$, be a factor of $\boldsymbol{Z}_pG_n$ as above. To determine $\zeta(\boldsymbol{Z}G_n\,;s)$, we have only to treat $\zeta(\Lambda\,;s)$, because $\zeta(\boldsymbol{Z}_pG_p\,;s)$ has been determined in § 3.

Denote by $\hat{K}_d, \hat{K}_{dp}, \hat{R}_d$ and $\hat{R}_{dp}$ the p-adic completion of K_d, K_{dp}, R_d and R_{dp}, respectively. As in § 3, we write $\boldsymbol{Z}_p[\xi_d]\circ H=\boldsymbol{Z}_p[\xi_d]e_1\oplus\cdots\oplus\boldsymbol{Z}_p[\xi_d]e_q$ and let $N_0e_i=\boldsymbol{Z}_p[\xi_d]e_i\oplus\boldsymbol{Z}_p[\xi_d,\varepsilon_p]e_i$ and $N_1e_i=\boldsymbol{Z}_p[\xi_d]C_pe_i$, $1\leqq i\leqq q$. These are Λ-lattices in a natural way. There are 2^q isomorphism classes of full Λ-lattices in $\boldsymbol{Q}_p\Lambda$, which are represented by

$$L_{(\delta_1,\cdots,\delta_q)}=N_{\delta_1}e_1\oplus\cdots\oplus N_{\delta_1}e_q,\quad \text{where } \delta_i=0 \text{ or } 1.$$

There is a relation: $\Lambda=L_{(1,\dots,1)}\subseteqq L_{(\delta_1,\dots,\delta_q)}\subseteqq L_{(0,\dots,0)}$. We have $L_{(0,\dots,0)}=\mathfrak{A}\oplus\mathfrak{B}$ as $\boldsymbol{Z}_p$-orders, where $\mathfrak{A}=\boldsymbol{Z}_p[\xi_d]\circ H$ and $\mathfrak{B}=\boldsymbol{Z}_p[\xi_d,\varepsilon_p]\circ H$. Since the extensions $\boldsymbol{Q}(\varepsilon_d)/K_d$ and $\boldsymbol{Q}(\varepsilon_{dp})/K_{dp}$ are unramified at p, $\mathfrak{A}$ and $\mathfrak{B}$ are maximal $\boldsymbol{Z}_p$-orders (cf. [3, § 40]), and hence we may identify $\mathfrak{A}$ with $M_q(\hat{R}_d)$ and $\mathfrak{B}$ with $M_q(\hat{R}_{dp})$. Let π be a prime element of $\hat{R}_{dp}$. Then $\hat{R}_d/p\hat{R}_d \cong \hat{R}_{dp}/\pi\hat{R}_{dp}=F$ (say), and $|F|=p_d=p^{\varphi(d)/qg_d}$. Let us denote $P=p_d$.

LEMMA 4.2. *Let* $L=L_{(\delta_1,\dots,\delta_q)}$ *and let* $r=\sum_{i=1}^{q}\delta_i$. *Then*

i) $(L:\Lambda)=P^{q(q-r)}$

ii) $\mu(\mathrm{Aut}_\Lambda(L))^{-1}=\prod_{i=0}^{r-1}\frac{(P^q-P^i)^2}{P^r-P^i}$.

PROOF. i) $(L:\Lambda)=(\boldsymbol{Z}_p[\xi_d]\oplus \boldsymbol{Z}_p[\xi_d,\varepsilon_p]:\boldsymbol{Z}_p[\xi_d]C_p)^{q-r}$
$=|\boldsymbol{Z}_p[\xi_d]/p\boldsymbol{Z}_p[\xi_d]|^{q-r}=P^{q(q-r)}$.

ii) Let ω be the primitive q-th root of unity in $\boldsymbol{Z}_p$ for which $\tau e_i=\omega^{i-1}e_i$. Let $Y_k=\sum_{i=0}^{q-1}\omega^{-ki}\xi_d^{r^i}$, where $k\in\boldsymbol{Z}$, then $\tau Y_k=\omega^k Y_k\tau$. Since d is square-free and coprime to p, ε_d is a generator of a normal basis for $\boldsymbol{F}_p(\varepsilon_d)/\boldsymbol{F}_p$, and so $\sum_{i=0}^{q-1}\bar{\omega}^{-ki}\varepsilon_d{}^{r^i}\neq 0$ in $\boldsymbol{F}_p(\varepsilon_d)$. Thus we see that Y_k is a unit in $\boldsymbol{Z}_p[\xi_d]$. Then there is an isomorphism between

$$\left\{((a_{ij}),(b_{ij}))\in M_q(\hat{R}_d)\oplus M_q(\hat{R}_{dp})\;\middle|\;\begin{matrix} a_{ij}\equiv b_{ij} \bmod \pi\hat{R}_{dp} \text{ if } \delta_j=1, \text{ in particular,}\\ a_{ij},b_{ij}\in\pi\hat{R}_{dp} \text{ if } \delta_i=0 \text{ and } \delta_j=1\end{matrix}\right\}$$

and $\mathrm{End}_\Lambda(L)$, induced by

$$((a_{ij}),(b_{ij}))\longmapsto f: f(e_i)=\Big(\sum_{j=1}^{q}Y_{i-j}a_{ij}e_j,\ \sum_{j=1}^{q}Y_{i-j}b_{ij}e_j\Big)\in\mathfrak{A}\oplus\mathfrak{B},\quad 1\leqq j\leqq q.$$

Hence we see that

$$\begin{aligned}\mu(\mathrm{Aut}_\Lambda(L))^{-1}&=\mu(\mathrm{Aut}_\Lambda(\mathfrak{A}\oplus\mathfrak{B}))^{-1}(\mathrm{Aut}_\Lambda(\mathfrak{A}\oplus\mathfrak{B}):\mathrm{Aut}_\Lambda(L))\\ &=\frac{|GL_q(F)|^2}{|GL_r(F)||GL_{q-r}(F)|^2P^{2r(q-r)}}\\ &=\prod_{i=0}^{r-1}\frac{(P^q-P^i)^2}{P^r-P^i}.\end{aligned}$$

Let $\mathfrak{X}=M_q(F)$ and, for each $X\in\mathfrak{X}$, let $\Delta(X)=\{A\in M_q(\hat{R}_d)|A \bmod pM_q(\hat{R}_d)=X\}$. To simplify the notation, denote by $\int_{\Delta(X)}$ the integral $\int_{\Delta(X)\cap GL_q(\hat{R}_d)}\|x\|^s_{M_q(\hat{R}_d)}d^*x$. Then we have

LEMMA 4.3. *Let* $L=L_{(\delta_1,\dots,\delta_q)}$ *and let* $r=\sum_{i=1}^{q}\delta_i$. *Then*

$$\int_{(L:\Lambda)\cap Q_q\Lambda^*}\|x\|^s_{Q_{pd}}d^*x=\sum_{X\in\mathfrak{X}_r}\Big(\int_{\Delta(X)}\Big)^2,$$

where $\mathfrak{X}_r=\{(x_{ij})\in\mathfrak{X}|x_{ij}=0$ *for* $r+1\leqq i\leqq q$, $1\leqq j\leqq q\}$.

PROOF. $\{L:\Lambda\}=\{x\in\Lambda \mid Lx\subseteqq\Lambda\}$, since $1\in L$,

$$=\left\{x\in\Lambda \,\middle|\, \frac{\Phi_p(\sigma)}{p}e_jx\in p\mathfrak{A} \text{ and } \left(1-\frac{\Phi_p(\sigma)}{p}\right)e_jx\in(1-\varepsilon_p)\mathfrak{B} \text{ if } \delta_j=0\right\}.$$

Every element of Λ can be written as

$$\sum_{1\leq i,j\leq q} e_i\xi_d{}^j\sigma^j z_{ij}(\sigma)\,,\quad z_{ij}(\sigma)\in(\boldsymbol{Z}_p[\xi_d]C_p)^H$$
$$=(\sum_{i,j} e_i\xi_d{}^j z_{ij}(1)\,,\ \sum_{i,j} e_i\xi_d{}^j\varepsilon_p{}^j z_{ij}(\varepsilon_p))\in\mathfrak{A}\oplus\mathfrak{B}$$

Hence $\{L:\Lambda\}$ may be identified with

$$\left\{((x_{ij}),(y_{ij}))\in M_q(\hat{R}_d)\oplus M_q(\hat{R}_{dp}) \,\middle|\, \begin{array}{ll} x_{ij}\equiv y_{ij} \bmod \pi\hat{R}_{dp} & \text{for } 1\leqq i,j\leqq q \\ x_{kj},\, y_{kj}\in\pi\hat{R}_{dp} & \text{for } r+1\leqq k\leqq q \end{array}\right\}$$
$$=\dot{\bigcup_{X\in\mathfrak{X}_r}}\Delta(X)\oplus\Delta'(X)\,,$$

where $\Delta'(X)=\{B\in M_q(\hat{R}_{dp}) \mid B \bmod \pi M_q(\hat{R}_{dp})=X\}$. Thus we see that

$$\int_{\{L:\Lambda\}\cap Q_p\Lambda^*}||x||^s_{Q_p\Lambda}d^*x$$
$$=\sum_{X\in\mathfrak{X}_r}\left[\int_{\Delta(X)\cap GL_q(\hat{R}_d)}||x||^s_{M_q(\hat{R}_d)}d^*x\int_{\Delta'(X)\cap GL_q(\hat{R}_{dp})}||x||^s_{M_q(\hat{R}_{dp})}d^*x\right].$$

Since $\hat{R}_d/p\hat{R}_d\cong\hat{R}_{dp}/\pi\hat{R}_{dp}$, we have

$$\int_{\{L:\Lambda\}\cap Q_p\Lambda^*}||x||^s_{Q_p\Lambda}d^*x=\sum_{X\in\mathfrak{X}_r}\left(\int_{\Delta(X)}\right)^2.$$

Each $X\in\mathfrak{X}$ becomes the standard form $X_h=\begin{pmatrix} \begin{matrix}1 & & \\ & \ddots & \\ 0 & & 1\end{matrix}{}^{h} & 0 \\ & 0 \end{pmatrix}$, for some $0\leqq h\leqq q$, by elementary transformations. Therefore there exist $A, B\in GL_q(F)$ such that $AXB=X_h$. Let $\tilde{A}, \tilde{B}\in GL_q(\hat{R}_d)$ such that $\tilde{A} \bmod pM_q(\hat{R}_d)=A$ and $\tilde{B} \bmod pM_q(\hat{R}_d)=B$. Then we have $\tilde{A}\Delta(X)\tilde{B}=\Delta(X_h)$. From this it follows that $\int_{\Delta(X)}=\int_{\Delta(X_h)}$.

LEMMA 4.4.

$$\int_{\Delta(X_h)}=\frac{P^{-q(q-h)s}}{\prod_{i=0}^{h-1}(P^q-P^i)\prod_{i=h}^{q-1}(1-P^{i-qs})}.$$

PROOF. Let

$$E=\overbrace{\left(\begin{array}{ccc|c} 1+pR & & & \\ & pR & & R \\ & & 1+pR & \\ \hline & pR & & GL_{q-h}(R) \end{array}\right)}^{h} \quad (R=\hat{R}d)$$

$$=\left\{(x_{ij})\in GL_q(\hat{R}_d)\;\middle|\; \begin{array}{l} x_{ii}\equiv 1 \bmod p\hat{R}_d \quad \text{for} \quad 1\leqq i\leqq h \\ x_{ij}\in p\hat{R}_d \quad \text{for } 1\leqq j\leqq h \text{ and } i\neq j \\ (x_{ij})_{h+1\leqq i,j\leqq q}\in GL_{q-h}(\hat{R}_d) \end{array}\right\}.$$

Then E acts on $\Delta(X_h)\cap GL_q(\hat{K}_d)$ by left multiplication. As a full set of representatives of $E\backslash\Delta(X_h)\cap GL_q(\hat{K}_d)$, we can choose the set of matrices $(x_{ij})\in GL_q(\hat{K}_d)$ such that

i) for $1\leqq j\leqq h$, $x_{jj}=1$

ii) for $h+1\leqq j\leqq q$, $x_{jj}=p^{m_j}$, where $m_j\geqq 1$

iii) for $1\leqq j\leqq h$ and $i\neq j$, and for $h+1\leqq j\leqq q$ and $i>j$, $x_{ij}=0$

iv) for $h+1\leqq j\leqq q$ and $i<j$, x_{ij} ranges over all representatives of $pR_d/p^{m_j}R_d$, where m_j, $h+1\leqq j\leqq q$, are as in ii). If m_j, $h+1\leqq j\leqq q$, are given, there are $\prod_{j=h+1}^{q} P^{(m_j-1)(j-1)}$ matrices of the form

$$\left(\begin{array}{ccc|ccc} 1 & & 0 & & & \\ & \ddots & & & * & \\ & & 1 & & & \\ \hline & & & p^{m_{h+1}} & & \\ & 0 & & & \ddots & \\ & & & 0 & & p^{m_q} \end{array}\right)$$

among the above $\{(x_{ij})\}$. Thus we have

$$\int_{\Delta(X_h)}=\mu(E)\sum_{\substack{m_i\geqq 1\\ h+1\leqq i\leqq q}}\prod_{i=h+1}^{q}\left[\left(\frac{P^{m_i}}{P}\right)^{i-1}P^{-qm_is}\right]$$

$$=\frac{P^{-q(q-h)s}}{\prod_{i=0}^{h-1}(P^q-P^i)\prod_{i=h}^{q-1}(1-P^{i-qs})}.$$

PROPOSITION 4.5.

$$\zeta(\boldsymbol{Z}_p[\xi_d]\circ G_p\,;\,s)=\sum_{r=0}^{q}\sum_{h=0}^{r}\left[{}_qC_r\prod_{i=h}^{r-1}\frac{(P^q-P^i)^2}{P^r-P^i}\times\left(\prod_{i=0}^{h-1}\frac{P^q-P^i}{P^h-P^i}\right)\times\frac{P^{-q(q+r-2h)s}}{\prod_{i=h}^{q-1}(1-P^{i-qs})^2}\right].$$

PROOF. Let $r \geqq h$ be integers. Then there are $n_{r,h}=\prod_{i=0}^{h-1}\frac{P^r-P^i}{P^h-P^i}$ distinct F-subspaces of dimension h contained in an F-space of dimension r, and there are $m_h=\prod_{i=0}^{h-1}(P^q-P^i)$ ways of permutations of q vectors in an F-space V of dimension h which span V. Then, in $\mathfrak{X}_r$, there are $n_{r,h}m_h$ matrices with standard form X_h for each $0 \leqq h \leqq r$. Let $L=L_{(\delta_1,\dots,\delta_q)}$ and let $r=\sum_{i=1}^{q}\delta_i$. Then, by force of (1.2), (4.2) and (4.3), we have

$$Z(A,L;s)=\prod_{i=0}^{r-1}\frac{(P^q-P^i)^2}{P^r-P^i}P^{q(q-r)s}\left[\sum_{h=0}^{r}\left\{n_{r,h}m_h\left(\int_{A(X_h)}\right)^2\right\}\right]$$

$$=\sum_{h=0}^{r}\left[\prod_{i=h}^{r-1}\frac{(P^q-P^i)^2}{P^h-P^i}\times\left(\prod_{i=0}^{h-1}\frac{P^p-P^i}{P^h-P^i}\right)\times\frac{P^{-q(q+r-2h)s}}{\prod_{i=1}^{q-1}(1-P^{i-qs})^2}\right], \quad \text{by (4.4).}$$

Thus we have

$$\zeta(A;s)=\sum_{r=0}^{q}{}_qC_r Z(A, L_{(\underbrace{1,\dots,1}_{r},0\dots0)};s)$$

$$=\sum_{r=0}^{q}\sum_{h=0}^{r}\left[{}_qC_r\prod_{i=h}^{r-1}\frac{(P^q-P^i)^2}{P^r-P^i}\times\left(\prod_{i=0}^{h-1}\frac{P^q-P^i}{P^h-P^i}\right)\times\frac{P^{-q(q+r-2h)s}}{\prod_{i=h}^{q-1}(1-P^{i-qs})^2}\right].$$

Let us recall the polynomial $G_n(X)$ defined in (3.4). By the proof of (3.4), we may view $G_n(X)=\sum_{\sigma\in S_n}p^{c_\sigma}X^{e_\sigma}$ as a polynomial both in p and X. From this point of view, we will write $G_n(p,X)$ instead of $G_n(X)$. Put $G_0(p,X)=1$. Then, combining (4.1), (3.7) and (4.5), we have

THEOREM 4.6. *Let q be a prime and let n be a square-free integer coprime to q. Let $C_n\cdot C_q$ be the semidirect product of C_n by C_q in which C_q acts faithfully on the subgroup C_p of C_n for every $p|n$. Then*

$$\zeta(\boldsymbol{Z}(C_n\cdot C_q);s)=\zeta_{\boldsymbol{Z}}(s)\zeta_{\boldsymbol{Z}[\epsilon_q]}(s)\left(\prod_{\substack{d|n\\ d\neq 1}}\prod_{i=0}^{q-1}\zeta_{R_d}(qs-i)\right)(1-q^{-s}+q^{1-2s})$$

$$\times\prod_{p|n}\left(F_{p,1}(s)\prod_{\substack{d|n/p\\ d\neq 1}}(F_{p,d}(s))^{g_d}\right),$$

$$F_{p,1}(s)=\sum_{k=0}^{q}\left[{}_qC_k(1+(p-1)p^{-s})^{q-k}G_{q-k}(p,p^{k-qs})\prod_{i=0}^{k-1}((1-p^{-s})(1-p^{i-qs}))\right],$$

and for $d\neq 1$,

$$F_{p,d}(s)=\sum_{r=0}^{q}\sum_{h=0}^{r}\left[{}_qC_r\prod_{i=h}^{r-1}\frac{(p_d{}^q-p_d{}^i)^2}{p_d{}^r-p_d{}^i}\prod_{i=0}^{h-1}\left(\frac{p_d{}^q-p_d{}^i}{p_d{}^h-p_d{}^i}(1-p_d{}^{i-qs})^2\right)\times p_d{}^{-q(q+r-2h)s}\right],$$

where for each $p|n$ *and* $1 \neq d|n/p$, g_d *is the number of distinct prime ideals over* (p) *in* R_d *and* $p_d = p^{\varphi(d)/qg_d}$.

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