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FIBER SHAPE CATEGORIES

By

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0. Introduction.

For any metric compactum B, we define categories M_B, R_B and FR_B whose objects are all maps af compacta to B, respectively. The purpose of this paper is to study the categories and shape fibrations. In particular, we show the following.

(1) There is a category isomorphism $S_B: M_B \to R_B$ such that $S_B(p: E \to B) = p: E \to B$ for each object $p: E \to B$ of M_B.

(2) Let $p: E \to B$ and $p': E' \to B$ be maps between compacta. Then the following are equivalent.

(i) p is isomorphic to p' in M_B.

(ii) p is isomorphic to p' in R_B.

(iii) p is isomorphic to p' in FR_B.

(3) Let $p: E \to B$ and $p': E' \to B$ be objects of FR_B and let $f: p \to p'$ be a morphism in FR_B. If B has a finite closed cover $\{B_i\}_{i=1,2,\cdots n}$ such that for each $i=1,2,\cdots n$ the restriction $f|p^{-1}(B_i): p|p^{-1}(B_i) \to p'|p'^{-1}(B_i)$ is an isomorphism in FR_{B_i}, then $f: p \to p'$ is an isomorphism in FR_B, where $p|p^{-1}(B_i): p^{-1}(B_i) \to B_i$ denotes the restriction of p to $p^{-1}(B_i)$.

(4) Let $p: E \to B$ and $p': E' \to B$ be shape fibrations between compacta. Then a morphism $f: p \to p'$ of FR_B is an isomorphism in FR_B if and only if f induces a strong shape equivalence $T(f): E \to E'$.

(5) Let $p: E \to B$ and $p': E' \to B$ be shape fibrations between compacta. Suppose that B is a connected ANR or B is a continuum with a finite closed cover consisting of FAR's. Then a morphism $f: p \to p'$ of FR_B is an isomorphism in FR_B if and only if for some $b_0 \in B$, the restriction $T(f|p^{-1}(b_0)): p^{-1}(b_0) \to p'^{-1}(b_0)$ of $T(f)$ to $p^{-1}(b_0)$ is a strong shape equivalence.

Throughout this paper, all spaces are metrizable and all maps are continuous. By an ANR (resp. AR), we denote an ANR (resp. AR) for the class of metrizable spaces. We mean by N the set of positive integers, by I the unit interval $[0,1]$ and by Q the Hilbert cube. Let f and g be maps from a space X into the compactum (Y, d). The *sup-metric* d is given by

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$$d(f, g)=\sup \{d(f(x), g(x)) | x \in X\}.$$

Let E, E' and B be compacta contained in AR's X, X' and Y, respectively. Suppose that $\tilde{p}: X \to Y$ and $\tilde{p}': X' \to Y$ are extensions of maps $p: E \to B$ and $p': E' \to B$, respectively. A fundamental sequence (see [1]) $\underline{f}=\{f_n, E, E'\}_{X,X'}$, is a *fiber fundamental sequence over* B [7] if for any $\varepsilon>0$ and any neighborhood U' of E' in X' there is a neighborhood U of E in X and a positive integer n_0 such that for each $n \geqq n_0$ there is a homotopy $F: U \times I \to U'$ such that $F(x, 0)=f_{n_0}(x)$, $F(x, 1)=f_n(x)$ for $x \in U$ and $d(\tilde{p}' F(x, t), \tilde{p}(x))<\varepsilon$ for $x \in U$, $t \in I$. A fiber fundamental sequence over B $\underline{f}=\{f_n, E, E'\}_{X,X'}$ is *fiber homotopic* to a fiber fundamental sequence over B $\underline{g}=\{g_n, E, E'\}_{X,X'}$ ($\underline{f} \underset{B}{\simeq} \underline{g}$) if for any $\varepsilon>0$ and any neighborhood U' of E' in X' there is a neighborhood U of E in X and a positive integer n_0 such that for each $n \geqq n_0$ there is a homotopy $K: U \times I \to U'$ such that $K(x, 0)=f_n(x)$, $K(x, 1)=g_n(x)$ for $x \in U$ and $d(\tilde{p}' K(x, t), \tilde{p}(x))<\varepsilon$ for $x \in U$, $t \in I$. A map $p: E \to B$ is *fiber shape equivalent* to a map $p': E' \to B$ if there are fiber fundamental sequences over B $\underline{f}=\{f_n, E, E'\}_{X,X'}$ and $\underline{g}=\{g_n, E', E\}_{X',X}$ such that $\underline{g}\underline{f} \underset{B}{\simeq} \underline{1}_E$ and $\underline{f}\underline{g} \underset{B}{\simeq} \underline{1}_{E'}$, where $\underline{1}_E$ denotes a fiber fundamental sequence over B induced by the identity $1_E: E \to E$. Such $\underline{f}$ is called a *fiber shape equivalence*. A map $p: E \to B$ is *shape shrinkable* [7] if p induces a fiber shape equivalence from p to the identity $1_B: B \to B$. Note that $p: E \to B$ is shape shrinkable iff p is a hereditary shape equivalence (see [7, Corollary 3.5]). We denote by M_B the category whose objects are all maps of compacta to B and whose morphisms are fiber homotopy classes of fiber fundamental sequences over B.

1. $F(p, p')$-maps, $F(p, p')$-homotopies and $WF(p, p')$-homotopy classes.

For a subset E of a space X, E is *unstable* in X [13] if there is a homotopy $H: X \times I \to X$ such that $H(x, 0)=x$, $H(x, t) \in X-E$ for $x \in X$, $0<t \leqq 1$. Let $p: E \to B$ and $p': E' \to B$ be maps between compacta and let E and E' be subsets of compacta X and X', respectively. A map $f: X-E \to X'-E'$ is an *$F(p, p')$-map* if for each $b \in B$ and each neighborhood W' of $p'^{-1}(b)$ in X' there is a neighborhood W of $p^{-1}(b)$ in X such that $f(W-E) \subset W'-E'$. $F(p, p')$-maps $f, g: X-E \to X'-E'$ are *$F(p, p')$-homotopic* ($f \underset{F(p,p')}{\simeq} g$) if there is a homotopy $H: (X-E) \times I \to X'-E'$ such that $H(x, 0)=f(x)$, $H(x, 1)=g(x)$ for $x \in X-E$ and for each $b \in B$ and each neighborhood W' of $p'^{-1}(b)$ in X' there is a neighborhood W of $p^{-1}(b)$ in X such that $H((W-E) \times I) \subset W'-E'$. Such a homotopy $H: (X-E) \times I \to X'-E'$ is called an *$F(p, p')$-homotopy*. Consider $E \times I$ as a closed subset of $X \times I$ and a map $p\pi: E \times I \to B$, where $\pi: E \times I \to E$ is the projection. Then a homotopy $H: (X-E) \times I \to X'-E'$ is an $F(p, p')$-homotopy iff H is an $F(p\pi, p')$-map. $X-E$ and $X'-E'$ are said to be of the *same* $F(p, p')$-

homotopy type $(X-E \underset{F(p,p')}{\simeq} X'-E')$ if there is an $F(p,p')$-map $f: X-E \to X'-E'$ and an $F(p',p)$-map $g: X'-E' \to X-E$ such that $gf \underset{F(p,p)}{\simeq} 1_{(X-E)}$ and $fg \underset{F(p',p')}{\simeq} 1_{(X'-E')}$, where $1_{(X-E)}$ denotes the identity of $X-E$. Such an $F(p,p')$-map $f: X-E \to X'-E'$ is called an *$F(p,p')$-homotopy equivalence.* $F(p,p')$-maps $f, g: X-E \to X'-E'$ are *$WF(p,p')$-homotopic* $(f \underset{WF(p,p')}{\simeq} g)$ if for any finite open cover $\{W_i'\}_{i=1,2,\cdots n}$ of E' in X' such that for each $b \in B$, $p'^{-1}(b) \subset W_i'$ for some i, there is a finite open cover $\{W_j\}_{j=1,2,\cdots m}$ of E in X such that for each $b \in B$, $p^{-1}(b) \subset W_j$ for some j and a homotopy $H: (X-E) \times I \to X'-E'$ such that $H(x,0)=f(x)$, $H(x,1)=g(x)$ for $x \in X-E$ and for each $i=1,2,\cdots m$, $H((W_j-E) \times I) \subset W_i'$ for some $i=1,2,\cdots n$. $X-E$ and $X'-E'$ are said to be of the *same $WF(p,p')$-homotopy type* $(X-F \underset{WF(p,p')}{\simeq} X'-E')$ if there is an $F(p,p')$-map $f: X-E \to X'-E'$ and an $F(p',p)$-map $g: X'-E' \to X-E$ such that $gf \underset{WF(p,p)}{\simeq} 1_{(X-E)}$ and $fg \underset{WF(p',p')}{\simeq} 1_{(X'-E')}$. Such an $F(p,p')$-map $f: X-E \to X'-E'$ is called a *$WF(p,p')$-homotopy equivalence.*

2. Categories R_B and FR_B.

In this section, we define categories R_B and FR_B. We show that there is a category isomorphism $S_B: M_B \to R_B$ and some applications are given.

LEMMA 2.1 ([10, Lemma 3]). *Let X and X' be compact AR's containing E as an unstable closed subset. Then there is a map $\varphi(X,X'): X \to X'$ such that*

$$(*) \quad \varphi(X,X')|E = 1_E \quad \text{and} \quad \varphi(X,X')(X-E) \subset X'-E.$$

If $\varphi_1, \varphi_2: X \to X'$ satisfy the condition $(*)$, then there is a homotopy $H: X \times I \to X'$ such that $H(x,0)=\varphi_1(x)$, $H(x,1)=\varphi_2(x)$ for $x \in X$ and $H(x,t)=x$ for $x \in E$, $t \in I$ and $H((X-E) \times I) \subset X'-E$. In particular, for any map $p: E \to B$ $\varphi(X,X')|X-E: X-E \to X'-E$ is an $F(p,p)$-map and $H|(X-E) \times I: (X-E) \times I \to X'-E$ is an $F(p,p)$-homotopy.

For any compactum B, we shall define categories R_B and FR_B as follows. For a compactum E, we denote by $m(E)$ the set of compact AR's containing E as an unstable subset. Let $p: E \to B$ and $p': E' \to B$ be maps between compacta and let $X_1, X_2 \in m(E)$ and $X_1', X_2' \in m(E')$. An $F(p,p')$-map $f: X_1-E \to X_1'-E'$ is *$WF(p,p')$-equivalent* to an $F(p,p')$-map $g: X_2-E \to X_2'-E'$ if $\varphi(X_1', X_2')|X_1'-E' \circ f \underset{WF(p,p')}{\simeq} g \circ \varphi(X_1, X_2)|X_1-E$, where $\varphi(X_1, X_2)$ and $\varphi(X_1', X_2')$ are maps satisfying the condition $(*)$ of Lemma 2.1. An $F(p,p')$-map $f: X_1-E \to X_1'-E'$ is *$F(p,p')$-equivalent* to an $F(p,p')$-map $g: X_2-E \to X_2'-E'$ if $\varphi(X_1', X_2')|X_1'-E' \circ f \underset{F(p,p')}{\simeq} g \circ \varphi(X_1, X_2)|X_1-E$. Objects of R_B are maps of compacta to B. For objects $p: E \to B$ and $p': E' \to B$ of R_B, morphisms from p to p' in R_B are $WF(p,p')$-equivalence classes of collections of $F(p,p')$-maps $f: X-E \to X'-E'$, $X \in m(E)$, $X' \in m(E')$. Obviously, R_B forms a category. Similarly, Objects of FR_B are maps of compacta to B and for objects $p:E\to B$ and $p':E'\to B$ of FR_B, morphisms from p to p' in FR_B are $F(p,p')$-equivalence classes of collections of $F(p,p')$-maps $f:X-E\to X'-E'$, $X\in m(E)$, $X'\in m(E')$. Then FR_B forms a category.

The proof of the following theorem is analogous to one of [10, Theorem 1], but more informations will be used.

THEOREM 2.2. *There is a category isomorphism $S_B:M_B\to R_B$ such that $S_B(p:E\to B)=p:E\to B$ for each object $p:E\to B$ of M_B.*

PROOF. Let $p:E\to B$, $p':E'\to B$ be objects of M_B and consider E, E' and B as closed subsets of the Hilbert cube Q. Suppose that $\tilde{p}:Q\to Q$ and $\tilde{p}':Q\to Q$ are extensions of p and p', respectively. Choose a sequence $\varepsilon_1>\varepsilon_2>\varepsilon_3>,\cdots,$ of positive numbers such that $\lim_{i\to\infty}\varepsilon_i=0$ and decreasing sequences $\{U_n\}, \{V_n\}$ of compact ANR-neighborhoods of E, E' in Q, respectively such that $\bigcap_{n=1}^{\infty}U_n=E$, $\bigcap_{n=1}^{\infty}V_n=E'$. Let $\underline{U}=\{U_k, i_k^{k+1}, k\in N\cup\{0\}\}$ be an inverse sequence such that U_0 is a one point set, $i_0^1:U_1\to U_0$ is the constant map and $i_k^{k+1}:U_{k+1}\to U_k$ $(k\geqq 1)$ is the inclusion. Similarly, we obtain an inverse sequence $\underline{V}=\{V_k, j_k^{k+1}, k\in N\cup\{0\}\}$. Consider the infinite telescope (e.g. see [10, p. 74]) $T(\underline{U})=\bigcup_{k=0}^{\infty}M_k(\underline{U})$, where $M_k(\underline{U})$ denotes the mapping cylinder obtained by $i_k^{k+1}:U_{k+1}\to U_k$, i.e., $M_k(\underline{U})$ is obtained by identifying points $(x,1)\in U_{k+1}\times\{1\}$ and $i_k^{k+1}(x)=x\in U_k$ for $x\in U_{k+1}$ in a topological sum $U_{k+1}\times I\cup U_k$, and $T(\underline{U})$ is obtained by identifying each point of $U_k\times\{0\}$ in $M_{k-1}(\underline{U})$ and the corresponding point of U_k in $M_k(\underline{U})$. Let $N(\underline{U})=T(\underline{U})\cup E$ be an AR having the same topology as in [10, p. 74]. Note that $T(\underline{U})\cong T'(\underline{U})=C(U_1)\cup\bigcup_{j=2}^{\infty}U_j\times[1/j+1, 1/j]\subset Q\times(0,1]$ and $(N(\underline{U}), E)\cong(T'(\underline{U})\cup E\times\{0\}, E\times\{0\})\subset Q\times[0,1]$, where $C(U_1)$ is a cone over $U_1\times\{1/2\}$ with a vertex $(v,1)$, $v\in Q$ in $Q\times[1/2,1]$. Similarly, we obtain $T(\underline{V})$ and $N(\underline{V})$. Suppose that $f=\{f_n, E, E'\}_{Q,Q}$ is a fiber fundamental sequence over B. Inductively, we can find a sequence $0=n_0<n_1<n_2<n_3<,\cdots,$ of integers such that for $n\geqq n_i$, there is a homotopy $H_{n_i,n}:U_{n_i}\times I\to V_i$ such that $H_{n_i,n}(x,0)=f_{n_i}(x)$, $H_{n_i,n}(x,1)=f_n(x)$ for $x\in U_{n_i}$ and $d(\tilde{p}'H_{n_i,n}(x,t), \tilde{p}(x))<\varepsilon_i$ for $x\in U_{n_i}$, $t\in I$. Define a map $s(f):T(\underline{U})\to T(\underline{V})$ as follows. For each $k=0,1,2,\cdots$, consider the subset $\bigcup_{i=n_k}^{n_{k+1}-1}M_i(\underline{U})$ and $M_k(\underline{V})$ of $T(\underline{U})$ and $T(\underline{V})$, respectively. Define a map $s(f)_k:\bigcup_{i=n_k}^{n_{k+1}-1}M_i(\underline{U})\to M_k(\underline{V})$ by

$$s(f)_k(x,t)=\begin{cases} f_{n_k}i_{n_k}^{j+1}(x), & \text{for } (x,t)\in M_j(\underline{U}),\ j=n_k, n_k+1,\cdots,n_{k+1}-2, \\ (f_{n_{k+1}}(x), 2t), & \text{for } 0\leqq t\leqq 1/2,\ (x,t)\in M_{n_{k+1}-1}(\underline{U}), \\ H_{n_k,n_{k+1}}(x, 2-2t), & \text{for } 1/2\leqq t\leqq 1,\ (x,t)\in M_{n_{k+1}-1}(\underline{U}), \end{cases}$$

where $f_0: U_0 \to V_0$ is the constant map. Define $s(f)$ by $s(f)|\bigcup_{i=n_k}^{n_{k+1}-1} M_i(\underline{U}) = s(f)_k$ for each $k=0,1,2,\cdots$. By the construction of $s(f)$, it is an $F(p,p')$-map. Note that $N(\underline{U}) \in m(E)$, $N(\underline{V}) \in m(E')$. To complete the proof, we need the following lemma. By the lemma, we can define $S_B([f])$ as the $WF(p,p')$-equivalence class containing $s(f)$, where $[f]$ denotes the fiber homotopy class containing f and we can conclude that S_B is a category isomorphism from M_B to R_B. The proof of the lemma is similar to one of [10, Lemma 5], hence we omit it.

LEMMA 2.3. *Let $[p: E \to B, p': E' \to B]$ be the set of fiber homotopy classes of fiber fundamental sequences from p to p' and $[T(\underline{U}), T(\underline{V})]_{WF(p,p')}$ the set of $WF(p,p')$-homotopy classes of $F(p,p')$-maps from $T(\underline{U})$ to $T(\underline{V})$. Then s induces a $1:1$ correspondence from $[p: E \to B, p': E \to B]$ onto $[T(\underline{U}), T(\underline{V})]_{WF(p,p')}$.*

THEOREM 2.4. *Let $p: E \to B$ and $p': E' \to B$ be maps between compacta and let $X \in m(E)$, $X' \in m(E')$. If an $F(p,p')$-map $f: X-E \to X'-E'$ is a $WF(p,p')$-homotopy equivalence, then there is an $F(p,p')$-map $g: X-E \to X'-E'$ such that $f \underset{WF(p,p')}{\simeq} g$ and g is an $F(p,p')$-homotopy equivalence. In particular, the following are equivalent.*

(1) *p is isomorphic to p' in M_B.*
(2) *If $X \in m(E)$ and $X' \in m(E')$, then $X-E \underset{WF(p,p')}{\simeq} X'-E'$.*
(3) *There are $X \in m(E)$ and $X' \in m(E')$ such that $X-E \underset{WF(p,p')}{\simeq} X'-E'$.*
(4) *p is isomorphic to p' in R_B.*
(5) *If $X \in m(E)$ and $X' \in m(E')$, then $X-E \underset{F(p,p')}{\simeq} X'-E'$.*
(6) *There are $X \in m(E)$ and $X' \in m(E')$ such that $X-E \underset{F(p,p')}{\simeq} X'-E'$.*
(7) *p is isomorphic to p' in FR_B.*

PROOF. Suppose that an $F(p,p')$-map $f: X-E \to X'-E'$ is a $WF(p,p')$-homotopy equivalence. Embed E and E' into the Hilbert cube Q as Z-sets, respectively. By Lemma 2.1, Theorem 2.2 and the proof of [7, Theorem 3.1], there is a homeomorphism $h: Q-E \to Q-E'$ which is an $F(p,p')$-map and $\varphi(Q,X')|Q-E \circ h \circ \varphi(X,Q)|X-E \underset{WF(p,p')}{\simeq} f$, where $\varphi(Q,X')$ and $\varphi(X,Q)$ are maps satisfying the condition $(*)$ of Lemma 2.1. Set $g = \varphi(Q,X')|Q-E' \circ h \circ \varphi(X,Q)|X-E$. Clearly g satisfies the condition of Theorem 2.4. The rest of the proof follows from this result, Lemma 2.1 and Theorem 2.2.

COROLLARY 2.5. *A* map *$p: E \to B$ between compacta is shape shrinkable if and only if for any $X \in m(E)$, $Y \in m(B)$ and for any extension $\tilde{p}: X \to Y$ of p such that $p(X-E) \subset Y-B$, $\tilde{p}|X-E: X-E \to Y-B$ is an $F(p, 1_B)$-homotopy equivalence, where 1_B denotes the identity of B.*

REMARK 2.6. Note that if B is a one point set, the categories M_B and FR_B are the same as shape category (see [1]) and strong (or fine) shape category (see [5], [6] and [10]), respectively.

3. The Category FR_B.

Let $p: E\to B$ and $p': E'\to B$ be objects of R_B (resp. FR_B) and let $f: p\to p'$ be a morphism in R_B (resp. FR_B). For any closed subset C of B, we shall define a morphism $f|p^{-1}(C): p|p^{-1}(C)\to p'|p'^{-1}(C)$ in R_C (resp. FR_C), where $p|p^{-1}(C)$ is the restriction map of p, i.e., $p|p^{-1}(C): p^{-1}(C)\to C$. We need the following lemma. We omit the proof.

LEMMA 3.1. *Let A be a compactum and B be a closed subset of A. Suppose $X\in m(A)$ and $Y\in m(B)$. Then there are maps $r(X, Y): X\to Y$ and $i(Y, X): Y\to X$ such that*

$$(**)\quad r(X, Y)|B=1_B \text{ and } r(X, Y)|(X-A)\subset Y-B,$$

$$(***)\quad i(Y, X)|B=1_B \text{ and } i(Y, X)|(Y-B)\subset X-A.$$

If $r, r': X\to Y$ satisfy the condition (**), then there is a homotopy $H: X\times I\to Y$ such that

$$H(x, 0)=r(x) \text{ and } H(x, I)=r'(x) \text{ for } x\in X,$$

$$H(x, t)=x \text{ for } x\in B \text{ and } t\in I,\quad H((X-A)\times I)\subset Y-B.$$

Similarly, if $i, i': Y\to X$ satisfy the condition (***), then there is a homotopy $K: Y\times I\to X$ such that

$$K(y, 0)=i(y),\quad K(y, 1)=i'(y) \text{ for } y\in Y,$$

$$K(y, t)=y \text{ for } y\in B \text{ and } t\in I,\quad K((Y-B)\times I)\subset X-A.$$

Suppose that $X\in m(E)$, $X'\in m(E')$, $Y\in m(p^{-1}(C))$ and $Y'\in m(p'^{-1}(C))$ and an $F(p, p')$-map $f: X-E\to X'-E'$ determines the morphism $f: p\to p'$. Then the composition $r(X', Y')|X'-E'\circ f\circ i(Y, X)|Y-p^{-1}(C): Y-p^{-1}(C)\to Y'-p'^{-1}(C)$ is an $F(p|p^{-1}(C), p'|p'^{-1}(C))$-map, where $r(X', Y')$ and $i(Y, X)$ are maps satisfying the condition (**) and (***) of Lemma 3.1, respectively. We define the restriction $f|p^{-1}(C): p|p^{-1}(C)\to p'|p'^{-1}(C)$ by the $WF(p|p^{-1}(C), p'|p'^{-1}(C))$-equivalence class (resp. the $F(p|p^{-1}(C), p'|p'^{-1}(C))$-equivalence class) containing $r(X', Y')|X'-E'\circ f\circ i(Y, X)|Y-p^{-1}(C)$.

PROPOSITION 3.2. *Let $p: E\to B$ and $p': E'\to B$ be objects of R_B (resp. FR_B). If a morphism $f: p\to p'$ in R_B (resp. FR_B) is an isomorphism, then for any closed subset C of B, $f|p^{-1}(C)$ is an isomorphism in R_C (resp. FR_C).*

THEOREM 3.3. *Let $p:E\to B$ and $p':E'\to B$ be objects of FR_B and let $f:p\to p'$ be a morphism in FR_B. If B has a finite closed cover $\{B_i\}_{i=1,2,\cdots n}$ such that for each $i=1,2,\cdots n$ the restriction $f|p^{-1}(B_i):p|p^{-1}(B_i)\to p'|p'^{-1}(B_i)$ is an isomorphism in FR_{B_i}, then $f:p\to p'$ is an isomorphism in FR_B.*

To prove Theorem 3.3, we need the following lemma.

LEMMA 3.4. *Let E and E' be closed subsets of compacta X and X', respectively and let $p:E\to B$ and $p':E'\to B$ be maps between compacta. Suppose that A is a closed subset of X, $X'\in m(E')$ and $G:(A-E)\times I\to X'-E'$ is an $F(p|A\cap E,p')$-homotopy, where $p|A\cap E:A\cap E\to B$. If there is an extension $\tilde{g}:(X-E)\times\{0\}\to X'-E'$ of $G|(A-E)\times\{0\}$ which is an $F(p,p')$-map, then there is an extension $\tilde{G}:(X-E)\times I\to X'-E'$ of G and $\tilde{g}$ such that $\tilde{G}$ is an $F(p,p')$-homotopy.*

PROOF. Since $X'-E'$ is an ANR, there is a neighborhood U of $A-E$ in $X-E$ and an extension $G':(X-E)\times\{0\}\cup U\times I\to X'-E'$ of G such that $G'|(X-E)\times\{0\}=\tilde{g}$. For each $x\in A-E$, choose a neighborhood U_x of x in U such that

(1) $U_x\subset\{y\in U|d(y,x)<d(x,E)/2\}$ and

$U_x\subset\{y\in U|d(G(x,t),G'(y,t))<d(x,E)$ for each $t\in I\}$.

Set $V=\bigcup_{x\in A-E}U_x$. Then V is a neighborhood of $A-E$ in $X-E$. Choose a map $r:X-E\to I$ such that $r(x)=0$ for $x\in(X-E)-V$ and $r(x)=1$ for $x\in A-E$. Define a homotopy $\tilde{G}:(X-E)\times I\to X'-E'$ by $\tilde{G}(x,t)=G'(x,r(x)t)$ for $x\in X-E$ and $t\in I$. To complete the proof, we must show that $\tilde{G}$ is an $F(p,p')$-homotopy. Suppose that $b\in B$ and W' is a neighborhood of $p'^{-1}(b)$ in X'. Choose a neighborhood W'' of $p'^{-1}(b)$ in X' such that $\mathrm{Cl}_{X'}W''\subset W'$. Let $\varepsilon_1=d(\mathrm{Cl}_{X'}W'',X'-W')>0$. Since $\tilde{g}$ is an $F(p,p')$-map and G is an $F(p|A\cap E,p')$-homotopy, there is a neighborhood W_1 of $p^{-1}(b)$ in X such that

(2) $\tilde{g}(W_1-E)\subset W''-E'$ and $G(((A\cap W_1)-E)\times I)\subset W''-E'$.

Let $\varepsilon_2=\mathrm{Min}\{d(X-W_1,p^{-1}(b)),\varepsilon_1\}>0$. Choose a neighborhood $W_2\subset W_1$ of $p^{-1}(b)$ in X such that $d(y,p^{-1}(b))<\varepsilon_2/2$ for all $y\in W_2$. Then we show that $\tilde{G}((W_2-E)\times I)\subset W'-E'$. If $y\in W_2-E-V$, by the construction of $\tilde{G}$ and by (2), $\tilde{G}(y,t)=\tilde{g}(y)\subset W''-E'\subset W'-E'$. Suppose $y\in(W_2-E)\cap V$. Then there is U_x for some $x\in A-E$ such that $U_x\ni y$. By (1) we have

(3) $d(x,y)\leqq d(x,E)/2\leqq d(y,E)\leqq d(y,p^{-1}(b))<\varepsilon_2/2$.

By (3) we have

(4) $d(x,p^{-1}(b))\leqq d(x,y)+d(y,p^{-1}(b))<\varepsilon_2/2+\varepsilon_2/2=\varepsilon_2$.

Therefore $x\in W_1$. By (2), $G(x,t)\in W''-E'$ for $t\in I$. By (1) and (4),

(5) $d(G(x,t),G'(y,t))<d(x,E)\leqq d(x,p^{-1}(b))<\varepsilon_2\leqq\varepsilon_1$.

Hence $G'(y,t)\in W'-E'$. By the construction of $\tilde{G}$, we conclude that $\tilde{G}(y,t)\in W'-E'$ for each $t\in I$. Thus $\tilde{G}$ is an $F(p,p')$-homotopy. This completes the proof.

PROOF OF THEOREM 3.3. It is enough to give the proof of the case $n=2$. The case $n\geqq 3$ is proved by induction. We may assume $B_1\cap B_2=B_0\neq\phi$. If $B_0=\phi$, the proof is trivial.

Embed E and E' into the Hilbert cube Q. For each $i=1,2$, choose a decreasing sequence $\underline{U}^i=\{U_j^i\}_{j=1,2,\dots}$ of compact ANR-neighborhoods of $p^{-1}(B_i)$ in Q such that

(1) $p^{-1}(B_i)=\bigcap_{j=1}^{\infty}U_j^i$ and $U_j^0=U_j^1\cap U_j^2$ is a compact ANR for each $i=1,2$ and $j=1,2,\cdots$.

Set $U_j=U_j^1\cup U_j^2$, $\underline{U}=\{U_j\}_{j=1,2,\dots}$ and $\underline{U}^0=\{U_j^0\}_{j=1,2,\dots}$. Then $\underline{U}$ is a decreasing sequence of compact ANR-neighborhoods of E in Q such that

(2) $N(\underline{U})\supset N(\underline{U}^i)$ for each $i=1,2$ and

(2) $N(\underline{U})\in m(E)$, $N(\underline{U}^i)\in m(p^{-1}(B_i))$ for each $i=1,2$ and $N(\underline{U}^0)=N(\underline{U}^1)\cap N(\underline{U}^2)\in m(p^{-1}(B_0))$ (see the proof of Theorem 2.2).

Similarly we obtain $N(\underline{V}),N(\underline{V}^i)$ for each $i=1,2$ and $N(\underline{V}^0)$ such that

(4) $N(\underline{V})\supset N(\underline{V}^i)$ for each $i=1,2$ and

(5) $N(\underline{V})\in m(E')$, $N(\underline{V}^i)\in m(p'^{-1}(B_i))$ for each $i=1,2$ and $N(\underline{V}^0)=N(\underline{V}^1)\cap N(\underline{V}^2)\in m(p'^{-1}(B_0))$.

By Lemma 2.1, there is an $F(p,p')$-map $f\colon T(\underline{U})=N(\underline{U})-E\to T(\underline{V})=N(\underline{V})-E'$ which is contained in the $F(p,p')$-equivalence class $f\colon p\to p'$. Now by the following lemma (Lemma 3.5), we may assume that

(6) $f(T(\underline{U}^i))\subset T(\underline{V}^i)$ for each $i=1,2$.

By Proposition 3.2, $f|T(\underline{U}^0)\colon T(\underline{U}_0)\to T(\underline{V}^0)$ is an $F(p|p^{-1}(B_0),p'|p'^{-1}(B_0))$-homotopy equivalence. Hence there is an $F(p'|p'^{-1}(B_0),\ p|p^{-1}(B_0))$-map $g_0\colon T(\underline{V}^0)\to T(\underline{U}^0)$ and an $F(p'|p'^{-1}(B_0),p'|p'^{-1}(B_0))$-homotopy $H_0\colon T(\underline{V}^0)\times I\to T(\underline{V}^0)$ such that

(7) $g_0f|T(\underline{U}^0)\underset{F(p|p^{-1}(B_0),\,p|p^{-1}(B_0))}{\simeq}1_{T(\underline{U}^0)}$ and

(8) $H_0(x,0)=fg_0(x)$ for $x\in T(\underline{V}^0)$ and $H_0(x,t)=x$ for $x\in T(\underline{V}^0)$ and $1/2\leqq t\leqq 1$.

By Lemma 3.4 and the same way as in Brown [2, 7.4.1], for each $i=1,2$ there is an $F(p'|p'^{-1}(B_i), p|p^{-1}(B_i))$-maps $g_i: T(\underline{V}^i)\to T(\underline{U}^i)$ and an $F(p'|p'^{-1}(B_i), p'|p'^{-1}(B_i))$-homotopy $H_i: T(\underline{V}^i)\times I\to T(\underline{V}^i)$ such that

(9) $g_i|T(\underline{V}^0)=g_0$,

(10) $H_i(x, 0)=fg_i(x)$ for $x\in T(\underline{V}^i)$, $H_i(x, t)=x$ for $x\in T(\underline{V}^i)$ and $1/2\leqq t\leqq 1$ and

(11) $H_i|T(\underline{V}^0)\times I=H_0$.

By (9) we can define a map $g: T(\underline{V})\to T(\underline{U})$ by

$$(12)\quad g(x)=\begin{cases} g_1(x) & \text{for } x\in T(\underline{V}^1) \\ g_2(x) & \text{for } x\in T(\underline{V}^2). \end{cases}$$

Then g is an $F(p', p)$-map and by (10) and (11) we have

$$(13)\quad fg \underset{F(p',p')}{\simeq} 1_{T(\underline{V})}.$$

Note that $g|T(\underline{V}^i): T(\underline{V}^i)\to T(\underline{U}^i)$ is an $F(p'|p'^{-1}(B_i), p|p^{-1}(B_i))$-homotopy equivalence. By the same argument as above, there is an $F(p, p')$-map $f': T(\underline{U})\to T(\underline{V})$ such that

$$(14)\quad gf' \underset{F(p,p)}{\simeq} 1_{T(\underline{U})}.$$

By (13) and (14),

$$(15)\quad f \underset{F(p,p')}{\simeq} fgf' \underset{F(p,p')}{\simeq} f'.$$

Hence $fg \underset{F(p',p')}{\simeq} 1_{T(\underline{V})}$ and $gf \underset{F(p,p)}{\simeq} 1_{T(\underline{U})}$, which implies that f is an $F(p, p')$-homotopy equivalence. Thus the morphism $f: p\to p'$ is an isomorphism in FR_B. This completes the proof.

LEMMA 3.5. *Let $f: T(\underline{U})\to T(\underline{V})$ be an $F(p, p')$-map. Then there is an $F(p, p')$-map $g: T(\underline{U})\to T(\underline{V})$ such that $g(T(\underline{U}^i))\subset T(\underline{V}^i)$ for each $i=1, 2$ and $g \underset{F(p,p')}{\simeq} f$.*

PROOF. Since $N(\underline{V}^i)$ is an AR for each $i=0, 1, 2$, there is a retraction $r_i': N(\underline{V})\to N(\underline{V}^i)$. Choose a map $\alpha_i: N(\underline{V})\to I$ such that $\alpha_i^{-1}(0)=N(\underline{V}^i)$ for each $i=0, 1, 2$. Since $N(\underline{V}^i)\in m(p'^{-1}(B_i))$ for each $i=0, 1, 2$, there is a homotopy $H_i: N(\underline{V}^i)\times I\to N(\underline{V}^i)$ such that $H_i(x, 0)=x$ for $x\in N(\underline{V}^i)$ and $H_i(x, t)\in T(\underline{V}^i)$ for $x\in N(\underline{V}^i)$ and $0<t\leqq 1$. Define a map $r_i: N(\underline{V})\to N(\underline{V}^i)$ for each $i=0, 1, 2$ by $r_i(x)=H_i(r_i'(x), \alpha_i(x))$ for $x\in N(\underline{V})$. Then $r_i|N(\underline{V}^i)=1_{N(\underline{V}^i)}$ and $r_i(T(\underline{V}))\subset T(\underline{V}^i)$ for each $i=0, 1, 2$. Similarly, for each $i=0, 1, 2$ there is a homotopy $K_i: N(\underline{V})\times I\to N(\underline{V})$ such that $K_i(x, 0)=x$, $K_i(x, 1)=r_i(x)$ for $x\in N(\underline{V})$, $K_i(x, t)=x$ for $x\in N(\underline{V}^i)$ and $t\in I$ and $K_i(T(\underline{V})\times I)\subset T(\underline{V})$. Define

a homotopy $\varphi_0: T(\underline{U}^0)\times I\to T(\underline{V})$ by $\varphi_0(x,t)=K_0(f(x),t)$ for $x\in T(\underline{U}^0)$ and $t\in I$. Then $\varphi_0(x,0)=f(x)$, $\varphi_0(x,1)=r_0 f(x)\in T(\underline{V}^0)$ for $x\in T(\underline{U}^0)$ and φ_0 is an $F(p|p^{-1}(B_0),p')$-homotopy. By Lemma 3.4, there is an $F(p,p')$-map $g': T(\underline{U})\to T(\underline{V})$ such that $g'|T(\underline{U}_0)=r_0 f|T(\underline{U}^0)$ and $g' \underset{F(p,p')}{\simeq} f$. Note $g'(T(\underline{U}^0))\subset T(\underline{V}^0)$. Define a homotopy $\varphi_i: T(\underline{U}^i)\times I\to T(\underline{V})$ for each $i=1,2$ by $\varphi_i(x,t)=K_i(g'(x),t)$ for $x\in T(\underline{U}^i)$ and $t\in I$. Then $\varphi_i(x,0)=g'(x)$, $\varphi_i(x,1)=r_i g'(x)\in T(\underline{V}^i)$ for $x\in T(\underline{U}^i)$ and $\varphi_i(x,t)=g'(x)$ for $x\in T(\underline{U}^0)$ and $t\in I$. Also φ_i is an $F(p|p^{-1}(B_i),p')$-homotopy. Define a homotopy $\varphi: T(\underline{U})\times I\to T(\underline{V})$ by

$$\varphi(x,t)=\begin{cases}\varphi_1(x,t) & \text{for } x\in T(\underline{U}^1) \text{ and } t\in I\\ \varphi_2(x,t) & \text{for } x\in T(\underline{U}^2) \text{ and } t\in I.\end{cases}$$

Then $\varphi(x,0)=g'(x)$ for $x\in T(\underline{U})$ and $\varphi(T(\underline{U}^i))\subset T(\underline{V}^i)$ for each $i=1,2$ and φ is an $F(p,p')$-homotopy. Define a map $g: T(\underline{U})\to T(\underline{V})$ by $g(x)=\varphi(x,1)$ for $x\in T(\underline{U})$. Then g satisfies the condition of Lemma 3.5.

By Corollary 2.5 and Theorem 3.3, we have the following.

COROLLARY 3.6. *Let $p: E\to B$ be a map between compacta. If there is a finite closed cover $\{B_i\}_{i=1,2,\cdots n}$ of B such that $p|p^{-1}(B_i): p^{-1}(B_i)\to B_i$ is shape shrinkable for each $i=1,2,\cdots n$, then $p: E\to B$ is shape shrinkable.*

COROLLARY 3.7. *Let $p: E\to B$ and $p': E'\to B$ be maps between compacta and let $f: E\to E'$ be a fiber map over B (i.e., $p'f=p$). If there is a finite closed cover $\{B_i\}_{i=1,2,\cdots n}$ of B such that for each $i=1,2,\cdots n$, $f|p^{-1}(B_i): p^{-1}(B_i)\to p'^{-1}(B_i)$ is a fiber homotopy equivalence over B_i, then f induces an isomorphism $f: p\to p'$ in FR_B. In particular, f is a fiber shape equivalence over B.*

REMARK 3.8. In the statement of Corollary 3.7, we cannot conclude that f is a fiber homotopy equivalence over B. Define a map $p:[0,3]\to[0,2]$ by $p|[0,1]=1_{[0,1]}$, $p([1,2])=1$ and $p(t)=t-1$ for $t\in[2,3]$. Let $B_1=[0,1]$ and $B_2=[1,2]$. It is clear that p is a fiber map from p to the identity $1_{[0,2]}$ and for each $i=1,2$, $p|p^{-1}(B_i): p^{-1}(B_i)\to B_i$ is a fiber homotopy equivalence over B_i. But there is no fiber map $g:[0,2]\to[0,3]$ over $[0,2]$.

4. Shape fibrations and Strong shape equivalences.

We denote by s-Sh the strong (or fine) shape category (see [5], [6] and [10]). Let $p: E\to B$ and $p': E'\to B$ be objects of FR_B and $f: p\to p'$ be a morphism in FR_B. Choose $X\in m(E)$, $X'\in m(E')$ and an $F(p,p')$-maps $f: X-E\to X'-E'$ contained in the $F(p,p')$-equivalence class $f: p\to p'$. Since every $F(p,p')$-map is a proper map, the morphism $T(f): E\to E'$ of s-Sh induced by the proper map $f: X-E\to X'-E'$ is independent of the choices of $X \in m(E)$, $X' \in m(E')$ and $f: X-E \to X'-E'$. Clearly there is a functor $T: FR_B \to s\text{-}Sh$ such that $T(p: E \to B) = E$ for each object $p: E \to B$ of FR_B.

In this section, we show the following theorem which is a more general result than [7, Theorem 2.3].

THEOREM 4.1. *Let $p: E \to B$ and $p': E' \to B$ be shape fibrations between compacta* (see [11]). *Then a morphism $f: p \to p'$ of FR_B is an isomorphism in FR_B if and only if $T(f): E \to E'$ is an isomorphism in s-Sh.*

First, we need the following.

LEMMA 4.2. *Let $p: E \to B$ be a shape fibration between compacta and let $\tilde{p}: X \to Y$ be an extension of p, where X and Y are AR's containing E and B, respectively. Suppose that $\varepsilon > 0$ and U (resp. V) is a neighborhood of E (resp. B) in X (resp. Y). Then there is $\delta > 0$ and a neighborhood U_1 (resp. V_1) of E (resp. B) in X (resp. Y) satisfying the following property; for any space Z and a closed subset A of Z, any maps $h: (Z \times \{0\}) \cup (A \times I) \to U_1$ and $H: Z \times I \to V_1$ such that $d(\tilde{p}h, H|(Z \times \{0\}) \cup (A \times I)) < \delta$, then there is an extension $\tilde{H}: Z \times I \to U$ of h such that $d(\tilde{p}\tilde{H}, H) < \varepsilon$. Such a pair $(U_1, V_1; \delta)$ is called a lifting pair for $(U, V; \varepsilon)$.*

SKETCH OF THE PROOF OF LEMMA 4.2. Observe [11, Theorem 2] and [12, Proposition 1]. The lemma is proved by the same way as in Allaud and Fadell [A fiber homotopy extension theorem, Trans. A.M.S. Soc. 104 (1962), 239-251, Theorem (2.1) and Theorem (2.4)] and shape theoretic consideration.

LEMMA 4.3. *Let X be a compact ANR. Then for any $\varepsilon > 0$ there is $\alpha(\varepsilon) > 0$ such that for any space Z and a closed subset A of Z, any $\alpha(\varepsilon)$-near maps $f, g: Z \to X$ and a homotopy $H: A \times I \to X$ such that $H(z, 0) = f(z)$, $H(z, 1) = g(z)$ for $z \in A$ and* $\operatorname{diam} H(\{z\} \times I) < \alpha(\varepsilon)$ *for $z \in A$, then there is an extension $F: Z \times I \to X$ of H such that $F(z, 0) = f(z)$, $F(z, 1) = g(z)$ for $z \in Z$ and* $\operatorname{diam} F(\{z\} \times I) < \varepsilon$ *for $z \in Z$.*

PROOF OF THEOREM 4.1. It is enough to give the proof of sufficiency. Choose $X \in m(E)$, $X' \in m(E')$ and $Y \in m(B)$ which are convenient AR's (i.e., An AR X is *convenient* if for each compactum A in X and each neighborhood U of A in X there is a compact ANR $M \subset U$ with $A \subset \operatorname{Int} M$). Let $\tilde{p}: X \to Y$ and $\tilde{p}': X' \to Y$ be extensions of p and p', respectively. Since $Y \in m(B)$, we may assume $\tilde{p}(X-E) \subset Y-B$ and $\tilde{p}'(X'-E') \subset Y-B$. Suppose that $f: X-E \to X'-E'$ is an $F(p, p')$-map which is contained in the $F(p, p')$-equivalence class $f: p \to p'$. Since f is a proper homotopy equivalence, there is a proper map $g: X'-E' \to X-E$ and a proper homotopy $H: (X'-$

$E')\times I\to X'-E'$ such that gf is properly homotopic to 1_{X-E} and $H(x',0)=x'$, $H(x',1)=fg(x')$ for $x'\in X'-E'$.

We will construct decreasing sequences $\{C_n\}_{n=1,2,\dots}$ and $\{D_n\}_{n=1,2,\dots}$ of compact ANR's, a decreasing sequence $\{\varepsilon_n\}_{n=1,2,\dots}$ of positive numbers and sequences $\{g_n\}_{n=1,2,\dots}$, $\{G_n\}_{n=1,2,\dots}$ and $\{R_n\}_{n=1,2,\dots}$ of maps satisfying the following properties (1)~(9).

(1) $X\supset C_1\supset \operatorname{Int} C_1\supset C_2\supset\cdots\supset E$, $X'\supset D_1\supset \operatorname{Int} D_1\supset D_2\supset\cdots\supset E'$ and $E=\bigcap_{n=1}^{\infty} C_n$, $E'=\bigcap_{n=1}^{\infty} D_n$.

(2) $\lim_{n\to\infty}\varepsilon_n=0$.

(3) $g_{2n-1}: D_{2n}-\operatorname{Int} D_{2n+1}\to C_{2n-1}-E$ for $n=1,2,\cdots$,

$g_{2n}: D_{2n+1}-\operatorname{Int} D_{2n+2}\to C_{2n-3}-E$ for $n=2,3,\cdots$.

(4) $G_{2n-1}:(D_{2n}-\operatorname{Int} D_{2n+1})\times I\to C_{2n-1}-E$ for $n=1,2,\cdots$,

$G_{2n}:(D_{2n+1}-\operatorname{Int} D_{2n+2})\times I\to C_{2n-3}-E$ for $n=2,3,\cdots$.

(5) $R_{2n-1}:(D_{2n}-\operatorname{Int} D_{2n+1})\times[0,2]\to D_{2n-1}-E'$ for $n=1,2,\cdots$,

$R_{2n}:(D_{2n+1}-\operatorname{Int} D_{2n+2})\times[0,2]\to D_{2n-3}-E'$ for $n=2,3,\cdots$.

(6) $G_{2n-1}(x',0)=g_{2n-1}(x')$, $G_{2n-1}(x',1)=g(x')$ for $x'\in D_{2n}-\operatorname{Int} D_{2n+1}$
and $G_{2n}(x',0)=g_{2n}(x')$, $G_{2n}(x',1)=g(x')$ for $x'\in D_{2n+1}-\operatorname{Int} D_{2n+2}$
and $G_{2n-1}|(\operatorname{Bd} D_{2n+1})\times I=G_{2n}|(\operatorname{Bd} D_{2n+1})\times I$,
$G_{2n}|(\operatorname{Bd} D_{2n+2})\times I=G_{2n+1}|(\operatorname{Bd} D_{2n+2})\times I$.

(7) $R_{2n-1}(x',0)=x'$, $R_{2n-1}(x',2)=fg_{2n-1}(x')$ for $x'\in D_{2n}-\operatorname{Int} D_{2n+1}$,
$R_{2n}(x',0)=x'$, $R_{2n}(x',2)=fg_{2n}(x')$ for $x'\in D_{2n+1}-\operatorname{Int} D_{2n+2}$
and $R_{2n-1}|(\operatorname{Bd} D_{2n+1})\times[0,2]=R_{2n}|(\operatorname{Bd} D_{2n+1})\times[0,2]$, $R_{2n}|(\operatorname{Bd} D_{2n+2})\times[0,2]=R_{2n+1}|(\operatorname{Bd} D_{2n+2})\times[0,2]$.

(8) $d(\tilde{p}g_{2n-1}(x'),\tilde{p}'(x'))<\varepsilon_{2n-1}$ for $x'\in D_{2n}-\operatorname{Int} D_{2n+1}$
and $d(\tilde{p}g_{2n}(x'),\tilde{p}'(x'))<\varepsilon_{2n-3}$ for $x'\in D_{2n+1}-\operatorname{Int} D_{2n+2}$.

(9) $d(\tilde{p}'R_{2n-1}(x',t),\tilde{p}'(x'))<\varepsilon_{2n-1}$ for $x'\in D_{2n}-\operatorname{Int} D_{2n+1}$, $t\in[0,2]$
and $d(\tilde{p}'R_{2n}(x',t),\tilde{p}'(x'))<\varepsilon_{2n-3}$ for $x'\in D_{2n+1}-\operatorname{Int} D_{2n+2}$, $t\in[0,2]$.

Let D_1 (resp. B_1) be compact ANR-neighborhood of E' (resp. B) in X' (resp. Y) such that $\tilde{p}'(D_1)\subset B_1$ and $\varepsilon_1>0$. Since $p': E'\to B$ is a shape fibration, by Lemma 4.2, there is a compact ANR-neighborhood D_1' (resp. B_1') of E' (resp. B) in X' (resp. Y) and $\delta_1>0$ such that

(10) $D_1\supset D_1'$, $B_1\supset B_1'$, $\tilde{p}'(D_1')\subset B_1'$ and $(D_1', B_1'; \delta_1)$ is a lifting pair for $(D_1, B_1; \varepsilon_1/2)$, $\delta_1<\varepsilon_1/2$.

Since B_1' is a compact ANR, there is a positive number $\alpha(\delta_1)$ satisfying the condition of Lemma 4.3 and $\delta_1>\alpha(\delta_1)>0$. Since $f: X-E\to X'-E'$ is an $F(p,p')$-map, we can easily see that there is a compact ANR-neighborhood C_1 of E in X such that

(11) $\tilde{p}(C_1)\subset B_1'$, $f(C_1-E)\subset D_1'-E'$ and $d(\tilde{p}'f(x), \tilde{p}(x))<\alpha(\delta_1)/2$ for $x\in C_1-E$.

Since $p: E\to B$ is a shape fibration, by Lemma 4.2 there is a compact ANR-neighborhood C_2 (resp. B_2) of E (resp. B) in X (resp. Y) and a positive number $\varepsilon_2<\alpha(\delta_1)$ such that

(12) $\tilde{p}(C_2)\subset B_2$, $C_1\supset C_2$, $B_1'\supset B_2$, $(C_2, B_2; \varepsilon_2)$ is a lifting pair for $(C_1, B_1'; \alpha(\delta_1)/2)$ and $d(\tilde{p}'f(x), \tilde{p}(x))<\varepsilon_2$ for $x\in C_2-E$.

Since g and H are proper maps respectively, we can choose a compact ANR-neighborhood D_2 of E' in X' such that

(13) $D_1\supset \operatorname{Int} D_1\supset D_2$,

(14) $g(D_2-E')\subset C_2-E$ and $\tilde{p}g(D_2-E')\subset B_2-B$,

(15) $H((D_2-E')\times I)\subset D_1'-E'$ and $\tilde{p}'H((D_2-E')\times I)\subset B_2-B$.

By (12) and (14), we have

(16) $d(\tilde{p}'H(x',1), \tilde{p}g(x'))=d(\tilde{p}'fg(x'), \tilde{p}g(x'))<\varepsilon_2$ for $x'\in D_2-E'$.

Choose a compact ANR-neighborhood D_3 of E' in X' with $D_3\subset \operatorname{Int} D_2$. By (12), (16) and $X\in m(E)$, there is a homotopy $G_1: (D_2-\operatorname{Int} D_3)\times I\to C_1-E$ such that

(17) $G_1(x',1)=g(x')$ for $x'\in D_2-\operatorname{Int} D_3$ and $d(\tilde{p}G_1, \tilde{p}'H|(D_2-\operatorname{Int} D_3)\times I)<\alpha(\delta_1)/2<\varepsilon_1$.

Define a map $g_1: D_2-\operatorname{Int} D_3\to C_1-E$ by $g_1(x')=G_1(x',0)$ for $x'\in D_2-\operatorname{Int} D_3$. Then we have

(18) $d(\tilde{p}g_1(x'), \tilde{p}'(x'))=d(\tilde{p}G_1(x',0), \tilde{p}'H(x',0))<\alpha(\delta_1)/2<\varepsilon_1$.

Define a homotopy $L_1: (D_2-\operatorname{Int} D_3)\times[0,2]\to D_1'-E'$ by

(19) $$L_1(x',s)=\begin{cases} H(x',s) & \text{for } x'\in D_2-\operatorname{Int} D_3,\ 0\leqq s\leqq 1, \\ fG_1(x',2-s) & \text{for } x'\in D_2-\operatorname{Int} D_3,\ 1\leqq s\leqq 2. \end{cases}$$

By (11), (17) and (19),

(20) $$\begin{aligned} d(\tilde{p}'L_1(x',s), \tilde{p}'L_1(x',2-s)) &= d(\tilde{p}'H(x',s), \tilde{p}'fG_1(x',s)) \\ &\leqq d(\tilde{p}'H(x',s), \tilde{p}G_1(x',s))+d(\tilde{p}G_1(x',s), \tilde{p}'fG_1(x',s)) \\ &<\alpha(\delta_1)/2+\alpha(\delta_1)/2=\alpha(\delta_1), \quad \text{where } 0\leqq s\leqq 1. \end{aligned}$$

By the choice of $\alpha(\delta_1)$, there is a homotopy $K_1:(D_2-\text{Int } D_3)\times[0,2]\times I\to B_1'$ such that

$$(21)\quad K_1(x',s,t)=\tilde{p}'L_1(x',s) \text{ for } t\leqq 1-s \text{ or } t\leqq s-1 \text{ and } d(\tilde{p}'(x'),K_1(x',s,1))<\delta_1<\varepsilon_1/2 \text{ for } x'\in D_2-\text{Int } D_3,\ 0\leqq s\leqq 2.$$

Define a map $L_1':(D_2-\text{Int } D_3)\times(\{0\}\times I\cup[0,2]\times\{0\}\cup\{2\}\times I)\to D_1'-E'$ by

$$(22)\quad L_1'(x',s,t)=\begin{cases} L_1(x',0) & \text{for } s=0,\ 0\leqq t\leqq 1, \\ L_1(x',s) & \text{for } 0\leqq s\leqq 2,\ t=0 \\ L_1(x',2) & \text{for } s=2,\ 0\leqq t\leqq 1. \end{cases}$$

By (21) and (22), $\tilde{p}'L_1'=K_1|(D_2-\text{Int } D_3)\times(\{0\}\times I\cup[0,2]\times\{0\}\cup\{2\}\times I)$. By (10), there is a homotopy $M_1:(D_2-\text{Int } D_3)\times[0,2]\times I\to D_1-E$ such that

$$(23)\quad M_1|(D_2-\text{Int } D_3)\times(\{0\}\times I\cup[0,2]\times\{0\}\cup\{2\}\times I)=L_1' \text{ and } d(\tilde{p}'M_1,K_1)<\varepsilon_1/2.$$

Define a homotopy $R_1:(D_2-\text{Int } D_3)\times[0,2]\to D_1-E'$ by

$$(24)\quad R_1(x',s)=M_1(x',s,1) \text{ for } x'\in D_2-\text{Int } D_3,\ 0\leqq s\leqq 2.$$

Then $R_1(x',0)=x'$, $R_1(x',2)=fg_1(x')$ for $x'\in D_2-\text{Int } D_3$. By (21), (23) and (24), we have

$$(25)\quad d(\tilde{p}'R_1(x',s),\tilde{p}'(x')) \leqq d(\tilde{p}'M_1(x',s,1),K_1(x',s,1))+d(K_1(x',s,1),\tilde{p}'(x')) <\varepsilon_1/2+\varepsilon_1/2=\varepsilon_1 \text{ for } x'\in D_2-\text{Int } D_3,\ 0\leqq s\leqq 2.$$

If we continue the process as above, then we can construct decreasing sequences $\{C_n\}$, $\{D_n\}$ of compact ANR's, a decreasing sequence $\{\varepsilon_n\}$ of positive numbers and sequences $\{g_{2n-1}\}_{n=1,2,\cdots}$, $\{G_{2n-1}\}_{n=1,2,\cdots}$ and $\{R_{2n-1}\}_{n=1,2,\cdots}$ of maps satisfying the conditions (1)~(9).

Next, for each $n=2,3,\cdots$, we will construct maps g_{2n}, G_{2n} and R_{2n} satisfying the conditions (3)~(9). Define a map $G_{2n}':(D_{2n+1}-\text{Int } D_{2n+2})\times\{1\}\cup(\text{Bd } D_{2n+1}\cup \text{Bd } D_{2n+2})\times I\to C_{2n-1}-E$ by

$$(26)\quad G_{2n}'(x',t)=\begin{cases} g(x') & \text{for } x'\in D_{2n+1}-\text{Int } D_{2n+2},\ t=1, \\ G_{2n-1}(x',t) & \text{for } x'\in \text{Bd } D_{2n+1},\ t\in I, \\ G_{2n+1}(x',t) & \text{for } x'\in \text{Bd } D_{2n+2},\ t\in I. \end{cases}$$

By (16) and (17), we have

$$(27)\quad d(\tilde{p}'H|(D_{2n+1}-\text{Int } D_{2n+2})\times\{1\}\cup(\text{Bd } D_{2n+1}\cup\text{Bd } D_{2n+2})\times I,\tilde{p}G_{2n}') <\alpha(\delta_{2n-1})/2<\varepsilon_{2n-1}<\varepsilon_{2n-2}.$$

By (12), (27) and $X\in m(E)$, there is a homotopy $G_{2n}:(D_{2n+1}-\text{Int } D_{2n+2})\times I\to C_{2n-3}-E$

such that

(28) $G_{2n}|(D_{2n+1}-\text{Int}\,D_{2n+2})\times\{1\}\cup(\text{Bd}\,D_{2n+1}\cup\text{Bd}\,D_{2n+2})\times I=G'_{2n}$ and
$d(\tilde{p}G_{2n}, \tilde{p}'H|(D_{2n+1}-\text{Int}\,D_{2n+2})\times I)<\alpha(\delta_{2n-3})/2$.

Define a map $g_{2n}: D_{2n+1}-\text{Int}\,D_{2n+2}\to C_{2n-3}-E$ by $g_{2n}(x')=G_{2n}(x',0)$ for $x'\in D_{2n+1}-\text{Int}\,D_{2n+2}$. Clearly, g_{2n} and G_{2n} satisfy the conditions as we wanted. Similarly, we obtain $R_{2n}:(D_{2n+1}-\text{Int}\,D_{2n+2})\times[0,2]\to D_{2n-3}-E'$ which fulfills our requirement.

Clearly we obtain maps $g': X'-E'\to X-E$, $G:(X'-E')\times I\to X-E$ and $R:(X'-E')\times[0,2]\to X'-E'$ such that

$$g'|D_{2n}-\text{Int}\,D_{2n+1}=g_{2n-1} \quad \text{for } n=1,2,\cdots,$$
$$g'|D_{2n+1}-\text{Int}\,D_{2n+2}=g_{2n} \quad \text{for } n=2,3,\cdots,$$
$$G|(D_{2n}-\text{Int}\,D_{2n+1})\times I=G_{2n-1} \quad \text{for } n=1,2,\cdots,$$
$$G|(D_{2n+1}-\text{Int}\,D_{2n+2})\times I=G_{2n} \quad \text{for } n=2,3,\cdots,$$
$$R|(D_{2n}-\text{Int}\,D_{2n+1})\times[0,2]=R_{2n-1} \quad \text{for } n=1,2,\cdots,$$
$$R|(D_{2n+1}-\text{Int}\,D_{2n+2})\times[0,2]=R_{2n} \quad \text{for } n=2,3,\cdots.$$

and $G(x',0)=g'(x')$, $G(x',1)=g(x')$ for $x'\in X'-E'$, $R(x',0)=x'$, $R(x',2)=fg'(x')$ for $x'\in X'-E'$ (because $X\in m(E)$, $X'\in m(E')$). By (1)~(9), we can conclude that g' is an $F(p',p)$-map such that g' is properly homotopic to g and $fg' \underset{F(p',p')}{\simeq} 1_{X'-E'}$. Note that g' is a proper homotopy equivalence. To complete the proof, we apply the same process to $g': X'-E'\to X-E$ instead of $f: X-E\to X'-E'$. Thus there is an $F(p,p')$-map $f': X-E\to X'-E'$ such that $g'f' \underset{F(p,p)}{\simeq} 1_{X-E}$. Then $f \underset{F(p,p')}{\simeq} fg'f' \underset{F(p,p')}{\simeq} f'$, which implies $fg' \underset{F(p',p')}{\simeq} 1_{X'-E'}$ and $g'f \underset{F(p,p)}{\simeq} 1_{X-E}$. Hence the morphism $f: p\to p'$ is an isomorphism in FR_B. This completes the proof.

As a special case of Theorem 4.1, we have the next corollary.

COROLLARY 4.4. Let $p: E\to B$ *be a map between compacta. Then the following are equivalent.*

(1) p *is a shape fibration and a strong shape equivalence.*

(1) p *is shape shrinkable.*

PROOF. (1)→(2) follows from Theorem 4.1. (2)→(1) follows from [7, Corollary 3.6] and Corollary 2.5.

For a map $p: E\to B$, $S^*(p)$ denotes the morphism of *s-Sh* induced by p. By the similar way as the proof of Theorem 4.1, we have the following proposition.

PROPOSITION 4.5. *Let $p: E\to B$ and $p': E'\to B$ be maps between compacta and let $f: E\to E'$ be a morphism in s-Sh such that $S^*(p)=S^*(p')f$. If $p': E'\to B$ is a shape fibration, then there is a morphism $g: p\to p'$ in FR_B such that $T(g)=f$.*

By Theorem 4.1 and Proposition 4.5, we have

COROLLARY 4.6. *Let $p: E\to B$ and $p': E'\to B$ be shape fibrations between compacta. If a morphism $f: E\to E'$ in s-Sh is an isomorphism such that $S^*(p)=S^*(p')f$, then there is an isomorphism $g: p\to p'$ in FR_B such that $T(g)=f$.*

5. Applications.

In this section, some applications are given. First, we obtain the following theorem by Theorem 3.3 and Theorem 4.1.

THEOREM 5.1. *Let $p: E\to B$ and $p': E'\to B$ be shape fibrations between compacta and let $f: p\to p'$ be a morphism in FR_B. If there is a finite closed cover $\{B_i\}_{i=1,2,\cdots,n}$ of B such that $T(f|p^{-1}(B_i)): p^{-1}(B_i)\to p'^{-1}(B_i)$ is an isomorphism in s-Sh for each $i=1,2,\cdots,n$, then $f: p\to p'$ is an isomorphism in FR_B.*

It is well-known that if $p: E\to B$ is a Hurewicz fibration and B is contractible, the inclusion $i: p^{-1}(b)\to E$ is a homotopy equivalence for each $b\in B$. Note that if B is an FAR and a Z-set in Q, there is a decreasing sequence $Q\supset B_1\supset B_2\supset\cdots$, of compact neighborhoods of B in Q such that $\bigcap_{i=1}^{\infty} B_i=B$, each B_i is homeomorphic to Q. The proof of the following proposition is similar to one of Theorem 4.1. We omit it.

PROPOSITION 5.2. (cf. [12, Theorem 1]). *Let $p: E\to B$ be a shape fibration between compacta. If B is an FAR, the inclusion $i: p^{-1}(b)\to E$ induces an isomorphism in s-Sh for each $b\in B$.*

PROPOSITION 5.3. *Let $p: E\to B$ and $p': E'\to B$ be shape fibrations between compacta and let B be an FAR. Then a morphism $f: p\to p'$ of FR_B is an isomorphism in FR_B if and only if for some $b_0\in B$, the restriction $T(f|p^{-1}(b_0)): p^{-1}(b_0)\to p'^{-1}(b_0)$ of $T(f)$ to $p^{-1}(b_0)$ is an isomorphism in s-Sh.*

PROOF. It is enough to give the proof of sufficiency. There is a commutative diagram

$$\begin{array}{ccc} p^{-1}(b_0) & \xrightarrow{T(f|p^{-1}(b_0))} & p'^{-1}(b_0) \\ {\scriptstyle S^*(i)}\downarrow & & \downarrow{\scriptstyle S^*(i')} \\ E & \xrightarrow{T(f)} & E' \end{array}$$

in s-Sh, where i and i' are the inclusion maps. By Proposition 5.2, i and i' induce isomorphisms in s-Sh. Hence $T(f): E \to E'$ is an isomorphism in s-Sh. By Theorem 4.1, $f: p \to p'$ is an isomorphism in FR_B.

THEOREM 5.4. *Let $p: E \to B$ and $p': E' \to B$ be shape fibrations between compacta. Suppose that B is a continuum with a finite closed cover consisting of FAR's. Then a morphism $f: p \to p'$ of FR_B is an isomorphism in FR_B if and only if for some $b_0 \in B$, the restriction $T(f|p^{-1}(b_0)): p^{-1}(b_0) \to p'^{-1}(b_0)$ of $T(f)$ to $p^{-1}(b_0)$ is an isomorphism in s-Sh.*

PROOF. It is enough to give the proof of sufficiency. Let $\{B_i\}$ be a finite closed cover consisting of FAR's. Since B is connected, by Proposition 5.3, we conclude that the restriction $f|p^{-1}(B_i): p|p^{-1}(B_i) \to p'|p'^{-1}(B_i)$ of f to $p^{-1}(B_i)$ is an isomorphism in FR_{B_i} for each i. By Theorem 3.3 $f: p \to p'$ is an isomorphism in FR_B.

COROLLARY 5.5. *Let $p: E \to B$ and $p': E' \to B$ be shape fibrations between compacta and let B be a connected ANR. Then a morphism $f: p \to p'$ of FR_B is an isomorphism in FR_B if and only if the restriction $T(f|p^{-1}(b_0)): p^{-1}(b_0) \to p'^{-1}(b_0)$ is an isomorphism in s-Sh for some $b_0 \in B$.*

PROOF. Define maps $p \times 1_Q: E \times Q \to B \times Q$, $p' \times 1_Q: E' \times Q \to B \times Q$ by $(p \times 1_Q)(e, q) = (p(e), q)$, $(p' \times 1_Q)(e', q) = (p'(e'), q)$ for $e \in E, e' \in E'$ and $q \in Q$. Note that $p \times 1_Q$ and $p' \times 1_Q$ are shape fibrations. Choose an $F(p, p')$-map $f: X - E \to X' - E'$ which is contained in the $F(p, p')$-equivalence class $f: p \to p'$. Define a map $f \times 1_Q: (X \times Q - E \times Q) \to (X' \times Q - E' \times Q)$ by $(f \times 1_Q)(x, q) = (f(x), q)$ for $x \in X \times Q - E \times Q$, $q \in Q$. Clearly, the map $f \times 1_Q: (X \times Q - E \times Q) \to (X' \times Q - E' \times Q)$ determines a morphism $f \times 1_Q: p \times 1_Q \to p' \times 1_Q$ of $FR_{B \times Q}$. Since B is a compact ANR, $B \times Q$ is a compact Q-manifold. Clearly, $B \times Q$ has a finite closed cover consisting of FAR's. By Theorem 5.4, $f \times 1_Q: p \times 1_Q \to p' \times 1_Q$ is an isomorphism in $FR_{B \times Q}$. By Proposition 3.2, $f: p \to p'$ is an isomorphism in FR_B.

By Theorem 5.4, Corollary 5.5 and [7, Corollary 3.6], we have the following.

COROLLARY 5.6. *Let $p: E \to B$ be a map between compacta. Suppose that B is* *an ANR or B has a finite closed cover consisting of FAR's. Then the following are equivalent.*

(1) *p is a cell-like shape fibration.*

(2) *p is shape shrinkable.*

REMARK 5.7. In the statement of Corollary 5.6, the assumption about B cannot be omitted. Edwards and Hastings [6, pp. 196–200] give an example of a cell-like shape fibration which fails to be a shape equivalence. Also, we cannot omit the condition "shape fibration" of (1). It is well-known that there is a map $p: E \to Q$ of continuum E to the Hilbert cube Q which is cell-like and not a shape equivalence (see [15]).

COROLLARY 5.8 (cf., [8, Theorem 2.1]). *Let $p: E \to B$ and $p': E' \to B$ be shape fibrations between compacta and let B be a connected ANR or a continuum with a finite closed cover consisting of FAR's. Suppose that $f = \{f_n, E, E'\}_{X,X'}$ is a fiber fundamental sequence over B and one of the following properties* (1) *or* (2) *is satisfied;*

(1) *for some $b_0 \in B$, $\mathrm{Fd}(p^{-1}(b_0)) \leq 1$.*

(2) *for some $b_0 \in B$, $\mathrm{Fd}(p^{-1}(b_0)) < \infty$, $p^{-1}(b_0)$ has finite components and each component is pointed* 1*-movable.*

Then f is a fiber shape equivalence over B it and only if the restriction $f|p^{-1}(b_0) = \{f_n, p^{-1}(b_0), p'^{-1}(b_0)\}_{X,X'}$ of f to $p^{-1}(b_0)$ is a shape equivalence.

PROOF. It is enough to give the proof of sufficiency. Let $f: p \to p'$ be a morphism in FR_B induced by f, i.e., $f: p \to p'$ is an $F(p, p')$-equivalence class containing $s(f)$ (see the proof of Theorem 2.2 for the notation $s(f)$). Then the property (1) or (2) implies that $T(f|p^{-1}(b_0)): p^{-1}(b_0) \to p'^{-1}(b_0)$ is an isomorphism in s-Sh (see [5, Theorem 8.3, Theorem 6.4 and Theorem 7.3] and [4, Theorem 3.6]). Hence f is an isomorphism in FR_B by Theorem 5.4 and Corollary 5.5. Thus f is a fiber shape equivalence over B.

REMARK 5.9. Chapman and Siebenmann (Finding a boundary for a Hilbert cube manifold, Acta Math., 137 (1976), 171–208) have asked the following question: Is each weak proper homotopy equivalence a proper homotopy equivalence? The positive answer would give a stronger result than Corollary 5.8; in fact we could omit the assumptions (1), (2) in Corollary 5.8.

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