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SHAPE FIBRATIONS AND FIBER SHAPE EQUIVALENCES, II

By

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0. Introduction.

In [2] and [3], Coram and Duvall introduced the notion of approximate fibrations and they characterized this in terms of movability conditions for maps. Mardešić and Rushing [11] defined shape fibrations and showed that for compact ANR's, those agree with approximate fibrations. In [8], we defined fiber fundamental sequences and fiber shape equivalences.

In this paper, we show that fiber fundamental sequences have shape theoretic properties analogous to the homotopy theoretic properties of fiber maps. In particular, we prove the following:

(1) Let E and B be compacta and let B be an FAR. Then a map $p:E\to B$ is a shape fibration if and only if p is shape trivial.

(2) A proper map $p:E\to B$ between locally compact, separable metric ANR's is an approximate fibration if and only if p is locally shape trivial.

(3) Let $p:E\to B$ be a shape fibration from a compactum E to a connected compact ANR B and let $p':E'\to B$ be an approximate fibration between compact ANR's. Then a fiber map $f:E\to E'$ over B is a fiber shape equivalence over B if and only if for some $b_0\in B$, $f|p^{-1}(b_0):p^{-1}(b_0)\to p'^{-1}(b_0)$ is a shape equivalence.

It is assumed that all spaces are metrizable and all maps are continuous. If x and y are points of a metric space, $d(x, y)$ denotes the distance from x to y. For maps $f, g: X\to Y$ of compacta, $d(f, g)=\sup\{d(f(x), g(x))|x\in X\}$. We denote by I the unit interval $[0, 1]$ and by Q the Hilbert cube. A proper map $p:E\to B$ between locally compact, separable metric ANR's is an *approximate fibration* [2] if given an open cover $\mathcal{U}$ of B, a space X and maps $h:X\to E$, $H:X\times I\to B$ such that $ph=H_0$, then there is a homotopy $\tilde{H}:X\times I\to E$ such that $\tilde{H}_0=h$ and $p\tilde{H}$ and H are $\mathcal{U}$-close, where $H_t(x)=H(x, t)$. Let $\underset{\sim}{E}=(E_i, q_{ij})$ and $\underset{\sim}{B}=(B_i, r_{ij})$ be inverse sequences of compacta and let $\underset{\sim}{p}=(p_i)$ be a sequence of maps $p_i: E_i\to B_i$. $\underset{\sim}{p}:\underset{\sim}{E}\to\underset{\sim}{B}$ is a *level map* if for any i and $j\geqq i$, $p_iq_{ij}=r_{ij}p_j$. A map $p:E\to B$ between compacta is a *shape fibration* [11] if there is a level map $\underset{\sim}{p}:\underset{\sim}{E}\to\underset{\sim}{B}$ of compact ANR-sequences with invlim $\underset{\sim}{E}=E$, invlim $\underset{\sim}{B}=B$ and invlim $\underset{\sim}{p}=p$ satisfying the following property; for each i and

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$\varepsilon>0$ there is $j\geqq i$ and $\delta>0$ such that for any space X and any maps $h: X\to E_j$, $H: X\times I\to B_j$ with $d(p_jh, H_0)<\delta$, there is a homotopy $\tilde{H}: X\times I\to E_i$ such that $d(\tilde{H}_0, q_{ij}h)<\varepsilon$ and $d(p_i\tilde{H}, r_{ij}H)<\varepsilon$. Such a pair (E_j, δ) is called a *lifting pair* for (E_i, ε). Let E and E' and B be compacta in the Hilbert cube Q and let $\tilde{p}: Q\to Q$ and $\tilde{p}': Q\to Q$ be extensions of maps $p: E\to B$ and $p': E'\to B$, respectively. A fundamental sequence (see [1]) $f=\{f_n, E, E'\}_{Q,Q}$ is a *fiber fundamental sequence over* B [8] if for any $\varepsilon>0$ and any neighborhood U' of E' in Q there is a neighborhood U of E in Q and a positive integer n_0 such that for each $n\geqq n_0$ there is a homotopy $F: U\times I\to U'$ such that $F_0=f_{n_0}|U$, $F_1=f_n|U$ and $d(\tilde{p}'F(x, t), \tilde{p}(x))<\varepsilon$, $x\in U$, $t\in I$. A fiber fundamental sequence $f=\{f_n E, E'\}_{Q,Q}$ over B is *fiber homotopic* to a fiber fundamental sequence $g=\{g_n, E, E'\}_{Q,Q}$ over B ($f\underset{B}{\simeq} g$) if for any $\varepsilon>0$ and any neighborhood U' of E' in Q there is a neighborhood U of E in Q and a positive integer n_0 such that for each $n\geqq n_0$ there is a homotopy $K: U\times I\to U'$ such that $K_0=f_n|U$, $K_1=g_n|U$ and $d(\tilde{p}'K(x, t), \tilde{p}(x))<\varepsilon$, $x\in U$, $t\in I$. A map $p: E\to B$ over B is *fiber shape equivalent* to a map $p': E'\to B$ over B if there are fiber fundamental sequences over B $f=\{f_n, E, E'\}_{Q,Q}$ and $g=\{g_n, E', E\}_{Q,Q}$ such that $gf\underset{B}{\simeq} \underline{1}_E$ and $fg\underset{B}{\simeq} \underline{1}_{E'}$, where $\underline{1}_E$ denotes a fiber fundamental sequence over B induced by the identity $1_E: E\to E$. Such f is called a *fiber shape equivalence over* B. A map $p: E\to B$ is *shape trivial* if p is fiber shape equivalent to the projection $\pi: p^{-1}(b)\times B\to B$ for some $b\in B$. A map $p: E\to B$ is *shape shrinkable* if p induces a fiber shape equivalence from p to the identity 1_B. Obviously if a map $p: E\to B$ is shape shrinkable, p is shape trivial. Note that a map $p: E\to B$ is shape shrinkable iff p is a hereditary shape equivalence (see [8, Corollary 3.5]).

1. Fiber weak dominations preserve shape fibrations.

In [8, Corollary 3.6], we show that fiber shape equivalences preserve shape fibrations. In this section, we show that fiber weak dominations preserve shape fibrations. Also, we give a condition for a map, which implies that the map is an approximate fibration. First, we have the following. (see [8] for the definition of fiber weak domination.)

PROPOSITION 1.1. *Let $p: E\to B$ and $p': E'\to B$ be maps between compacta and $p: E\to B$ be a shape fibration. Let $f=\{f_n, E, E'\}_{Q,Q}$ be a fiber fundamental sequence over B which is a weak domination in shape category* (see [4]). *Then f is a fiber weak domination if and only if $p': E'\to B$ is a shape fibration.*

PROOF. By [8, Lemma 2.1], it is enough to give the proof of necessity. Let $\tilde{p}: Q\to Q$ and $\tilde{p}': Q\to Q$ be extensions of p and p', respectively. By induction, we can choose decreasing sequences of compact ANR-neighborhoods $\{E_n\}$, $\{E_n'\}$ and $\{B_n\}$ of E, E' and B, respectively such that $\bigcap_{n=1}^{\infty} E_n=E$, $\bigcap_{n=1}^{\infty} E_n'=E'$ and $\bigcap_{n=1}^{\infty} B_n=B$ and $\tilde{p}(E_n)\subset B_n$, $\tilde{p}'(E_n')\subset B_n$. Let $\varepsilon>0$ and i be a positive integer. Since $\underline{f}$ is a fiber fundamental sequence, there is $j_0\geqq i$ such that

(1) for any $k\geqq j_0$, there is a homotopy $G_{j_0,k}: E_{j_0}\times I\to E_i'$ such that $G_{j_0,k}(x,0)=f_{j_0}(x)$, $G_{j_0,k}(x,1)=f_k(x)$ for $x\in E_{j_0}$ and

(2) $d(\tilde{p}(x), \tilde{p}'G_{j_0,k}(x,t))<\varepsilon/6$ for $x\in E_{j_0}$, $t\in I$.

Since $p: E\to B$ is a shape fibration, there is $j_1\geqq j_0$ and $\delta_1>0$ such that (E_{j_1}, δ_1) is a lifting pair for $(E_{j_0}, \varepsilon/6)$. Since $\underline{f}$ is a fiber weak domination, there is $j_2\geqq j_1$ and a map $g: E_{j_2}'\to E_{j_1}$ and a homotopy $K: E_{j_2}'\times I\to E_i'$ such that

(3) $d(\tilde{p}g(x), \tilde{p}'(x))<\delta_2=\delta_1/2$, $x\in E_{j_2}'$,

(4) $K(x,0)=x$, $K(x,1)=f_{j_0}g(x)$, $x\in E_{j_2}'$ and

(5) $d(\tilde{p}'(x), \tilde{p}'K(x,t))<\varepsilon/6$, $x\in E_{j_2}'$, $t\in I$.

Let $h: X\to E_{j_2}'$ be a map and let $H: X\times I\to B_{j_2}$ be a homotopy such that

(6) $d(\tilde{p}'h(x), H(x,0))<\delta_2(<\varepsilon/6)$ for $x\in X$.

By (3) and (6), we have

(7) $d(\tilde{p}gh(x), H(x,0))\leqq d(\tilde{p}gh(x), \tilde{p}'h(x))+d(\tilde{p}'h(x), H(x,0))<\delta_2+\delta_2=\delta_1$.

By [11, Proposition 1], we may assume that there is a homotopy $H': X\times I\to E_{j_0}$ such that

(8) $H'(x,0)=gh(x)$, $x\in X$ and

(9) $d(H(x,t), \tilde{p}H'(x,t))<\varepsilon/6$, $x\in X$, $t\in I$.

Define a homotopy $H'': X\times I\to E_i'$ by $H''=f_{j_0}H'$. Then, by (2) and (9),

(10)
$$\begin{aligned} d(\tilde{p}'H''(x,t), H(x,t)) &= d(\tilde{p}'f_{j_0}H'(x,t), H(x,t)) \\ &\leqq d(\tilde{p}'f_{j_0}H'(x,t), \tilde{p}H'(x,t))+d(\tilde{p}H'(x,t), H(x,t)) \\ &<\varepsilon/6+\varepsilon/6=\varepsilon/3, \quad x\in X,\ t\in I. \end{aligned}$$

By (4), (5), (6) and (8),

(11) $K(h(x),0)=h(x)$, $K(h(x),1)=f_{j_0}gh(x)=f_{j_0}H'(x,0)=H''(x,0)$,

(12)
$$\begin{aligned} d(\tilde{p}'K(h(x),t), H(x,0)) &\leqq d(\tilde{p}'K(h(x),t), \tilde{p}'h(x))+d(\tilde{p}'h(x), H(x,0)) \\ &<\varepsilon/6+\varepsilon/6=\varepsilon/3, \quad x\in X,\ t\in I. \end{aligned}$$

Since X is a metric space, there is a map $s: X\to(0,1]$ such that

(13) $\operatorname{diam} H(x\times[0, s(x)])<\varepsilon/3$.

By (11), we can define a homotopy $\tilde{H}: X\times I\to E_i'$ by

$$(14)\quad \tilde{H}(x,t)=\begin{cases} K(h(x), 2t/s(x)), & 0\leqq t\leqq s(x)/2, \\ H''(x, 2t-s(x)), & s(x)/2\leqq t\leqq s(x), \\ H''(x,t), & s(x)\leqq t\leqq 1. \end{cases}$$

Then by (10), (12) and (13), we have $\tilde{H}(x,0)=h(x)$ and $d(\tilde{p}'\tilde{H}(x,t)), H(x,t))<\varepsilon$ for $x\in X$, $t\in I$. Hence $p': E'\to B$ is a shape fibration. This completes the proof.

COROLLARY 1.2. *Fiber shape dominations preserve shape fibrations.*

In the following proposition, the implication (1)→(2) is a special case of M. Jani [7, Theorem 5.2]. Here we give the direct proof.

PROPOSITION 1.3. *Let E and B be compacta and let B be an FAR. Then, for a map $p: E\to B$ the following are equivalent.*

(1) *p is a shape fibration.*
(2) *p is shape trivial.*

PROOF. (2)→(1) follows from Corollary 1.2. We shall prove (1)→(2). Embed B into the Hilbert cube Q as a Z-set. Since $p: E\to B$ is a shape fibration and B is an FAR, we may assume that there are compact ANR neighborhoods $\{B_i\}_{i=0,1,2,\cdots}$ and $\{E_i\}_{i=0,1,2,\cdots}$ of B and E in Q, respectively such that

(1) each B_i is homeomorphic to Q, $B_0\supset B_1\supset B_2\supset\cdots$, $E_0\supset E_1\supset E_2\supset\cdots$ and $\bigcap_{i=0}^{\infty} B_i=B$, $\bigcap_{i=0}^{\infty} E_i=E$,
(2) $\tilde{p}(E_i)\subset B_i$, where $\tilde{p}: Q\to Q$ is an extension of p, and $\underline{p}=(\tilde{p}|E_i): \underline{E}=(E_i)\to\underline{B}=(B_i)$ has the HLP and $(i+1)$ is a lifting index for i (see [11, Theorem 2]).

Let $F_i=(\tilde{p}|E_i)^{-1}(b)$. $b\in B$. Then $\bigcap_{i=0}^{\infty} B_i\times F_i=B\times p^{-1}(b)$. Now we shall show that for $k=0,1,2,\cdots$, there are maps $f_k: B_{4k+2}\times F_{4k+2}\to E_{4k}$, $g_k: E_{4(k+1)}\to B_{4k+2}\times F_{4k+2}$ and homotopies $H_k: E_{4(k+1)}\times I\to E_{4k}$, $K_k: B_{4(k+1)+2}\times F_{4(k+1)+2}\times I\to B_{4k+2}\times F_{4k+2}$ such that

(3) $\tilde{p}f_k(y,x)=y$, $(y,x)\in B_{4k+2}\times F_{4k+2}$,

$\pi_{4k+2}g_k(x)=\tilde{p}(x)$, $x\in E_{4(k+1)}$, where $\pi_{4k+2}: B_{4k+2}\times F_{4k+2}\to B_{4k+2}$ is the projection,

(4) $H_k(x,0)=x$, $H_k(x,1)=f_kg_k(x)$, $x\in E_{4(k+1)}$,

(5) $K_k(z,0)=z$, $K_k(z,1)=g_kf_{k+1}(z)$, $z\in B_{4(k+1)+2}\times F_{4(k+1)+2}$,

(6) $\tilde{p}H_k(x,t)=\tilde{p}(x)$, $x\in E_{4(k+1)}$, $t\in I$ and
$\pi_{4k+2}K_k(z,t)=\pi_{4(k+1)+2}(z)$, $z\in B_{4(k+1)+2}\times F_{4(k+1)+2}$, $t\in I$.

Since $B_i \cong Q$, there is a homotopy $G_i : B_i \times I \to B_i$ such that

(7) $G_i(y, 0) = y$, $G_i(y, 1) = b$, $y \in B_i$.

Let $l_k : B_{4k+2} \times F_{4k+2} \to E_{4k+2}$ and $L_k : B_{4k+2} \times F_{4k+2} \times I \to B_{4k+2}$ be defined by

(8) $l_k(y, x) = x$, $(y, x) \in B_{4k+2} \times F_{4k+2}$ and

(9) $L_k((y, x), t) = G_{4k+2}(y, t)$, $(y, x) \in B_{4k+2} \times F_{4k+2}$, $t \in I$.

Then we have

(10) $L_k((y, x), 1) = G_{4k+2}(y, 1) = b = \tilde{p} l_k(y, x)$.

By HLP, there is a homotopy $\tilde{L}_k : B_{4k+2} \times F_{4k+2} \times I \to E_{4k+1}$ such that

(11) $\tilde{L}_k((y, x), 1) = l_k(y, x) = x$, $(y, x) \in B_{4k+2} \times F_{4k+2}$ and

(12) $\tilde{p} \tilde{L}_k = L_k$.

We define $f_k : B_{4k+2} \times F_{4k+2} \to E_{4k+1} \subsetneq E_{4k}$ by

(13) $f_k(y, x) = \tilde{L}_k(y, x, 0)$.

Then $\tilde{p} f_k(y, x) = \tilde{p} \tilde{L}_k(y, x, 0) = L_k(y, x, 0) = y$. Let r_k be the identity of $E_{4(k+1)}$ and $R_k : E_{4(k+1)} \times I \to B_{4(k+1)}$ be defined by

(14) $R_k(x, t) = G_{4(k+1)}(\tilde{p}(x), t)$, $x \in E_{4(k+1)}$, $t \in I$.

Then we have

(15) $R_k(x, 0) = G_{4(k+1)}(\tilde{p}(x), 0) = \tilde{p}(x) = \tilde{p} r_k(x)$.

Hence, there is a homotopy $\tilde{R}_k : E_{4(k+1)} \times I \to E_{4k+3}$ such that

(16) $\tilde{R}_k(x, 0) = r_k(x) = x$, $x \in E_{4(k+1)}$ and

(17) $\tilde{p} \tilde{R}_k = R_k$.

We define $g_k : E_{4(k+1)} \to B_{4k+3} \times F_{4k+3} \subsetneq B_{4k+2} \times F_{4k+2}$ by

(18) $g_k(x) = (\tilde{p}(x), \tilde{R}_k(x, 1))$.

Then $\pi_{4k+2} g_k(x) = \tilde{p}(x)$. Also, we shall construct homotopies $H_k : E_{4(k+1)} \times I \to E_{4k}$. Let $\varphi : E_{4(k+1)} \times (I \times 0 \cup 1 \times I \cup I \times 1) \to E_{4k+1}$ be defined by

(19) $\varphi(x, t, 0) = \tilde{R}_k(x, t)$, $\varphi(x, t, 1) = \tilde{L}_k(\tilde{p}(x), \tilde{R}_k(x, 1), t)$ and

(20) $\varphi(x, 1, s) = \tilde{R}_k(x, 1)$.

Then $\varphi(x, 0, 0) = \tilde{R}_k(x, 0) = x$, $\varphi(x, 0, 1) = \tilde{L}_k(\tilde{p}(x), \tilde{R}_k(x, 1), 0) = f_k g_k(x)$. Since $B_{4k+1} \cong Q$, there is a map $\Phi : E_{4(k+1)} \times I \times I \to B_{4k+1}$ such that

(21) $\Phi | E_{4(k+1)} \times (I \times 0 \cup 1 \times I \cup I \times 1) = \tilde{p} \varphi$ and

(22) $\Phi(x, 0, s)=\tilde{p}(x),\ s\in I$.

By HLP, there is a homotopy $\tilde{\Phi}: E_{4(k+1)}\times I\times I\to E_{4k}$ such that

(23) $\tilde{\Phi}|E_{4(k+1)}\times(I\times 0\cup 1\times I\cup I\times 1)=\varphi$ and

(24) $\tilde{p}\tilde{\Phi}=\Phi$.

Define $H_k: E_{4(k+1)}\times I\to E_{4k}$ by

(25) $H_k(x, s)=\tilde{\Phi}(x, 0, s)$.

Obviously, H_k satisfies (4) and (6). By the same way as above, we obtain $K_k: B_{4(k+1)+2}\times F_{4(k+1)+2}\times I\to B_{4k+2}\times F_{4k+2}$ satisfying (5) and (6).

Now, choose a decreasing sequence $\{\varepsilon_k\}_{k=0,1,2,\cdots}$ of positive numbers such that $\lim_{k\to\infty}\varepsilon_k=0$. By (3), (5), and (6), we can easily see that for $k=0,1,2,\cdots$ there are compact ANR neighborhoods $\tilde{F}_{4k+2}$ of F_{4k+2} in Q and extensions $\tilde{f}_k: B_{4k+2}\times\tilde{F}_{4k+2}\to E_{4k}$ and $\tilde{K}_k: B_{4(k+1)+2}\times\tilde{F}_{4(k+1)+2}\times I\to B_{4k+2}\times\tilde{F}_{4k+2}$ of f_k and K_k, respectively such that $\bigcap_{k=0}^{\infty}\tilde{F}_{4k+2}=\bigcap_{k=0}^{\infty}F_{4k+2}=p^{-1}(b)$ and

(26) $d(\tilde{p}\tilde{f}_k(y, x), y)<\varepsilon_k,\quad (y, x)\in B_{4k+2}\times\tilde{F}_{4k+2}$ and

(27) $\tilde{K}_k(z, 0)=z,\quad \tilde{K}_k(z, 1)=g_k\tilde{f}_{k+1}(z)$ and

$d(\tilde{\pi}_{4k+2}\tilde{K}_k(z, t),\ \tilde{\pi}_{4(k+1)+2}(z))<\varepsilon_k,\ z\in B_{4(k+1)+2}\times\tilde{F}_{4(k+1)+2},\ t\in I$, where

$\tilde{\pi}_{4k+2}: B_{4k+2}\times\tilde{F}_{4k+2}\to B_{4k+2}$ is the projection.

By (3), (4), (6), (26), (27) and Borsuk's homotopy extension theorem, we can easily see that there are fiber fundamental sequences over B $\underline{f}=\{f_n, B\times p^{-1}(b), E\}_{Q\times Q, Q}$ and $\underline{g}=\{g_n, E, B\times p^{-1}(b)\}_{Q, Q\times Q}$ induced by $\{\tilde{f}_k\}_{k=0,1,2,\cdots}$ and $\{g_k\}_{k=0,1,2,\cdots}$, respectively such that $\underline{g}\underline{f}\underset{B}{\simeq}\underline{1}_{B\times p^{-1}(b)}$ and $\underline{f}\underline{g}\underset{B}{\simeq}\underline{1}_E$. This completes the proof.

In [3], D. Coram and P. Duvall introduced the notion of movability for maps and they characterized approximate fibrations in terms of the movability condition. A proper map $p: E\to B$ between locally compact, separable metric spaces is *completely movable* [3] if for each $b\in B$ and each neighborhood U of the fiber $F_b=p^{-1}(b)$ in E there is a neighborhood V of F_b in U such that if $F_c=p^{-1}(c)$ is any fiber in V and W is any neighborhood of F_c in V, then there is a homotopy $H: V\times I\to U$ such that $H(x, 0)=x$, $H(x, 1)\in W$ for $x\in V$ and $H(x, t)=x$ for $x\in F_c$, $t\in I$.

THEOREM [3, Theorem 3.9]. *For a proper map $p: E\to B$ between locally compact, separable metric ANR's, then p is an approximate fibration if and only if p is completely movable.*

Now, we give another condition. A proper map $p: E \to B$ between locally compact, separable metric spaces is *locally shape trivial* if for each $b \in B$ there is a compact neighborhood V of b in B such that $p|p^{-1}(V): p^{-1}(V) \to V$ is shape trivial. We have the following.

THEOREM 1.4. *For a proper map $p: E \to B$ between locally compact, separable metric ANR's, then p is an approximate fibration if and only if p is locally shape trivial.*

PROOF. Suppose that p is an approximate fibration. Define a map $p \times 1_Q: E \times Q \to B \times Q$ by $(p \times 1_Q)(e, q) = (p(e), q)$ for $e \in E$, $q \in Q$, where $Q = \prod_{i=1}^{\infty} I_i$ ($I_i = [0, 1]$) and consider the map $p: E \to B$ as the restriction $p \times 1_Q | E \times \{0\}: E \times \{0\} \to B \times \{0\}$, where $0 = (0, 0, \cdots) \in Q$. Choose any point $b \in B$. Since $B \times Q$ is a Q-manifold, there is a compact neighborhood W of $(b, 0)$ in $B \times Q$ such that W is homeomorphic to Q. Since $p \times 1_Q$ is an approximate fibration and W is of trivial shape and the restriction of a shape fibration is a shape fibration, by Proposition 1.3 $p \times 1_Q | (p \times 1_Q)^{-1}(W): (p \times 1_Q)^{-1}(W) \to W$ is shape trivial. By [8, Proposition 1.7], the restriction $p \times 1_Q | (p \times 1_Q)^{-1}(W \cap B \times \{0\}): (p \times 1_Q)^{-1}(W \cap B \times \{0\}) \to W \cap B \times \{0\}$ is shape trivial. Note that $(p \times 1_Q)^{-1}(b, 0) = p^{-1}(b) \times \{0\}$. This implies that p is locally shape trivial.

Conversely, suppose that p is locally shape trivial. We shall show that p is completely movable. Let $b_0 \in B$ and U be any neighborhood of F_{b_0} in E. Since p is locally shape trivial, there is a compact neighborhood B_1 of b_0 in B such that $p|p^{-1}(B_1): p^{-1}(B_1) \to B_1$ is shape trivial. By [8, Proposition 1.7], we may assume that $p^{-1}(B_1) \subset U$. Choose a neighborhood $B_2 \subset B_1$ of b_0 in B such that for any $c \in B_2$ there is a homotopy $H_c: B_2 \times I \to B_1$ such that $H_c(b, 0) = b$, $H_c(b, 1) = c$, for $b \in B_2$ and $H_c(c, t) = c$, for $t \in I$. Let $V = p^{-1}(B_2)$. Let F_c be any fiber in V and W be any neighborhood of F_c in V. Define a homotopy $G: V \times I \to B_1$ by $G(e, t) = H_c(p(e), t)$, for $e \in V$, $t \in I$. Since $p: E \to B$ is a proper map, there is a positive number $\varepsilon > 0$ such that $p^{-1}(B(c; \varepsilon)) \subset W$, where $B(c; \varepsilon) = \{b \in B | d(b, c) < \varepsilon\}$. By Corollary 1.2, $p|p^{-1}(B_1): p^{-1}(B_1) \to B_1$ is a shape fibration. By [12, Proposition 1 and Remark 1], there is a homotopy $H: V \times I \to U$ such that $H(e, 0) = e$, $d(pH(e, t), G(e, t)) < \varepsilon$, for $e \in V$, $t \in I$ and $H(e, t) = e$, for $e \in F_c$, $t \in I$. Clearly, $H(V \times \{1\}) \subset W$. Thus p is completely movable, hence p is an approximate fibration.

REMARK 1.5. In the statement of Theorem 1.4, if E is not an ANR, we can not replace approximate fibration by shape fibration. Let E be the continuum which consists of all points in the plane having the polar coordinates (r, θ) for which $r = 1$, $r = 2$ or $r = (2 + e^{\theta})/(1 + e^{\theta})$ and B be the unit circle in the plane. Define a map $p: E \to B$ by $p(r, \theta) = (1, \theta)$. Since p is a locally trivial fiber space with totally disconnected fibers, it is locally shape trivial. But, p is not a shape fibration (see [13, p. 641]).

2. Fiber shape equivalences and shape of fibers.

In this section, we prove the following theorem.

THEOREM 2.1. *Let $p: E\to B$ be a shape fibration from a compactum E to a connected compact ANR B and let $p': E'\to B$ be an approximate fibration between compact ANR's. Then a fiber map $f: E\to E'$ over B is a fiber shape equivalence over B if and only if for some $b_0\in B$, $f|p^{-1}(b_0): p^{-1}(b_0)\to p'^{-1}(b_0)$ is a shape equivalence.*

We need the following propositions.

PROPOSITION 2.2. *Let $p: E\to B$ and $p': E'\to B$ be shape fibrations between compacta and let $\underline{f}=\{f_n, E, E'\}_{Q,Q}$ be a fiber fundamental sequence over B. If B is a pointed 1-movable continuum and for some $b_0\in B$, $\underline{f}|p^{-1}(b_0)=\{f_n, p^{-1}(b_0), p'^{-1}(b_0)\}_{Q,Q}$ is a shape equivalence, then for any $b\in B$, $\underline{f}|p^{-1}(b)$ is a shape equivalence.*

SKETCH OF THE PROOF. By the same way as the proof of [10, Theorem 2], there are fundamental equivalences $\underline{g}=\{g_n, p^{-1}(b), p^{-1}(b_0)\}_{Q,Q}$ and $\underline{h}=\{h_n, p'^{-1}(b_0), p'^{-1}(b)\}_{Q,Q}$. Since $\underline{f}$ is a fiber fundamental sequence over B and $p': E'\to B$ is a shape fibration, by the constructions of $\underline{g}$ and $\underline{h}$ we can conclude that $\underline{h}\circ\underline{f}|p^{-1}(b_0)\circ\underline{g}$ is homotopic to $\underline{f}|p^{-1}(b)$. Hence $\underline{f}|p^{-1}(b)$ is a shape equivalence (cf. Proposition 1.3).

Note that if B is an FAR and a Z-set in the Hilbert cube Q, there is a decreasing sequence $B_1\supset B_2\supset B_3\supset\cdots$, of compact neighborhoods of B in Q such that $\bigcap_{i=1}^{\infty} B_i=B$ and each B_i is homeomorphic to Q. The proof of the following proposition is similar to one of Proposition 1.3 and Proposition 2.2. Hence we omit it.

PROPOSITION 2.3. *Let $p: E\to B$ and $p': E'\to B$ be shape fibrations between compacta and let $\underline{f}=\{f_n, E, E'\}_{Q,Q}$ be a fiber fundamental sequence over B. If B is an FAR and for some $b_0\in B$, $\underline{f}|p^{-1}(b_0)$ is a shape equivalence, then $\underline{f}$ is a fiber shape equivalence over B.*

PROPOSITION 2.4. *Let $p: E\to B$ and $p': E'\to B$ be shape fibrations between compacta and let $f: E\to E'$ be a fiber map over B. Suppose that B is a continuum with a finite closed cover $\{B_i\}_{i=1,2,\cdots n}$ consisting of FAR's such that for each subset $\{i_1, i_2, \cdots i_s\}\subset\{1, 2, \cdots n\}$, $\bigcap_{j=1}^{s} B_{i_j}$ is a pointed 1-movable continuum (it may be the empty set) and it has a finite fundamental dimension* (i.e., $\mathrm{Fd}(\bigcap_{j=1}^{s} B_{i_j})<\infty$). *If there is a* *point* $b_0 \in B$ *such that* $\mathrm{Fd}(p^{-1}(b_0))<\infty$, *the number of the components of* $p^{-1}(b_0)$ *is finite, each component is pointed* 1-*movable and* $f|p^{-1}(b_0): p^{-1}(b_0)\to p'^{-1}(b_0)$ *is a shape equivalence, then* f *is a* (*strong*) *shape equivalence.*

PROOF. Since B is a pointed 1-movable continuum, by Proposition 2.2 for any $b\in B$, $f|p^{-1}(b)$ is a shape equivalence. Hence for any $b\in B$, $\mathrm{Sh}(p^{-1}(b))=\mathrm{Sh}(p'^{-1}(b))=\mathrm{Sh}(p^{-1}(b_0))$. In particular, the number of the components of $p'^{-1}(b)$ is equal to one of $p^{-1}(b_0)$, each component of $p'^{-1}(b)$ is pointed 1-movable and $\mathrm{Fd}(p^{-1}(b))=\mathrm{Fd}(p'^{-1}(b))=\mathrm{Fd}(p^{-1}(b_0))<\infty$. By Proposition 2.3, $f|p^{-1}(B_i): p^{-1}(B_i)\to p'^{-1}(B_i)$ is a fiber shape equivalence over B_i for each $i=1,2,\cdots n$. By [8, Proposition 1.7], $f|p^{-1}(\bigcap_{j=1}^{s} B_{ij}): p^{-1}(\bigcap_{j=1}^{s} B_{ij})\to p'^{-1}(\bigcap_{j=1}^{s} B_{ij})$ is a (unpointed) shape equivalence. By Proposition 1.3, we easily see that $p'^{-1}(\bigcap_{j=1}^{s} B_{ij})$ has a finite components and each components of $p'^{-1}(\bigcap_{j=1}^{s} B_{ij})$ is pointed 1-movable. By [5, Theorem 3.6], $f|p^{-1}(\bigcap_{j=1}^{s} B_{ij})$ is a pointed shape equivalence for each $x\in p^{-1}(\bigcap_{j=1}^{s} B_{ij})$. Also, by Proposition 1.3 $\mathrm{Fd}(p^{-1}(\bigcap_{j=1}^{s} B_{ij}))=\mathrm{Fd}(p'^{-1}(\bigcap_{j=1}^{s} B_{ij}))<\infty$. By [6, Theorem 7.3], $f|p^{-1}(\bigcap_{j=1}^{s} B_{ij})$ is a strong shape equivalence. By the proof of [9, Theorem], we can conclude that f is a (strong) shape equivalence. This completes the proof.

PROOF OF THEOREM 2.1. Define maps $f\times 1_Q: E\times Q\to E'\times Q$, $p\times 1_Q: E\times Q\to B\times Q$ and $p'\times 1_Q: E'\times Q\to B\times Q$ by $(f\times 1_Q)(e,q)=(f(e),q)$, $(p\times 1_Q)(e,q)=(p(e),q)$ and $(p'\times 1_Q)(e',q)=(p'(e'),q)$ for $e\in E$, $e'\in E'$ and $q\in Q=\prod_{i=1}^{\infty} I_i$, where $I_i=[0,1]$. Consider the map $f: E\to E'$ as the restriction $f\times 1_Q|E\times\{0\}: E\times\{0\}\to E'\times\{0\}$, where $0=(0,0,\cdots)\in Q$. Note that $f\times 1_Q$ is a fiber map over $B\times Q$ and $B\times Q$ is a compact Q-manifold. By the triangulation theorem for Q-manifolds, we may assume that $B\times Q=K\times Q$, where K is a finite polyhedron. Then $K\times Q=\bigcup_{i=1}^{n}\{\Delta_i\times Q|\Delta_i$ is a simplex of $K\}$, for some positive integer n. Note that $f\times 1_Q|(p\times 1_Q)^{-1}(b_0,0)$ is a shape equivalence. By [12, Corollary 2], $(p'\times 1_Q)^{-1}(b_0,0)$ is a pointed FANR, hence $(p\times 1_Q)^{-1}(b_0,0)$ is so. Obviously $\mathrm{Fd}((p\times 1_Q)^{-1}(b_0,0))<\infty$. By Proposition 2.4, we can conclude that $f\times 1_Q: E\times Q\to E'\times Q$ is a shape equivalence. By [8, Theorem 2.3], $f\times 1_Q$ is a fiber shape equivalence over $B\times Q$. By [8, Proposition 1.7], $f: E\to E'$ is a fiber shape equivalence over B. This completes the proof.

By [8, Theorem 2.3, Corollary 3.5 and 3.6] and Theorem 2.1, we have following theorem.

THEOREM 2.5. *Let* $p: E\to B$ *be a map from a compactum* E *to a compact ANR* B. *Then the following are equivalent.*

(1) p *is a cell-like shape fibration.*

(2) *p is a shape fibration and a shape equivalence.*

(3) *p is shape shrinkable.*

(4) *p is a hereditary shape equivalence.*

REMARK 2.6. In the statement of Theorem 2.5, the assumption about B cannot be omitted. Edwards and Hastings (Springer Lecture Note, Vol. 542, Berlin, 1976, pp. 196–200) give an example of a cell-like shape fibration $p: E \to B$ which fails to be a shape equivalence.

REMARK 2.7. In the statement (1) of Theorem 2.5, we cannot replace shape fibration by strong shape equivalence. It is well-known that there is a cell-like map $p: E \to Q$ from a continuum E to the Hilbert cube Q which fails to be a shape equivalence (see [14]). By taking the cone $C(E)$ and $C(Q) \cong Q$ over E and Q, respectively, we have a map $C(p): C(E) \to C(Q) \cong Q$ which is a cell-like map. Clearly, $C(p)$ is a strong shape equivalence. But, $C(p)$ is not shape shrinkable. We know that if a map $p: E \to B$ is shape shrinkable, it is a strong shape equivalence (see [8, Corollary 3.3]).

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