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A GENERALIZATION OF FREE L-SPACES

By

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1. Introduction.

In [11] and [12] Nagami defined the notions of L-spaces and free L-spaces respectively and for these spaces, proved fundamental theorems of dimension theory including coincidence theorem for dim and Ind. The class of L-spaces contains every Lašnev space, and the class of free L-spaces is the minimal one which is countably productive and hereditary and which contains every L-space. On the other hand the author has recently defined patched spaces ([13]) and free patched spaces ([15]) and for these spaces, proved fundamental theorems of dimension theory. A patched space is a paracompact perfectly normal space expressed as the finite union of matrizable subsets; a free patched space is a space embedded in the countable product of patched spaces. As also shown in [15], the same theorems are valid even for the class of μ-spaces which includes the class of free patched spaces (cf. [15, Added in proof]). One of the most interesting results concerning free L-spaces is an embedding theorem ([12, Theorem 3.4]) asserting that every free L-space can be embedded in the countable product of almost metric spaces (or, more strongly, of almost polyhedral spaces). By the theorem every free L-space is found to be a free patched space (but the converse is not true by [13, Example 5.1 and Remark 5.2]).

In this paper we define the notion of free L*-spaces which form a countably productive and hereditary class including that of free L-spaces. The notion of L*-spaces is also defined as a generalization of L-spaces. In Section 2 we examine basic properties of these spaces. Free L-spaces will be redefined in terms of free L*-spaces. The major part of Section 3 is devoted to the proof of a closed-embedding theorem (Theorem 3.6), the main result of this paper, asserting that every free L*-space can be embedded as a closed set in the countable product of much more simple spaces called a.e. metrizable spaces. An a.e. metrizable space is, roughly speaking, a space which is metrizable except at discrete points. It is to be noted that we do not impose on the spaces any such "approaching" condition as that imposed on almost metric spaces (cf. [12, Definition 3.1]). The technique used in the proof of our embedding theorem differs from that of Nagami's embed-

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ding theorem. Recall that in [12] Nagami proved his embedding theorem by using a Kuratowski map to the nerve of a locally finite open covering; but a certain fact (see Remark 2.7 (4)) obstructs us in using the same method. Our proof is based on contractions (=one-to-one maps) onto metric spaces which are modifiable with respect to given σ-locally finite collections of open sets (see Lemma 3.7). This method of proof also makes it possible, unexpectedly, to strengthen "embedding theorem" to "closed-embedding theorem".

Our embedding theorem says that every free L*-space is a free patched space. Though the reverse implication is not known to hold (see Problem 4.10), these two spaces are close to each other in the sense that, as our embedding theorem also asserts, every subspace of the countable product of 2-patched spaces is a free L*-space.

In the last of Section 3, fundamental theorems of dimension theory will be established for free L*-spaces as corollaries to the embedding theorem. The last section consists of examples and problems. It will be shown that a free L*-space is not necessarily a stratifiable space (and hence not necessarily a free L-space); a much stronger example is also presented.

Throughout the present paper all spaces are assumed to be *Hausdorff* topological spaces and maps to be *continuous*. The symbol N is used to denote the positive integers.

2. Free L*-spaces and L*-spaces.

Conventions. Let $\mathcal{U}$ be a collection of subsets of a space X. The symbol $\mathcal{U}^*$ denotes the union of the members of $\mathcal{U}$. Let Y be a subset of X. The symbol $\mathcal{U}|Y$ means the collection of the form $\{U\cap Y: U\in\mathcal{U}\}$. $\mathcal{U}$ is called discrete if each point of X has a neighborhood meeting at most one member of $\mathcal{U}$. $\mathcal{U}$ is called σ-discrete (resp. σ-locally finite) if $\mathcal{U}$ is the union of at most countably many discrete (resp. locally finite) collections. A subset of X is called discrete if it is discrete as a collection consisting of point sets. The symbol Cl Y (or $\bar{Y}$) denotes the closure of Y. Let $\mathcal{U}_i$, $1\leqq i\leqq k$, be collections of subsets of X. The symbol $\bigwedge_{i=1}^k\mathcal{U}_i$ means the collection of the form $\{\bigcap_{i=1}^k U_i: U_i\in\mathcal{U}_i, 1\leqq i\leqq k\}$. We sometimes use the symbol $\mathcal{U}_1\wedge\mathcal{U}_2$ in place of $\bigwedge_{i=1}^2\mathcal{U}_i$.

DEFINITION 2.1. Let X be a space and F a closed set of X. Let $\mathcal{U}$ be an open covering of $X-F$. An open set U of X is called a *$\mathcal{U}$-saturated* neighborhood of F if $U=F\cup\mathcal{V}^*$ for some subcollection $\mathcal{V}$ of $\mathcal{U}$. An open neighborhood V of F is called a *subcanonical neighborhood* of F *with respect to* $\mathcal{U}$ if there exist a sequence $\{V_i: i\in N\}$ of $\mathcal{U}$-saturated neighborhoods of F and a sequence $\{\mathcal{U}_i: i\in N\}$ of subcollections of $\mathcal{U}$ such that $V_{i+1}\subset X-\mathcal{U}_i^*\subset V_i\subset V$ for each $i\in N$.

DEFINITION 2.2. Let X be a space. Let $\mathcal{F}$ be a σ-discrete collection of closed sets of X, and for each $F\in\mathcal{F}$ let $\mathcal{U}_F$ be an open covering of $X-F$. The pair $\{\mathcal{F}, \{\mathcal{U}_F : F\in\mathcal{F}\}\}$ is said to be a *free* L*-*structure* on X if for each $F\in\mathcal{F}, \mathcal{U}_F$ is *σ-locally finite* in $X-F$ and if for each $x\in X$ and each neighborhood U of x, there are finite subcollection $\{F_1, \cdots, F_k\}$ of $\mathcal{F}$ and subcanonical neighborhoods U_i of F_i with respect to $\mathcal{U}_{F_i}$, $1\leqq i\leqq k$, such that $x\in\bigcap_{i=1}^k F_i\subset\bigcap_{i=1}^k U_i\subset U$. A space is said to be a *free* L*-*space* if it is a paracompact space admitting a free L*-structure.

DEFINITION 2.3. Let X be a space. Let $\mathcal{C}$ be the collection of all closed sets of X, and for each $F\in\mathcal{C}$ let $\mathcal{U}_F$ be an open covering of $X-F$. The collection $\{\mathcal{U}_F : F\in\mathcal{C}\}$ is called an L*-*structure* on X if for each $F\in\mathcal{C}, \mathcal{U}_F$ is *σ-locally finite* in $X-F$ and if every open neighborhood of F is a subcanonical neighborhood with respect to $\mathcal{U}_F$. A space is called an L*-*space* if it is a paracompact σ-space admitting an L*-structure.

We state the definitions of canonicity, free L-spaces and L-spaces in order to compare the corresponding notions with each other.

DEFINITION 2.4 (Nagami [12, Definition 1.1]). Let X, F and $\mathcal{U}$ be the same as in Definition 2.1. Let Y be a subset of X and let $i\in N$. The collection $\mathcal{U}(Y, i)$ is defined inductively by $\mathcal{U}(Y, 1)=\{U\in\mathcal{U} : U\cap Y\neq\emptyset\}$ and $\mathcal{U}(Y, i)=\{U\in\mathcal{U} : U\cap\mathcal{U}(Y, i-1)^*\neq\emptyset\}$. An open neighborhood V of F is called a *canonical neighborhood* of F *with respect to* $\mathcal{U}$ if for each i, Cl $(\mathcal{U}(X-V, i)^*)$ does not meet F.

DEFINITION 2.5 (Nagami [12, Definition 1.2]). Let $X, \mathcal{F}$ and $\mathcal{U}_F, F\in\mathcal{F}$, be the same as in Definition 2.2. The pair $\{\mathcal{F}, \{\mathcal{U}_F : F\in\mathcal{F}\}\}$ is called a *free* L-*structure* on X if for each $x\in X$ and each neighborhood U of x, there are finite subcollection $\{F_1, \cdots, F_k\}$ of $\mathcal{F}$ and canonical neighborhoods U_i of F_i with respect to $\mathcal{U}_{F_i}, 1\leqq i\leqq k$, such that $x\in\bigcap_{i=1}^k F_i\subset\bigcap_{i=1}^k U_i\subset U$. A space is called a *free* L-*space* if it is a paracompact space admitting a free L-structure.

DEFINITION 2.6 (cf. Nagami [11, Definitions 1.1 and 1.2]). Let $X, \mathcal{C}$ and $\mathcal{U}_F$, $F\in\mathcal{C}$, be the same as in Definition 2.3. The collection $\{\mathcal{U}_F : F\in\mathcal{C}\}$ is called an L-*structure* on X if for each $F\in\mathcal{C}$, every open neighborhood of F is a canonical neighborhood with respect to $\mathcal{U}_F$. A space is called an L-*space* if it is a paracompact σ-space admitting an L-structure.

REMARKS 2.7. (1) Definition 2.6 is an equivalent alteration of the original one by virtue of Nagami [11, Theorem 1.3].

(2) A subcanonical neighborhood of F with respect to $\mathcal{U}$ includes a $\mathcal{U}$-saturated

subcanonical neighborhood of F with respect to $\mathcal{U}$.

(3) If V is a canonical neighborhood of F with respect to $\mathcal{U}$, then V is a subcanonical neighborhood of F with respect to the same $\mathcal{U}$; indeed take for V_i the set $F\cup\{U\in\mathcal{U}: U\notin\mathcal{U}(X-V, i)\}^*$ and for $\mathcal{U}_i$ the collection $\mathcal{U}(X-V, i)$. A trivial example shows that the converse is not true. Further, unlike the case of canonical neighborhoods, a subcanonical neighborhood of F with respect to $\mathcal{U}$ is not necessarily subcanonical with respect to every refinement of $\mathcal{U}$. (To see this we have only to note that if V is a subcanonical neighborhood of F with respect to $\mathcal{U}$, then V remains subcanonical with respect to $\mathcal{U}\cup\{X-F\}$).

(4) In Definition 2.2, the collection of all finite intersections of members of $\mathcal{F}$ forms a σ-discrete net. Hence every free L*-space is a paracompact perfectly normal space and, therefore, a hereditarily paracompact space. Thus each $\mathcal{U}_F$ has a locally finite refinement. But, as suggested by the latter half of (3) and as assured by Theorem 2.8 and Example 4.1, the σ-local finiteness imposed on $\mathcal{U}_F$ in Definition 2.2 can not be replaced by local finiteness. This fact obstructs us in an analogous proof of our embedding theorem (Theorem 3.6) to that of Nagami's embedding theorem.

As a relation between free L-spaces and free L*-spaces, we have the following result, the proof of which is partly implicit in that of [11, Theorem 1.3].

THEOREM 2.8. *A space X is a free L-space if and only if X is a free L*-space with a free L*-structure $\{\mathcal{F}, \{\mathcal{U}_F: F\in\mathcal{F}\}\}$ such that for each $F\in\mathcal{F}, \mathcal{U}_F$ is locally finite in $X-F$.*

PROOF. Let X be a free L-space with a free L-structure $\{\mathcal{E}, \{\mathcal{U}_E: E\in\mathcal{E}\}\}$. Note that if U is a canonical neighborhood of E with respect to $\mathcal{U}_E$, then U is canonical with respect to every refinement of $\mathcal{U}_E$. Since X is hereditarily paracompact, we can assume that each $\mathcal{U}_E$ is locally finite in $X-E$. Now the 'only if'-part follows immediately from the first statement of Remark 2.7 (3). To show the 'if'-part let X be a free L*-space with a free L*-structure $\{\mathcal{F}, \{\mathcal{U}_F: F\in\mathcal{F}\}\}$ such that for each $F\in\mathcal{F}, \mathcal{U}_F$ is locally finite in $X-F$. Let $\mathcal{V}_F$ be the collection of all $\mathcal{U}_F$-saturated neighborhoods of F. Note that $\mathcal{V}_F$ is closure-preserving in X, and $\{X-V: V\in\mathcal{V}_F\}$ is closure-preserving in $X-F$, that is, in the terminology of [11], $\mathcal{V}_F$ is closure-preserving in both sides. Let $x\in X-F$. If $x\in\bigcap\{\bar{V}: V\in\mathcal{V}_F\}$, define $W_F(x)=(X-F)-\{X-V: x\in V, V\in\mathcal{V}_F\}^*$. If $x\notin\bigcap\{\bar{V}: V\in\mathcal{V}_F\}$, define $W_F(x)=X-(\{\bar{V}: x\in X-\bar{V}, V\in\mathcal{V}_F\}^*\cup\{X-V: x\in V, V\in\mathcal{V}_F\}^*)$. Then $W_F(x)$ is an open neighborhood of x not meeting F. Put $\mathcal{W}_F=\{W_F(x): x\in X-F\}$. We have only to show that if U is a subcanonical neighborhood of F with respect to $\mathcal{U}_F$, then U is a canonical neighborhood of F with respect to $\mathcal{W}_F$. By the definition of subcanonicity, there exist a sequence $\{U_i : i\in N\}$ of $\mathcal{U}_F$-saturated neighborhoods of F and a sequence $\{\mathcal{U}_i : i\in N\}$ of subcollections of $\mathcal{U}_F$ such that $U_{i+1}\subset X-\mathcal{U}_i^*\subset U_i\subset U$ for each i. By the definition of $W_F(x)$, if $W_F(x)\cap(X-U_i)\neq\emptyset$, then $W_F(x)\cap U_{i+1}=\emptyset$. This implies that U is canonical with respect to $\mathcal{W}_F$, which completes the proof.

Similarly we have:

THEOREM 2.9. *A space X is an L-space if and only if X is an L*-space with an L*-structure $\{\mathcal{U}_F : F\in C\}$ such that for each $F\in C$, $\mathcal{U}_F$ is locally finite in $X-F$.*

PROPOSITION 2.10. *Every subspace of a free L*-space is a free L*-space, and every countable product of free L*-spaces is a free L*-space.*

PROOF. The former statement is clear. To show the latter let $X_n, n\in N$, be free L*-spaces with free L*-structures $\{\mathcal{F}_n, \{\mathcal{U}_{F_n} : F_n\in\mathcal{F}_n\}\}$. Put $X=\prod_{n=1}^\infty X_n$. X is paracompact because every countable product of paracompact σ-spaces is a paracompact σ-space (Okuyama [16, Theorem 4.7]). Put $\mathcal{F}=\{p_n^{-1}(F_n) : F_n\in\mathcal{F}_n, n\in N\}$ and for each $F=p_n^{-1}(F_n)\in\mathcal{F}$, put $\mathcal{U}_F=\{p_n^{-1}(U) : U\in\mathcal{U}_{F_n}\}$, where $p_n : X\to X_n$ is the projection. It is then easy to check that $\{\mathcal{F}, \{\mathcal{U}_F : F\in\mathcal{F}\}\}$ is a free L*-structure on X. This completes the proof.

PROPOSITION 2.11. *Every closed subset and every open subset of an L*-space are L*-space.*

PROOF. The former statement is clear. To show the latter let X be an L*-space and G an open set of X. Using the regularity of X and the paracompactness of G, we can find a locally finite open covering $\{U_\alpha : \alpha\in A\}$ of G such that $\bar{U}_\alpha\subset G$ for each $\alpha\in A$, and further a closed covering $\{E_\alpha : \alpha\in A\}$ of G such that $E_\alpha\subset U_\alpha$ for each $\alpha\in A$. By the former statement, $\bar{U}_\alpha$ is an L*-space for each α. For each $n\in N$ define $A_n=\{\{\alpha_1, \cdots, \alpha_n\} : \alpha_1, \cdots, \alpha_n$ are mutually distinct n elements of $A\}$. Put $U(\alpha_1, \cdots, \alpha_n)=\bigcap_{j=1}^n U_{\alpha_j}-\{E_\alpha : \alpha\in A-\{\alpha_1, \cdots, \alpha_n\}\}^*$, $\mathcal{U}_n=\{U(\alpha_1, \cdots, \alpha_n) : \{\alpha_1, \cdots, \alpha_n\}\in A_n\}$ and $\mathcal{U}=\bigcup_{n=1}^\infty\mathcal{U}_n$. Then $\mathcal{U}$ is a locally finite open covering of G. Let F be a closed set of G. For each $\alpha\in A$ let $\mathcal{V}_\alpha$ be a σ-locally finite open covering of $\bar{U}_\alpha-F$ such that every open neighborhood of $F\cap\bar{U}_\alpha$ in $\bar{U}_\alpha$ is a subcanonical neighborhood with respect to $\mathcal{V}_\alpha$. For each $\{\alpha_1, \cdots, \alpha_n\}\in A_n$ put $\mathcal{V}(\alpha_1, \cdots, \alpha_n)=\bigwedge_{j=1}^n\mathcal{V}_{\alpha_j}|U(\alpha_1, \cdots, \alpha_n)$, $\mathcal{V}_n=\bigcup\{\mathcal{V}(\alpha_1, \cdots, \alpha_n) : \{\alpha_1, \cdots, \alpha_n\}\in A_n\}$ and $\mathcal{V}=\bigcup_{n=1}^\infty\mathcal{V}_n$. Then $\mathcal{V}$ is a σ-locally finite open covering of $G-F$. We shall show that every open neighborhood of F in G is a subcanonical neighborhood with respect to $\mathcal{V}$. It suffices to show that for each open neighborhood W of F in G, there are $\mathcal{V}$-saturated neighborhood H of F in G and a subcollection $\mathcal{W}$ of $\mathcal{V}$ such that $X-W\subset\mathcal{W}^*$ and $H\cap\mathcal{W}^*=\emptyset$.

For each α take a $\mathcal{V}_\alpha$-saturated neighborhood H_α of $F\cap \bar{U}_\alpha$ in $\bar{U}_\alpha$ and a subcollection $\mathcal{W}_\alpha$ of $\mathcal{V}_\alpha$ such that $\bar{U}_\alpha - W\subset \mathcal{W}_\alpha^*$ and $H_\alpha\cap \mathcal{W}_\alpha^*=\emptyset$. Now put $\mathcal{W}(\alpha_1, \cdots, \alpha_n) =\bigwedge_{j=1}^n \mathcal{W}_{\alpha_j}|U(\alpha_1, \cdots, \alpha_n)$, $\mathcal{W}_n=\cup\{\mathcal{W}(\alpha_1, \cdots, \alpha_n):\{\alpha_1, \cdots, \alpha_n\}\in A_n\}$ and $\mathcal{W}=\bigcup_{n=1}^\infty \mathcal{W}_n$. Also put $H(\alpha_1, \cdots, \alpha_n)=\bigcap_{j=1}^n H_{\alpha_j}\cap U(\alpha_1, \cdots, \alpha_n)$, $H_n=\cup\{H(\alpha_1, \cdots, \alpha_n):\{\alpha_1, \cdots, \alpha_n\}\in A_n\}$ and $H=\bigcup_{n=1}^\infty H_n$. It is easily checked that W and H satisfy the required properties; to check that $H\cap \mathcal{W}^*=\emptyset$, note that if $\{\alpha_1, \cdots, \alpha_n\}\in A_n$ and $\{\beta_1, \cdots, \beta_m\}\in A_m$ have no common element, then $U(\alpha_1, \cdots, \alpha_n)\cap U(\beta_1, \cdots, \beta_m)=\emptyset$. This completes the proof.

Whether every subset of an L*-space is an L*-space is unknown. Example 4.4 shows that even finite product of L*-spaces is not necessarily an L*-space.

A space is called a locally free L*-space (resp. a locally L*-space) if each point of the space has a neighborhood which is a free L*-space (resp. an L*-space) as a subspace. The following result is not so trivial:

PROPOSITION 2.12. *A paracompact locally free L*-space* (resp. *a paracompact locally L*-space*) *is a free L*-space* (resp. *an L*-space*).

PROOF. The latter statement has been essentially proved in the preceding proposition; indeed replace G in Proposition 2.11 by a given paracompact locally L*-space. To show the former statement, let X be a paracompact locally free L*-space. There exists a σ-discrete open covering $\mathcal{W}=\{W_\alpha:\alpha\in A\}$ of X such that for each $\alpha\in A$, $\bar{W}_\alpha$ is a free L*-space. Note that X is perfectly normal because it is a paracompact σ-space. Fix $\alpha\in A$ and put $W=W_\alpha$. We have only to construct a pair $\{\mathcal{F}, \{\mathcal{U}(F):F\in\mathcal{F}\}\}$ of σ-discrete collection $\mathcal{F}$ of closed sets of X and σ-locally finite open coverings $\mathcal{U}(F)$ of $X-F$, $F\in\mathcal{F}$, such that each member of $\mathcal{F}$ is included in $\bar{W}$ and such that for each point $x\in W$ and each neighborhood U of x in X, there are a finite subcollection $\{F_1, \cdots, F_k\}$ of $\mathcal{F}$ and subcanonical neighborhoods U_i of F_i with respect to $\mathcal{U}(F_i)$, $1\leqq i\leqq k$, such that $x\in\bigcap_{i=1}^k F_i\subset\bigcap_{i=1}^k U_i\subset U$. Write $W=\bigcup_{j=1}^\infty W_j$, where W_j are open sets of X such that $\bar{W}_j\subset W_{j+1}$, $j\in N$. Further write for each j, $X-\bar{W}_j=\bigcup_{m=1}^\infty W_{jm}$, where W_{jm} are open sets of X such that $X-W_{j+1}\subset W_{j1}$ and $\bar{W}_{jm}\subset W_{jm+1}$, $m\in N$. Let $\mathcal{U}(\bar{W}_j)$, $j\in N$, be the countable open covering of $X-\bar{W}_j$ defined by $\mathcal{U}(\bar{W}_j)=\{W_{j2}\}\cup\{W_{jm+2}-\bar{W}_{jm}:m\in N\}$. Clearly W_{j+1} is a (sub) canonical neighborhood of $\bar{W}_j$ with respect to $\mathcal{U}(\bar{W}_j)$. Let $\{\mathcal{E}, \{\mathcal{V}_E:E\in \mathcal{E}\}\}$ be a free L*-structure on $\bar{W}$. For each $E\in\mathcal{E}$, let $\mathcal{U}(E)$ be the σ-locally finite open covering of $X-E$ defined by $\mathcal{U}(E)=\{X-(\bar{W}_j\cup E):j\in N\}\cup(\bigcup_{j=1}^\infty \mathcal{V}_E|W_j)$. It is then easy to see that if G is a subcanonical neighborhood of E in $\bar{W}$ with respect to $\mathcal{V}_E$, then for every j, $G\cup(X-\bar{W}_j)$ is a subcanonical neighborhood of E in X with respect to $\mathcal{U}(E)$. Now put $\mathcal{F}=\mathcal{E}\cup\{\bar{W}_j:j\in N\}$ and consider the pair $\{\mathcal{F}, \{\mathcal{U}(F):F\in\mathcal{F}\}\}$. Clearly $\mathcal{F}$ is a σ-discrete collection of cloesd sets of X each member of which is included in $\bar{W}$, and for each $F \in \mathcal{F}$, $\mathcal{U}(F)$ is a σ-locally finite open covering of $X-F$. To show that the pair satisfies the required conditions, let x be a point in W and U a neighborhood of x. Since $\{\mathcal{E}, \{\mathcal{V}_E : E \in \mathcal{E}\}\}$ is a free L*-structure on $\bar{W}$, there exist a finite subcollection $\{E_1, \cdots, E_k\}$ of $\mathcal{E}$ and subcanonical neighborhoods G_i of E_i in $\bar{W}$ with respect to $\mathcal{V}_{E_i}$, $1 \leqq i \leqq k$, such that $x \in \bigcap_{i=1}^k E_i \subset \bigcap_{i=1}^k G_i \subset U$. Fix j so that $x \in W_j$, and note that for each $1 \leqq i \leqq k$, $(G_i \cup (X-\bar{W}_{j+1})) \cap W_{j+1} \subset G_i$. Now we have $x \in (\bigcap_{i=1}^k E_i) \cap \bar{W}_j \subset (\bigcap_{i=1}^k (G_i \cup (X-\bar{W}_{j+1}))) \cap W_{j+1} \subset U$, where as mentioned above $G_i \cup (X-\bar{W}_{j+1})$ and W_{j+1} are respectively subcanonical neighborhoods of E_i and $\bar{W}_j$ with respect to $\mathcal{U}(E_i)$ and $\mathcal{U}(\bar{W}_j)$, $1 \leqq i \leqq k$. This completes the proof.

REMARKS 2.13. It is also true that every paracompact locally free L-space is a free L-space. The proof is obtained from the above proof by replacing "free L*-" and "subcanonical" by "free L-" and "canonical" respectively and by modifying the definition of $\mathcal{U}(E)$ as follows: Write $X-E = \bigcup_{j=1}^{\infty} H_{E,j}$, where $H_{E,j}$ are open sets of X such that $\bar{H}_{E,j} \subset H_{E,j+1}$, $j \in N$. Put $\mathcal{H}_E = \{H_{E,2}\} \cup \{H_{E,j+2} - \bar{H}_{E,j} : j \in N\}$ and $\mathcal{W} = \{W_2\} \cup \{W_{j+2} - \bar{W}_j : j \in N\}$. Define $\mathcal{U}(E) = \{H_{E,2}\} \cup \{(H_{E,j+2} - \bar{H}_{E,j}) - \bar{W}_j : j \in N\} \cup (\mathcal{V}_E \wedge \mathcal{H}_E \wedge \mathcal{W})$.

3. A closed-embedding theorem and dimension for free L*-spaces.

Let X be a patched space, that is, a paracompact perfectly normal space expressed as the finite union of metrizable subsets. A finite disjoint covering of X by metrizable subsets is called a *patch* on X. (Some members of a patch may be empty sets.) $p(X)$ denotes the number $\inf\{|\Sigma| : \Sigma$ is a patch on $X\}$, where two vertical segments mean the cardinality. For a natural number n, an *n-patched space* is now defined to be a patched space X with $p(X) \leqq n$. As constructed in [13, Example 5.3] there exists, for each $n \geqq 2$, an n-patched space which is not an $(n-1)$-patched space.

The following lemma was pointed out by J. Chaber in a letter to the author (cf. [15, Lemma 4.6]).

LEMMA 3.1 (Chaber). *If X is an n-patched space, then X has a patch $\{X_i : 1 \leqq i \leqq n\}$ such that $\bigcup_{i=1}^j X_i$ is an open set of X for each $j = 1, \cdots, n$.*

PROOF. Let Σ be a patch on X with $|\Sigma| = n$. Write $\Sigma = \{M_k : 1 \leqq k \leqq n\}$ and put $\bar{\Sigma} = \{\bar{M}_k : 1 \leqq k \leqq n\}$. By Corson and Michael [3, Lemma 4.4] there exists for each k, a σ-locally finite collection $\mathcal{U}_k$ of open sets of $\bar{M}_k$ such that for each $x \in M_k$ and each neighborhood V of x in $\bar{M}_k$, there is some member $U \in \mathcal{U}_k$ with $x \in U \subset V$.

For each $i=1, 2, \cdots, n$, define $X_i=\{x\in X: x$ is contained in precisely i members of $\Sigma\}$. Then for each $j=1, 2, \cdots, n$, $\bigcup_{i=1}^{j} X_i$ is an open set of X and $\bigcup_{i=1}^{n} X_i=X$. To show that each X_i is metrizable, fix i and put $M(k_1, \cdots, k_i)=(\bigcap_{m=1}^{i} \bar{M}_{k_m})\cap X_i$ for $1\leqq k_1<k_2<\cdots<k_i\leqq n$. Clearly $\{M(k_1, \cdots, k_i): 1\leqq k_1<k_2<\cdots<k_i\leqq n\}$ is a disjoint open covering of X_i. Since $M(k_1, \cdots, k_i)\subset\{M_{k_m}: 1\leqq m\leqq i\}^*$ and $M(k_1, \cdots, k_i)\subset\bigcap_{m=1}^{i} \bar{M}_{k_m}$, $(\bigcup_{m=1}^{i} \mathcal{U}_{k_m})|M(k_1, \cdots, k_i)$ is a σ-locally finite base of $M(k_1, \cdots, k_i)$; thus $M(k_1, \cdots, k_i)$ is metrizable by the Nagata-Smirnov metrization theorem. Consequently X_i is metrizable, which completes the proof.

PROPOSITION 3.2. *Every 2-patched space is a free L*-space.*

PROOF. Let X be a 2-patched space. By the preceding lemma there exists a closed set M of X such that M and $X-M$ are both metrizable. Let $\mathcal{H}=\bigcup_{i=1}^{\infty} \mathcal{H}_i$ be a base of M such that for each i, $\mathcal{H}_i$ is discrete in M. By the collectionwise normality of X, we can find for each i, a discrete collection $\mathcal{G}_i$ of open sets of X such that $\mathcal{G}_i|M=\mathcal{H}_i$. Put $\mathcal{G}=\bigcup_{i=1}^{\infty} \mathcal{G}_i$ and write $\mathcal{G}=\{G_\alpha: \alpha\in A\}$. For each $\alpha\in A$ write $G_\alpha=\bigcup_{j=1}^{\infty} G_{\alpha j}$, where $G_{\alpha j}$ are open set of X such that $\bar{G}_{\alpha j}\subset G_{\alpha j+1}$, $j\in N$. We can find for each $\alpha\in A$ and $j\in N$, a countable open covering $\mathcal{U}(\bar{G}_{\alpha j})$ of $X-\bar{G}_{\alpha j}$ such that $G_{\alpha j+1}$ is a subcanonical neighborhood of $\bar{G}_{\alpha j}$ with respect to $\mathcal{U}(\bar{G}_{\alpha j})$ (see the construction of $\mathcal{U}(\bar{W}_j)$ in Proposition 2.12). On the other hand let $\mathcal{U}(M)$ be a σ-discrete base of $X-M$. Note that every open neighborhood of M is a subcanonical neighborhood with respect to $\mathcal{U}(M)$. By the perfect normality of X, we can assume that $\mathcal{U}(M)$ is σ-discrete in X. Write $\mathcal{U}(M)=\{V_\beta: \beta\in B\}$. For each $\beta\in B$, write $V_\beta=\bigcup_{k=1}^{\infty} F_{\beta k}$ where $F_{\beta k}$, $k\in N$, are closed sets of X. We can find for each $\beta\in B$ and $k\in N$, a countable open covering $\mathcal{U}(F_{\beta k})$ of $X-F_{\beta k}$ such that V_β is a subcanonical neighborhood of $F_{\beta k}$ with respect to $\mathcal{U}(F_{\beta k})$. Now define a σ-discrete collection $\mathcal{F}$ of closed sets of X by $\mathcal{F}=\{M\}\cup\{\bar{G}_{\alpha j}: \alpha\in A, j\in N\}\cup\{F_{\beta k}: \beta\in B, k\in N\}$, and consider the pair $\{\mathcal{F}, \{\mathcal{U}(F): F\in\mathcal{F}\}\}$. To show that the pair is a free L*-structure on X, let x be a point of X and U a neighborhood of x. In case $x\in X-M$, take $\beta\in B$ so that $x\in V_\beta\subset U$. Further take k so that $x\in F_{\beta k}$. Then we have $x\in F_{\beta k}\subset V_\beta\subset U$, where as stated above, V_β is a subcanonical neighborhood of $F_{\beta k}$ with respect to $\mathcal{U}(F_{\beta k})$. In case $x\in M$, take $\alpha\in A$ so that $x\in G_\alpha\cap M\subset U$. Further take j so that $x\in G_{\alpha j}$. Put $W=X-(\bar{G}_{\alpha j+1}-U)$; then W is an open neighborhood of M and hence, as noted above, a subcanonical neighborhood of M with respect to $\mathcal{U}(M)$. Now we have $x\in\bar{G}_{\alpha j}\cap M\subset G_{\alpha j+1}\cap W\subset U$, where as stated before, $G_{\alpha j+1}$ is a subcanonical neighborhood of $\bar{G}_{\alpha j}$ with respect to $\mathcal{U}(\bar{G}_{\alpha j})$. This completes the proof.

REMARK 3.3. As will be seen in Examples 4.5 and 4.1, a 2-patched space is not necessarily an L*-space, and also not necessarily a free L-space. The author does not know whether every n-patched space, $n\geqq 3$, is a free L*-space; the above proof seems not to be extended to induction.

DEFINITION 3.4. A space X is an *a.e. metrizable space* if X is a paracompact perfectly normal space including a discrete set D such that $X-D$ is metrizable.

An almost metric space defined by Nagami [12, Definition 3.1] is an a.e. metrizable space. More precisely a space is an almost metric space if and only if it is an a.e. metrizable L-space (cf [12, Lemma 3.2]). An a.e. metrizable space is clearly a 2-patched space. As will be seen in Example 4.2, there is an a.e. metrizable space which is not even a stratifiable space.

PROPOSITION 3.5. *An a.e. metrizable space is an L*-space.*

PROOF. Let X be an a.e. metrizable space with a discrete set D whose complement is metrizable. By the perfect normality, X admits a σ-locally finite net. Let F be a closed set of X. Let $\mathcal{G}$ be a σ-locally finite base of $X-(D\cup F)$, and take open sets V and W of X such that $D-F\subset V\subset\bar{V}\subset W\subset\bar{W}\subset X-F$. Put $\mathcal{U}=\{W\}\cup(\mathcal{G}|(X-\bar{V}))$. Then $\mathcal{U}$ is a σ-locally finite open covering of $X-F$; further it is clear that every open neighborhood of F is a subcanonical neighborhood with respect to $\mathcal{U}$. This completes the proof.

We can now state a closed-embedding theorem for free L*-spaces.

THEOREM 3.6. *The following five statements about a space X are equivalent.*

(1) *X is a free L*-space.*

(2) *X is embedded as a closed set in the countable product of a.e. metrizable spaces.*

(3) *X is embedded in the countable product of a.e. metrizable spaces.*

(4) *X is embedded in the countable product of 2-patched spaces.*

(5) *X is embedded in the countable product of L*-spaces.*

The implications (2)→(3)→(4) are clear. The implication (4)→(1) is a consequence of Proposition 2.10 and Proposition 3.2. The implications (3)→(5) and (5)→(1) follow from Proposition 3.5 and Proposition 2.10 respectively. Before verifying the implication (1)→(2), we need some preliminaries.

A space X is called *submetrizable* if there is a contraction (=one-to-one map) from X onto some metric space. Recall that a paracompact space whose square has a G_δ-diagonal is submetrizable ([1, Lemma 8.2]). Since every paracompact σ-

space has such a property ([16, Theorem 4.6]), every free L*-space is submetrizable. The following lemma plays a key role in constructing an embedding map.

LEMMA 3.7. *Let X be a submetrizable space and $\mathcal{U}$ a σ-locally finite collection of cozero sets of X. Then there exist a metric space M and a contraction f from X onto M such that $f(U)$ is an open set of M for every $U \in \mathcal{U}$.*

PROOF. In case $\mathcal{U}$ is σ-discrete, the lemma has been proved in [13, Lemma 3.1]. Thus we have only to check the following lemma.

LEMMA 3.8. *Let X be a space and $\mathcal{U}$ a σ-locally finite collection of cozero sets of X. Then there exists a σ-discrete collection $\mathcal{V}$ of cozero sets of X such that each member of $\mathcal{U}$ is the union of some members of $\mathcal{V}$.*

PROOF. Write $\mathcal{U}=\bigcup_{i=1}^{\infty}\mathcal{U}_i$, where each $\mathcal{U}_i$ is locally finite in X, and put $\mathcal{U}_{ij}=\{U_1\cap\cdots\cap U_j : U_1, \cdots, U_j$ are distinct j members of $\mathcal{U}_i\}$. Since $\mathcal{U}_{ij}$ is a locally finite covering of cozero sets of $\mathcal{U}_{ij}^*$, there is a σ-discrete covering $\mathcal{V}_{ij}$ of cozero sets of $\mathcal{U}_{ij}^*$ which refines $\mathcal{U}_{ij}$ (cf. [9, 2-27]). It is then clear that each member of $\mathcal{U}_i$ is the union of some members of $\bigcup_{j=1}^{\infty}\mathcal{V}_{ij}$. Since $\mathcal{U}_{ij}^*$ is a cozero set of X, we can write $\mathcal{U}_{ij}^*=\bigcup_{k=1}^{\infty} W_{ijk}$, where each W_{ijk} is a cozero set of X such that $\overline{W}_{ijk}\subset\mathcal{U}_{ij}^*$. Now put $\mathcal{V}_{ijk}=\mathcal{V}_{ij}|W_{ijk}$ and $\mathcal{V}=\cup\{\mathcal{V}_{ijk} : i, j, k\in N\}$. Then $\mathcal{V}$ is a σ-discrete collection of X consisting of cozero sets of X and satisfying the required condition. This completes the proof.

LEMMA 3.9. *Let X be a space and F a closed set of X. Let $\mathcal{U}$ and $\mathcal{V}$ be open coverings of $X-F$. If U and V are subcanonical neighborhoods of F with respect to $\mathcal{U}$ and $\mathcal{V}$ respectively, then $U\cap V$ is a subcanonical neighborhood of F with respect to $\mathcal{U}\wedge\mathcal{V}$. Further if $\mathcal{W}$ is an open covering of $X-F$ which is closed under finite intersections (that is, the intersection of any finite members of $\mathcal{W}$ is again a member of $\mathcal{W}$), then the intersection of finitely many subcanonical neighborhoods of F with respect to $\mathcal{W}$ is again subcanonical with respect to $\mathcal{W}$.*

PROOF. The first statement is easily checked, and the second is a consequence of the first with $\mathcal{U}=\mathcal{V}$; this completes the proof.

In contrast with the case of canonical neighborhoods, an easy example shows that even if U and V are subcanonical neighborhoods of F with respect to a common $\mathcal{U}$, $U\cap V$ is not necessarily subcanonical with respect to $\mathcal{U}$.

Proof of the implication (1)→(2) in Theorem 3.6. Let X be a free L*-space with a free L*-structure $\{\mathcal{F}, \{\mathcal{U}_F : F\in\mathcal{F}\}$. Write $\mathcal{F}=\bigcup_{i=1}^{\infty}\mathcal{F}_i$, where $\mathcal{F}_i$ is a discrete collection of closed sets of X, and write $\mathcal{F}_i=\{F(i,\alpha):\alpha\in A_i\}$. For each $i\in N$ let $\mathcal{V}_i=\{V(i,\alpha):\alpha\in A_i\}$ be a discrete collection of open sets of X such that $F(i,\alpha)\subset V(i,\alpha)$ for each $\alpha\in A_i$. For each $i\in N$ and $\alpha\in A_i$, let $G^j_{i\alpha}$, $j\in N$, be open sets of X such that $\mathrm{Cl}\, G^1_{i\alpha}\subset V(i,\alpha)$, $\mathrm{Cl}\, G^{j+1}_{i\alpha}\subset G^j_{i\alpha}$, $j\in N$, and $F(i,\alpha)=\bigcap_{j=1}^{\infty} G^j_{i\alpha}$.

We first construct for each $i\in N$ and $\alpha\in A_i$, an open covering $\mathcal{U}_{i\alpha}$ of $V(i,\alpha)-F(i,\alpha)$ satisfying the following conditions:

(i) *$\mathcal{U}_{i\alpha}$ is closed under finite intersections.*

(ii) *If U is a subcanonical neighborhood of $F(i,\alpha)$ in X with respect to $\mathcal{U}_{F(i,\alpha)}$, then for every $j\in N$, $U\cap G^j_{i\alpha}$ is a subcanonical neighborhood of $F(i,\alpha)$ in $V(i,\alpha)$ with respect to $\mathcal{U}_{i\alpha}$.*

(iii) *For every $j\in N$, $G^j_{i\alpha}$ is a $\mathcal{U}_{i\alpha}$-saturated subcanonical neighborhood of $F(i,\alpha)$ in $V(i,\alpha)$ with respect to $\mathcal{U}_{i\alpha}$.*

(iv) *$\mathcal{U}_{i\alpha}$ is σ-locally finite in $X-\mathcal{F}_i^*$.*

To construct this, put $\mathcal{G}_{i\alpha}=\{G^j_{i\alpha}-F(i,\alpha):j\in N\}\cup\{X-\mathrm{Cl}\, G^j_{i\alpha}:j\in N\}$. Then for each j, $G^j_{i\alpha}$ is a $\mathcal{G}_{i\alpha}$-saturated subcanonical neighborhood of $F(i,\alpha)$ with respect to $\mathcal{G}_{i\alpha}$. Hence by the first part of Lemma 3.9, if U is a subcanonical neighborhood of $F(i,\alpha)$ with respect to $\mathcal{U}_{F(i,\alpha)}$, then for every j, $U\cap G^j_{i\alpha}$ is a subcanonical neighborhood of $F(i,\alpha)$ with respect to $\mathcal{U}_{F(i,\alpha)}\wedge\mathcal{G}_{i\alpha}$. Now put $\mathcal{U}_{i\alpha}=(\bigcup_{k=1}^{\infty}\bigwedge_{m=1}^{k}\mathcal{U}_{i\alpha m})| V(i,\alpha)$, where $\mathcal{U}_{i\alpha m}=\mathcal{U}_{F(i,\alpha)}\wedge\mathcal{G}_{i\alpha}$ for every $m\in N$. Then $\mathcal{U}_{i\alpha}$ is a desired open covering of $V(i,\alpha)-F(i,\alpha)$ satisfying (i), (ii), (iii) and (iv); this completes the first construction.

We next find a σ-locally finite open covering $\mathcal{H}_c$ of X satisfying the following condition:

(v) *If x, y are distinct points of X, then there exists $H, H'\in\mathcal{H}_c$ such that $x\in H$, $y\in H'$ and $H\cap H'=\emptyset$.*

To find this, recall that X is submetrizable (see the remark preceding Lemma 3.7), that is, there is a contraction h from X onto some metric space S. Let $\mathcal{O}$ be a σ-locally finite base of S and put $\mathcal{H}_c=\{h^{-1}(O):O\in\mathcal{O}\}$. Then $\mathcal{H}_c$ is a σ-locally finite open covering of X satisfying (v).

Let us now consider for each $i\in N$, the σ-locally finite open covering $\mathcal{P}_i$ of $X-\mathcal{F}_i^*$ defined by

$\mathcal{P}_i=(\mathcal{H}_c|(X-\mathcal{F}_i^*))\cup(\cup\{\mathcal{U}_{i\alpha}:\alpha\in A_i\})\cup\{X-\{\mathrm{Cl}\, G^j_{i\alpha}:\alpha\in A_i\}^*:j\in N\}$. Using Lemma 3.7 together with the perfect normality of X, we obtain for each $i\in N$, a metric space M_i and a contraction g_i from $X-\mathcal{F}_i^*$ onto M_i such that for every $P\in\mathcal{P}_i$, $g_i(P)$ is an open set of M_i. Let X_i be the disjoint sum of A_i and M_i. The topology on X_i is given so that

(vi) *M_i is an open set of X_i and the original topology on M_i is not disturbed,* and

(vii) *each point* $\alpha\in A_i$ *has an open neighborhood base of the form* $\{\{\alpha\}\cup g_i(G-F(i,\alpha)): G$ *is a* $\mathcal{U}_{i\alpha}$*-saturated subcanonical neighborhood of* $F(i,\alpha)$ *in* $V(i,\alpha)$ *with respect to* $\mathcal{U}_{i\alpha}\}$.

The topology is well-defined by (i), by the latter half of Lemma 3.9, and by the fact that $\mathcal{U}_{i\alpha}\subset\mathcal{P}_i$.

Let $f_i: X\to X_i$ be the onto map defined by $f_i(x)=g_i(x)$ if $x\in X-\mathcal{F}_i^*$, and $f_i(x)=\alpha$ if $x\in F(i,\alpha)$. Clearly f_i is continuous.

ASSERTION 3.10. *For each* $i\in N$, X_i *is a.e. metrizable.*

PROOF. Fix $i\in N$. First note that for each j, $\{f_i(G^j_{i\alpha}): \alpha\in A_i\}$ is a discrete collection of open sets of X_i; the openness follows from (iii) and (vii), and the discreteness is a consequence of the fact that $f_i(X-\{\mathrm{Cl}\, G^1_{i\alpha}: \alpha\in A_i\}^*)$ and $f_i(\mathcal{U}^*_{i\alpha})$, $\alpha\in A_i$, are open sets of X_i (by (vi) and the definitions of $\mathcal{P}_i$ and g_i). Particularly A_i is a discrete set of X_i. By (vi) $X_i-A_i=M_i$ is metrizable. To show that X_i is Hausdorff, let x, y be distinct points of X_i. We have only to consider the case when $x=\beta, \beta\in A_i$, and $y\in M_i$. Take j so that $f_i^{-1}(y)\in X-\{\mathrm{Cl}\, G^j_{i\alpha}: \alpha\in A_i\}^*$. Then $f_i(G^j_{i\beta})$ and $f_i(X-\{\mathrm{Cl}\, G^j_{i\alpha}: \alpha\in A_i\}^*)$ are disjoint open sets of X_i containing x and y respectively; hence X_i is Hausdorff. We next show that X_i is regular at each point of A_i. (This is the most essential part of the proof depending on subcanonicity.) Let $\beta\in A_i$ and let V be an arbitrary neighborhood of β in X_i. By (iii) and (vii) there is a subcanonical neighborhood G of $F(i,\beta)$ in $V(i,\beta)$ with respect to $\mathcal{U}_{i\beta}$ such that $G\subset f_i^{-1}(V)\cap G^1_{i\beta}$. There exist by the definition of subcanonicity, a $\mathcal{U}_{i\beta}$-saturated subcanonical neighborhood V_G of $F(i,\beta)$ with respect to $\mathcal{U}_{i\beta}$ and a subcollection $\mathcal{U}_G$ of $\mathcal{U}_{i\beta}$ such that $V(i,\beta)-G\subset\mathcal{U}^*_G$ and $V_G\cap\mathcal{U}^*_G=\emptyset$. To show that $\mathrm{Cl}\, f_i(V_G)\subset f_i(G)$, let $x\in X_i-f_i(G)$. If $x\in f_i(V(i,\beta))$, then some member $U\in\mathcal{U}_G$ contains $f^{-1}(x)$; while since $\mathcal{U}_G\subset\mathcal{U}_{i\beta}\subset\mathcal{P}_i$, $f_i(U)$ is an open set of X_i. Consequently $f_i(U)$ is an open neighborhood of x not meeting $f_i(V_G)$. If $x\in X_i-f_i(V(i,\beta))$, then $f_i(X-\mathrm{Cl}\, G^1_{i\beta})$ is an open set of X_i containing x but not meeting $f_i(V_G)$. Hence X_i is regular at each point of A_i. To show that X_i is paracompact, let $\mathcal{D}$ be an open covering of X_i. There exists for each $\alpha\in A_i$, an open neighborhood $D(i,\alpha)$ of α included in some member of $\mathcal{D}$ and also included in $f_i(G^1_{i\alpha})$. By the regularity shown above, we can find for each $\alpha\in A_i$, an open neighborhood $W(i,\alpha)$ of α such that $\mathrm{Cl}\, W(i,\alpha)\subset D(i,\alpha)$. Put $T=X_i-\{\mathrm{Cl}\, W(i,\alpha): \alpha\in A_i\}^*$; then T is an open set of X_i because $\{f_i(G^1_{i\alpha}): \alpha\in A_i\}$ is discrete in X_i. Put $F=X_i-\{W(i,\alpha): \alpha\in A_i\}^*$. Since M_i is metrizable, there is a locally finite open covering $\mathcal{D}'$ of F which refines $\mathcal{D}|F$. Now $\{D(i,\alpha): \alpha\in A_i\}\cup(\mathcal{D}'|T)$ is a locally finite open covering of X_i which refines $\mathcal{D}$. Thus X_i is paracompact. The perfect normality of X_i follows from the fact that X_i is normal and that X_i is the countable union of the closed sets A_i and $X_i-\{f_i(G^j_{i\alpha}):\alpha\in A_i\}^*$, $j\in N$, each of which is metrizable and hence perfectly normal. This completes the proof of Assertion 3.10.

Let $\mathcal{U}_c$ be the σ-locally finite open covering of X defined by

$$\mathcal{U}_c=\mathcal{H}_c\cup\{X-\{\mathrm{Cl}\,G^j_{i\alpha}:\alpha\in A_i\}^*:\ j,i\in N\}\cup\{G^1_{i\alpha}:\alpha\in A_i,\ i\in N\}\,.$$

By Lemma 3.7 there exist a metric space X_c and a contraction f_c from X onto X_c such that for every $U\in\mathcal{U}_c$, $f_c(U)$ is an open set of X_c.

Now define a map $f:X\to X_c\times\prod_{i=1}^{\infty}X_i$ by

$$f(x)=(f_c(x),\ f_1(x),\ f_2(x),\ \cdots)\in X_c\times\prod_{i=1}^{\infty}X_i$$

for each $x\in X$. f is continuous because each factor is. f is one-to-one because f_c is. Further

Assertion 3.11. *$f:X\to X_c\times\prod_{i=1}^{\infty}X_i$ is an into homeomorphism.*

Proof. We have only to show that f is an open map to $f(X)$. Let $x\in X$ and U a neighborhood of x. By the definition of free L*-structures, we can find a finite subcollection $\{F_1,\cdots,F_k\}$ of $\mathcal{F}$ and subcanonical neighborhoods U_j of F_j with respect to $\mathcal{U}_{F_j}$, $1\leqq j\leqq k$, such that $x\in\bigcap_{j=1}^{k}F_j\subset\bigcap_{j=1}^{k}U_j\subset U$. Let $F_j=F(i(j),\alpha(j))\in\mathcal{F}_{i(j)}$, where $\alpha(j)\in A_{i(j)}$ and $i(j)\in N$. By (ii) and Remark 2.7 (2), there exists for each $j=1,2,\cdots,k$, a $\mathcal{U}_{i(j)\alpha(j)}$-saturated subcanonical neighborhood G_j of $F(i(j),\alpha(j))$ in $V(i(j),\alpha(j))$ with respect to $\mathcal{U}_{i(j)\alpha(j)}$ such that $G_j\subset U_j$. By (vii) $f_{i(j)}(G_j)$ is an open neighborhood of $\alpha(j)$ in $X_{i(j)}$ for each $j=1,2,\cdots,k$. Further since $f^{-1}_{i(j)}f_{i(j)}(G_j)=G_j$, we have $(\prod\{f_{i(j)}(G_j):1\leqq j\leqq k\}\times\prod\{X_i:i\in N-\{i(1),\cdots,j(k)\}\}\times X_c)\cap f(X)=f(\bigcap_{j=1}^{k}G_j)$. Consequently $f(\bigcap_{j=1}^{k}G_j)$ is an open neighborhood of $f(x)$ in $f(X)$ included in $f(U)$. Thus f is an open map to $f(X)$; this completes the proof.

Finally we show

Assertion 3.12. *$f(X)$ is a closed set of $X_c\times\prod_{i=1}^{\infty}X_i$.*

Proof. Let $y\in(X_c\times\prod_{i=1}^{\infty}X_i)-f(X)$. Let y_c and y_i, $i\in N$, be respectively the X_c-coordinate and the X_i-coordinate of y. Then one of the following two cases occurs; either $f_c^{-1}(y_c)\neq f_i^{-1}(y_i)$ for some i with $y_i\in X_i-A_i$, or $f_c^{-1}(y_c)\in X-f_i^{-1}(y_i)$ for some i with $y_i\in A_i$. In the first case we can find, by (v), H_c and H_i of $\mathcal{H}_c$ such that $f_c^{-1}(y_c)\in H_c$, $f_i^{-1}(y_i)\in H_i$ and $H_c\cap H_i=\emptyset$. By the definition of $\mathcal{U}_c$, $f_c(H_c)$ is an open neighborhood of y_c in X_c, and by the definition of $\mathcal{P}_i$, $f_i(H_i\cap(X-\mathcal{F}_i^*))$ is an open neighborhood of y_i in X_i. Hence $f_c(H_c)\times f_i(H_i\cap(X-\mathcal{F}_i^*))\times\prod\{X_n:n\in N-\{i\}\}$

is an open neighborhood of y not meeting $f(X)$. In the second case, let $f_i^{-1}(y_i)=F(i,\beta)$, $y_i=\beta\in A_i$. If $f_c^{-1}(y_c)\in X-\mathcal{G}_i^*$, take j so that $f_c^{-1}(y_c)\in X-\{\mathrm{Cl}\, G_{i\alpha}^j:\alpha\in A_i\}^*$. Then $f_c(X-\{\mathrm{Cl}\, G_{i\alpha}^j:\alpha\in A_i\}^*)\times f_i(G_{i\beta}^j)\times\prod\{X_n:n\in N,\ n\neq i\}$ is an open neighborhood of y not meeting $f(X)$. If $f_c^{-1}(y_c)\in F(i,\gamma)$ for some $\gamma\in A_i-\{\beta\}$, then $f_c(G_{i\gamma}^1)\times f_i(G_{i\beta}^1)\times\prod\{X_n: n\in N,\ n\neq i\}$ is an open neighborhood of y not meeting $f(X)$. Hence $f(X)$ is a closed set of $X_c\times\prod_{i=1}^\infty X_i$, which completes the proof of Assertion 3.12. We complete the proof of Theorem 3.6.

REMARK 3.13. (1) An analogous (but slightly modified) method allows us to embed every free L-space as a closed set in the countable product of almost metric spaces.

(2) It is to be noted that $\mathcal{H}_c, f_c$ and X_c are only needed to strengthen an embedding to a closed embedding; indeed the map $\prod_{i=1}^\infty f_i: X\to\prod_{i=1}^\infty X_i$ is an embedding map by itself.

(3) If $\dim X$ is not greater than n, then we can make $\dim X_i$, $i\in N$, and $\dim X_c$ not greater than n. This is possible by applying Pasynkov's factorization theorem ([18, Theorem 29]) to the maps $g_i: X-\mathcal{G}_i^*\to M_i$, $i\in N$, and $f_c: X\to X_c$ respectively.

Theorem 3.6 particularly says

COROLLARY 3.14. *Every free L*-space is a free patched space.*

This gives us fundamental theorems of dimension theory for free L*-spaces.

COROLLARY 3.15. *Let X be a free L*-space. Then the following four statements about X are equivalent.*

(1) $\dim X\leqq n$.

(2) *X is the image of a free L*-space X_0 with* $\dim X_0\leqq 0$ *by a closed map of* $\mathrm{ord}\leqq n+1$.

(3) *X is the union of $n+1$ subsets X_i, $1\leqq i\leqq n+1$, with* $\dim X_i\leqq 0$.

(4) $\mathrm{Ind}\, X\leqq n$.

PROOF. The equivalences of (1), (3) and (4) are direct consequences of Corollary 3.14 and [15, Theorem 1.3]. The implication (2)→(3) follows from Nagami [10, Lemma 4]. The implication (1)→(2) is essentially proved in [15, Theorem 1.3]: To outline this, first note that the following analogue of [15, Proposition 2.9] is valid by virtue of Theorem 3.6 and Proposition 2.10.

A space Y is a free L*-space if and only if it is the limit of an inverse system $\{Y_i, g_i: i\in N\}$ such that Y_1 is a metric space, each Y_i is a patched free L*-space and each $g_i: Y_{i+1}\to Y_i$ is an approximating contraction.

Now the proof of (1)→(2) of Corollary 3.15 is the same as the proof of (1)→(2) of [15, Theorem 1.3] under the replacements of "free patched space" and "patched space" by "free L*-space" and "patched free L*-space" respectively. This completes the proof of Corollary 3.15.

COROLLARY 3.16. *Let X be a free L*-space and Y a subset of X. Then there exists a G_δ-set Z of X such that $Y \subset Z$ and* $\dim Z = \dim Y$.

PROOF. This is immediate from Corollary 3.14 and [15, Theorem 1.4].

COROLLARY 3.17. *Every free L*-space is the perfect image of a free L*-space of* $\dim \leqq 0$.

PROOF. Let X be a free L*-space. By Theorem 3.6 X can be regarded as a (closed) subset of the countable product of 2-patched spaces X_i, $i \in N$. By [15, Proposition 2.5] there exists for each $i \in N$, an approximating contraction f_i from X_i onto a metric space Z_i. By a theorem of Morita [8], Z_i is the image of a metric space Y_i with $\dim Y_i \leqq 0$ by a perfect map g_i. Put $T_i = \{(y, x) \in Y_i \times X_i : g_i(y) = f_i(x)\} \subset Y_i \times X_i$, and let $r_i : T_i \to Y_i$, $s_i : T_i \to X_i$ be the restrictions to T_i of the projections. It follows from [15, Lemma 2.6] that r_i is an approximating contraction, and hence $\dim T_i \leqq 0$ by [15, Proposition 2.4]. By Nagami [10, Lemma 3] we have $\dim \prod_{i=1}^{\infty} T_i \leqq 0$. s_i is a perfect map because g_i is. Now define a perfect map $s : \prod_{i=1}^{\infty} T_i \to \prod_{i=1}^{\infty} X_i$ by $s((t_i)) = (s_i(t_i))$ for $(t_i) \in \prod_{i=1}^{\infty} T_i$. Put $T = s^{-1}(X)$ and $t = s|T$. Clearly t is a perfect map and $\dim T \leqq 0$. Since each T_i is a 2-patched space, T is a free L*-space by Theorem 3.6. This completes the proof.

4. Examples and problems.

EXAMPLE 4.1. *There is a free L*-space which is not a free L-space*:

Let S be Heath's butterfly space ([4]); S is the subset of the Euclidean plane of the form $S_1 \cup S_2$ where $S_1 = \{(x, 0) : x \text{ is irrational}\}$ and $S_2 = \{(x, y) : y > 0$ and both of x and y are rationals$\}$; the topology on S is given so that each point in S_2 has a usual neighborhood base in the Euclidean topology and so that each point $(x, 0)$ in S_1 has the neighborhood base of the form $\{U_n(x) : n \in N\}$ where $U_n(x) = \{(x', y') \in S : y' < |x - x'| < 1/n \text{ or } (x', y') = (x, 0)\}$. As was proved by Heath [4], S is then a cosmic space (=a regular space with a countable net) but not a stratifiable space. Since every free L-space is stratifiable (cf. [11, Theorem 1.7] and [12, Theorem 3.4]), S is not a free L-space. On the other hand since a cosmic space is paracompact and perfectly normal and since $\{S_1, S_2\}$ is a patch on S, we see that S is a 2-patched space and, therefore, a free L*-space by Proposition 3.2.

Further we obtain the following stronger example.

EXAMPLE 4.2. *There is a countable regular space X with a point p such that $X-\{p\}$ is metrizable but X is not stratifiable*:

Let S be Heath's space described above. Clearly S_1 is a closed set of S. For each $n\in N$ put $K_n=\{\pm m/n : m\in N\}\subset R$ and let $\mathcal{K}_n$ be the collection of all components of $R-K_n$, where R is the real line indentified with the x-axis of the Euclidean plane. Then $\mathcal{K}_n|S_1$ is a disjoint open covering of S_1. Let $\mathcal{D}_n$ be the upper semi-continuous decomposition on S defined by $\mathcal{D}_n=(\mathcal{K}_n|S_1)\cup\{\{s\} : s\in S_2\}$. Let T_n be its decomposition space and let $t_n : S\to T_n$ be the closed map naturally induced. Clearly each T_n is an a.e. metrizable space consisting of countable points. Note that for each point $s\in S$ and each neighborhood U of s in S, there exist $n\in N$ and an open set V of T_n such that $s\in t_n^{-1}(V)\subset U$. This implies that $\prod_{n=1}^{\infty} t_n : S\to\prod_{n=1}^{\infty} T_n$ is an into homeomorphism. Recall that stratifiability is a countably productive and hereditary property ([2, Theorems 2.3 and 2.4]). Since S is not stratifiable, T_n is not stratifiable for some n (in fact for every n). Fix such n. Since every para-compact locally stratifiable space is stratifiable ([2, Theorem 2.6]), we can find a point $p\in T_n$ and a neighborhood W of p in T_n such that $W-\{p\}$ is metrizable but W is not stratifiable.

REMARK 4.3. In answer to a question raised by Borges [1], Heath presented in [5] a countable regular space which is not stratifiable. His space is, however, nowhere first countable and, therefore, not a.e. metrizable.

EXAMPLE 4.4. *There are two L*-spaces whose product is not an L*-space*:

Let $X=\{p\}\cup N\subset\beta N$, where $p\in\beta N-N$ and βN is the Stone-Čech compactification of N. X is an a.e. metrizable space and, therefore, an L*-space. Note that X is not first countable at p. Consider the product of X with the unit interval $[0,1]$; the fact that the product is not an L*-space is essentially proved by Okuyama and Yasui [17, Theorem 3], but a proof is presented below for the reader's convenience. To show that $X\times[0,1]$ is not an L*-space, suppose the contrary. Then there is a σ-locally finite open covering $\mathcal{U}$ of $(X\times[0,1])-\{(p,0)\}$ such that every open neighborhood of $(p,0)$ is a subcanonical neighborhood with respect to $\mathcal{U}$. In particular every open neighborhood of $(p,0)$ includes a $\mathcal{U}$-saturated neighborhood of $(p,0)$. For each $n\in N$ let $\mathcal{U}_n$ be the subcollection of $\mathcal{U}$ consisting of all members of $\mathcal{U}$ which contain the point $(p,1/n)$. Note that $\mathcal{U}_n$ is a countable collection for each n, and write $\mathcal{U}_n=\{U_{nj}:j\in N\}$. Now let G be an arbitrary neighborhood of p in X. Take a $\mathcal{U}$-saturated neighborhood H of $(p,0)$ included in $G\times[0,1]$, and take $n\in N$ so that $(p,1/n)\in H$. Then for some $j\in N$, $(p,1/n)\in U_{nj}\subset H$, which implies $p\in p_X(U_{nj})\subset G$ where $p_X:X\times[0,1]\to X$ is the projection. Thus $\{p_X(U_{nj}):n,j\in N\}$ is a countable neighborhood base of p, which is a contradiction. Consequently $X\times[0,1]$ is not an L*-space.

EXAMPLE 4.5. *There is a 2-patched space which is not an L*-space*:

The product $X\times[0,1]$ above is such an example.

EXAMPLE 4.6. *There is an L*-space which is not a patched space*:

Lašnev [6] constructed a Lašnev space which is nowhere first countable. By Lemma 3.1 such a space is not a patched space. But by [11, Theorem 1.6] every Lašnev space is an L-space and, therefore, an L*-space.

EXAMPLE 4.7. *There is a free L-space which is not an L*-space*:

Let X be as in Example 4.4. Clearly X is (free) L-space, and hence $X\times[0,1]$ is a free L-space. But as proved there, $X\times[0,1]$ is not an L*-space.

In view of the fact that an L-space is an M_1-space ([11, Theorem 1.7]), we finally present the following example.

EXAMPLE 4.8. *There is an a.e. metrizable (and hence L*-) M_1-space which is not an L-space.*

Let X be the unit interval $[0,1]$. The topology on X is given so that each point in $X-\{0\}$ has a usual open neighborhood base in the Euclidean topology and so that the point 0 has an open neighborhood base $\mathcal{G}$ of the form

$$\{\bigcup_{n=k}^{\infty}(1/n-1/m(n),\ 1/n+1/m(n))\cup\{0\}:n\leqq m(n)\in N,\ k\in N\},$$

where $(\cdot,\cdot)$ denotes the open interval, and $m(n)$ is not fixed but varies freely on the integers not smaller than n. The space X is then an a.e. metrizable space. Further X is an M_1-space because $\mathcal{G}$ is a closure-preserving open neighborhood base of the point 0. To show that X is not an L-space, suppose the contrary and let $\mathcal{U}$ be an open covering of $X-\{0\}$ such that every open neighborhood of $\{0\}$ is a canonical neighborhood with respect to $\mathcal{U}$. We can assume that $\mathcal{U}$ is countable and locally finite in $X-\{0\}$. Write $\mathcal{U}=\{U_n:n\in N\}$, where $U_n\neq\emptyset$ for each $n\in N$. For each $n\in N$, take a point x_n in U_n so that $x_n\notin\{1,1/2,1/3,\cdots\}$. Put $W=X-\{x_n:$

$n\in N\}$. Since $\{x_n : n\in N\}$ is a closed set of $X-\{0\}$ not meeting $\{1, 1/2, 1/3, \cdots\}$, W is an open neighborhood of $\{0\}$ and, therefore, a canonical neighborhood of $\{0\}$ with respect to $\mathcal{U}$. Since $U_n\cap(X-W)\neq\emptyset$ for every $n\in N$, we consequently have $0\notin \mathrm{Cl}\,\mathcal{U}^*$. This means that 0 is an isolated point of X, which is a contradiction. Thus X is not an L-space.

PROBLEM 4.9. *Is every patched space a free L^*-space?*

A positive answer to this problem gives a positive answer to the following one.

PROBLEM 4.10. *Is every free patched space a free L^*-space?*

We conclude this paper by giving a partial answer to Problem 4.9 in the case of K_1-spaces.

DEFINITION 4.11 (van Douwen [19]). Let X be a space and Y a closed set of X. Y is called *K_1-embedded* in X if there is a function $k:\mathcal{T}(Y)\to\mathcal{T}(X)$ from the topology of Y into the topology of X such that

(1) $Y\cap k(V)=V$ for each $V\in\mathcal{T}(Y)$, and
(2) $k(V)\cap k(W)=\emptyset$ whenever $V\cap W=\emptyset$ and $V, W\in\mathcal{T}(Y)$.

A space X is called a *K_1-space* if every closed set of X is K_1-embedded in X.

The following version of Definition 4.11 is suitable for our purpose.

LEMMA 4.12. *Let X be a space and Y a closed set of X. Y is K_1-embedded in X if and only if there is a function $e:\mathcal{T}(Y)\to\mathcal{T}(X)$ such that*

(3) *$Y\cap e(V)=V$ for each $V\in\mathcal{T}(Y)$,*
(4) *$Y\cap \mathrm{Cl}\, e(V)=\mathrm{Cl}\, V$ for each $V\in\mathcal{T}(Y)$, and*
(5) *$e(V)\subset e(W)$ whenever $V\subset W$ and $V, W\in\mathcal{T}(Y)$.*

PROOF. Let $k:\mathcal{T}(Y)\to\mathcal{T}(X)$ be a function satisfying (1) and (2). We can assume that $k(V)\subset k(W)$ whenever $V\subset W$ and $V, W\in\mathcal{T}(Y)$. Then the function $e=k$ satisfies (3)—(5). Conversely if $e:\mathcal{T}(Y)\to\mathcal{T}(X)$ satisfies (3)—(5), then put $k(V)=e(V)-\mathrm{Cl}\, e(Y-\mathrm{Cl}\, V)$ for each $V\in\mathcal{T}(Y)$. This completes the proof.

Note that hereditarily normal spaces are just those spaces X in which every closed set Y admits a function $e:\mathcal{T}(Y)\to\mathcal{T}(X)$ satisfying (3) and (4).

PROPOSITION 4.13. *A patched K_1-space is a free L^*-space.*

This proposition is obtained inductively by using the following result together with Lemma 3.1.

PROPOSITION 4.14. *Let X be a paracompact perfectly normal space including a closed set Y such that Y is a free L^*-space and $X-Y$ is metrizable. If Y is K_1-embedded in X, then X is a free L^*-space.*

PROOF. Let $\{\mathcal{F}, \{\mathcal{U}_F : F \in \mathcal{F}\}\}$ be a free L^*-structure on Y. We can assume that for each $F \in \mathcal{F}, \mathcal{U}_F$ is σ-discrete in Y. Let $e : \mathcal{T}(Y) \to \mathcal{T}(X)$ be a function satisfying (3)—(5). For a moment fix $F \in \mathcal{F}$. We first construct a σ-discrete collection $\{r(U) : U \in \mathcal{U}_F\}$ of open sets of X satisfying the following conditions.

(6) $Y \cap r(U) = U$ for each $U \in \mathcal{U}_F$,

(7) $\mathrm{Cl}\, r(U) \cap \mathrm{Cl}\, r(V) = \emptyset$ whenever $\mathrm{Cl}\, U \cap \mathrm{Cl}\, V = \emptyset$ and $U, V \in \mathcal{U}_F$,

(8) $r(U) \subset e(U)$ for each $U \in \mathcal{U}_F$, and

(9) $\{r(U) : U \in \mathcal{U}_F\}$ is locally finite in $X-Y$.

To do this, write $\mathcal{U}_F = \bigcup_{i=1}^{\infty} \mathcal{U}_{F,i}$ where each $\mathcal{U}_{F,i}$ is discrete in X. By collectionwise normality we can find for each i, a discrete collection $\{g(U) : U \in \mathcal{U}_{F,i}\}$ of open sets of X such that $U \subset g(U)$ for each $U \in \mathcal{U}_{F,i}$. For each $i \geqq 2$ and $U \in \mathcal{U}_{F,i}$, put $h(U) = X - \{\mathrm{Cl}(g(V) \cap e(V)) : V \in \bigcup_{j=1}^{i-1} \mathcal{U}_{F,j},\ \mathrm{Cl}\, V \cap \mathrm{Cl}\, U = \emptyset\}^*$. It follows from (4) that $h(U)$ is an open set of X including $\mathrm{Cl}\, U$. Take an open set $s(U)$ of X such that $\mathrm{Cl}\, U \subset s(U) \subset \mathrm{Cl}\, s(U) \subset h(U)$. Write $Y = \bigcap_{i=1}^{\infty} Y_i$ by open sets Y_i such that $\mathrm{Cl}\, Y_{i+1} \subset Y_i$ for every i. Now define

$$r(U) = g(U) \cap e(U) \text{ if } U \in \mathcal{U}_{F,1}, \text{ and}$$

$$r(U) = g(U) \cap e(U) \cap s(U) \cap Y_i \text{ if } U \in \mathcal{U}_{F,i} \text{ and } i \geqq 2.$$

Then $\{r(U) : U \in \mathcal{U}_F\}$ satisfies (6)—(9).

Now let $\mathcal{S}$ be a σ-discrete base of $X-Y$. We can assume $\mathcal{S}$ is σ-discrete in X. Define a σ-discrete open covering $\mathcal{D}(F)$ of $X-F$ by

$$\mathcal{D}(F) = \{r(U) : U \in \mathcal{U}_F\} \cup \mathcal{S}.$$

ASSERTION. *If V is a subcanonical neighboorhood of F in Y with respect to $\mathcal{U}_F$, then $V \cup (X-Y)$ is a subcanonical neighborhood of F in X with respect to $\mathcal{D}(F)$.*

PROOF. By the definition of subcanonical neighborhoods, there are a sequence $\{V_i : i \in N\}$ of $\mathcal{U}_F$-saturated neighborhoods of F and a sequence $\{\mathcal{U}_i : i \in N\}$ of subcollections of $\mathcal{U}_F$ such that

(10) $V_{i+1} \subset Y - \mathcal{U}_i^* \subset V_i \subset V$ for every $i \in N$.

Then we have

(11) $F\cap\mathrm{Cl}\{r(U): U\in\mathcal{U}_i\}^*=\emptyset$ for each $i\in N$.

Indeed by (8) and (5)

$$\{r(U): U\in\mathcal{U}_i\}^*\subset\{e(U): U\in\mathcal{U}_i\}^*\subset e(\mathcal{U}_i^*),$$

but by (4) and (10)

$$\begin{aligned} F\cap\mathrm{Cl}\, e(\mathcal{U}_i^*) &= F\cap(Y\cap\mathrm{Cl}\, e(\mathcal{U}_i^*)) \\ &= F\cap\mathrm{Cl}\,\mathcal{U}_i^* \\ &= V_{i+1}\cap\mathrm{Cl}\,\mathcal{U}_i^*=\emptyset, \end{aligned}$$

which yields (11).

Put $\mathcal{V}_i=\{U\in\mathcal{U}_F: U\subset V_i\}$; then clearly $\mathcal{V}_{i+1}\subset\mathcal{V}_i$. Note that for each i,

(12) $\mathrm{Cl}\{r(U): U\in\mathcal{U}_i\}^*\cap\mathrm{Cl}\{r(U): U\in\mathcal{V}_{i+2}\}^*=\emptyset$.

In fact since $\mathrm{Cl}\,\mathcal{U}_i^*\cap\mathrm{Cl}\,\mathcal{V}_{i+2}^*=\emptyset$, it follows from (7) and (9) that

$$(\mathrm{Cl}\{r(U): U\in\mathcal{U}_i\}^*\cap\mathrm{Cl}\{r(U): U\in\mathcal{V}_{i+2}\}^*)\cap(X-Y)=\emptyset,$$

while by (4), (5), (8) and (10)

$$\begin{aligned} &(Y\cap\mathrm{Cl}\{r(U): U\in\mathcal{U}_i\}^*)\cap(Y\cap\mathrm{Cl}\{r(U): U\in\mathcal{V}_{i+2}\}^*) \\ &\subset(Y\cap\mathrm{Cl}\, e(\mathcal{U}_i^*))\cap(Y\cap\mathrm{Cl}\, e(\mathcal{V}_{i+2}^*)) \\ &=\mathrm{Cl}\,\mathcal{U}_i^*\cap\mathrm{Cl}\,\mathcal{V}_{i+2}^* \\ &\subset(Y-V_{i+1})\cap(Y-\mathcal{U}_{i+1}^*)=\emptyset, \end{aligned}$$

which yields (12). It follows from (11) and (12) that for each i,

(13) $\mathrm{Cl}\{r(U): U\in\mathcal{U}_i\}^*\cap(F\cup\mathrm{Cl}\{r(U): U\in\mathcal{V}_{i+2}\}^*)=\emptyset$.

Put $O_1=X$. By (13) we can take disjoint open sets P_1 and O_3 of X such that

$$\mathrm{Cl}\{r(U): U\in\mathcal{U}_1\}^*\subset P_1 \quad\text{and}$$
$$F\cup\mathrm{Cl}\{r(U): U\in\mathcal{V}_3\}^*\subset O_3.$$

It follows from (13) that

$$((X-O_3)\cup\mathrm{Cl}\{r(U): U\in\mathcal{U}_3\}^*)\cap(F\cup\mathrm{Cl}\{r(U): U\in\mathcal{V}_5\}^*)=\emptyset.$$

Next take disjoint open sets P_3 and O_5 of X such that

$$(X-O_3)\cup\mathrm{Cl}\{r(U): U\in\mathcal{U}_3\}^*\subset P_3 \quad\text{and}$$
$$F\cup\mathrm{Cl}\{r(U): U\in\mathcal{V}_5\}^*\subset O_5.$$

Repeating this process we obtain two sequences $\{P_{2i-1}: i\in N\}$ and $\{O_{2i-1}: i\in N\}$ of open sets of X such that for each i,

(14) $(X-O_{2i-1})\cup \mathrm{Cl}\{r(U): U\in\mathcal{U}_{2i-1}\}^*\subset P_{2i-1}$,

(15) $F\cup \mathrm{Cl}\{r(U): U\in\mathcal{V}_{2i-1}\}^*\subset O_{2i-1}$ and

(16) $P_{2i-1}\cap O_{2i+1}=\emptyset$.

Now define for each i,

$$D_{2i-1}=\{r(U): U\in\mathcal{V}_{2i-1}\}^*\cup(O_{2i-1}-Y)\cup F \quad \text{and}$$

$$\mathcal{D}_{2i-1}=\{r(U): U\in\mathcal{U}_{2i-1}\}\cup\{S\in\mathcal{S}: S\subset P_{2i-1}\}.$$

From (6), (10), (12), (14), (15) and (16) it follows that for each i,

(17) D_{2i-1} is a $\mathcal{D}(F)$-saturated neighborhood of F,

(18) $\mathcal{D}_{2i-1}$ is a subcollection of $\mathcal{D}(F)$, and

(19) $D_{2i+1}\subset X-\mathcal{D}^*_{2i-1}\subset D_{2i-1}\subset V\cup(X-Y)$.

This implies that $V\cup(X-Y)$ is a subcanonical neighborhood of F with respect to $\mathcal{D}(F)$. This completes the proof of Assertion.

We return to the proof of Proposition 4.14. Define $\mathcal{D}(Y)=\mathcal{S}$ and, as constructed in Proposition 3.2, let $\{\mathcal{E},\{\mathcal{D}(E): E\in\mathcal{E}\}\}$ be a pair of σ-discrete collection $\mathcal{E}$ of closed sets of X and countable open covers $\mathcal{D}(E)$, $E\in\mathcal{E}$, of $X-E$ such that for any point x in $X-Y$ and any member S of $\mathcal{S}$ with $x\in S$, there exists a member E of $\mathcal{E}$ such that $x\in E\subset S$ and S is a subcanonical neighborhood of E with respect to $\mathcal{D}(E)$.

Now define

$$\mathcal{K}=\mathcal{F}\cup\{Y\}\cup\mathcal{E},$$

and consider the pair $\{\mathcal{K},\{\mathcal{D}(K): K\in\mathcal{K}\}\}$. To show that the pair is a free L*-structure of X let $x\in X$ and let W be an open neighborhood of x. Since the case when $x\in X-Y$ is trivial, assume $x\in Y$. Then there exist a finite subcollection $\{F_1,\cdots,F_k\}$ of $\mathcal{F}$ and subcanonical neighborhoods U_i of F_i in Y with respect to $\mathcal{U}_{F_i}$, $1\leqq i\leqq k$, such that $x\in\bigcap_{i=1}^k F_i\subset\bigcap_{i=1}^k U_i\subset W\cap Y$. It follows from Assertion that $U_i\cup(X-Y)$ is a subcanonical neighborhood of F in X with respect to $\mathcal{D}(F_i)$. By the definition of subcanonical neighborhoods, we can find subcanonical neighborhoods W_i of F_i, $1\leqq i\leqq k$, with respect to $\mathcal{D}(F_i)$ such that $\overline{W}_i\subset U_i\cup(X-Y)$. Put $G=X-(\bigcap_{i=1}^k\overline{W}_i-W)$. Then G is an open neighborhood of Y and, therefore, a subcanonical neighborhood of Y with respect to $\mathcal{D}(Y)=\mathcal{S}$. Now we have

$$x\in Y\cap(\textstyle\bigcap_{i=1}^k F_i)\subset G\cap(\bigcap_{i=1}^k W_i)\subset W,$$

which implies that $\{\mathcal{K},\{\mathcal{D}(K): K\in\mathcal{K}\}\}$ is a free L*-structure on X. This completes the proof of Proposition 4.14 and, therefore, of Proposition 4.13.

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