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ON SOME SYSTEMS OF LINEAR OPERATORS CONNECTED WITH ARITHMETICAL INVERSION FORMULAS

By

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1. In the paper of W. P. Romanov [1] and the present author's work [2] the following operators played an important role:

$$L_{n}f(x)=\frac{1}{n}\sum_{k=0}^{n-1}f\left(x+\frac{k}{n}\right) \qquad (n=1,2,3,\cdots). \tag{1}$$

These operators are defined on the class of periodic functions $f(x)$ with period 1.

In this paper we shall investigate operators (1) from another point of view and establish their connection with the harmonical components of the function $f(x)$. We shall use the formal method of arithmetical inversion of series, mentioned by P. L. Čebyšev in 1851 (cf. [3; pp. 229-236]): If

$$\sum_{k=1}^{\infty}c_{nk}=A_{n} \qquad (n=1,2,3,\cdots) \tag{2}$$

then

$$c_{m}=\sum_{k=1}^{\infty}\mu(k)A_{mk} \qquad (m=1,2,3,\cdots), \tag{3}$$

where $\mu(k)$ is the Möbius function.

Čebyšev [3] was not based on this formal transformation. In fact the matter is quite difficult—equalities (3) are not always true even if the solution c_m of the system (2) does exist; they are true on the assumption

$$\sum_{m=1}^{\infty}|c_{m}|<\infty. \tag{4}$$

In case where the inequality (4) does not hold, the equations (2) may be solved in c_m but not uniquely.

2. A sufficient condition for the correctness of formulas (3) will be given by

THEOREM I. *If c_m and A_n $(m,n=1,2,3,\cdots)$ satisfy* (2) *and if there holds the inequality*

$$\sum_{m=1}^{\infty}2^{\nu(m)}|c_{m}|<\infty \tag{5}$$

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where $\nu(m)$ denotes the number of different prime divisors of m, then the formulas (3) *are true, and the series in these formulas are absolutely convergent.*

PROOF. We have, by (2) and a well-known property of $\mu(k)$, formal transformations

$$\sum_{k=1}^{\infty} \mu(k) A_{mk} = \sum_{k=1}^{\infty} \mu(k) \sum_{l=1}^{\infty} c_{mkl} = \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} \mu(k) c_{mkl}$$

$$= \sum_{n=1}^{\infty} c_{mn} \sum_{k|n} \mu(k) = c_m ,$$

since the intermediate double series is majorized by the series

$$\sum_{k=1}^{\infty} \sum_{l=1}^{\infty} |\mu(k) c_{mkl}| = \sum_{n=1}^{\infty} |c_{mn}| \sum_{k|n} |\mu(k)|$$

$$= \sum_{n=1}^{\infty} 2^{\nu(n)} |c_{mn}| \leqq \sum_{n=1}^{\infty} 2^{\nu(mn)} |c_{mn}| ,$$

which is convergent by the assumption (5).

Note that if in Theorem I the condition (5) is replaced by (4), then the theorem analogous to Theorem I cannot hold any longer. In this case we shall prove the following result (cf. [4]).

THEOREM II. *If the numbers c_m and A_n $(m, n=1, 2, 3, \cdots)$ satisfy* (2) *and if the condition* (4) *is fulfilled, then*

$$c_m = \lim_{N\to\infty} \sum_{d|[N]} \mu(d) A_{md} \qquad (n=1, 2, 3, \cdots) , \tag{6}$$

where $[N]$ denotes the least common multiple of the numbers $2, 3, \cdots, N$.

PROOF. Formal transformations will give

$$\sum_{d|[N]} \mu(d) A_{md} = \sum_{d|[N]} \mu(d) \sum_{k=1}^{\infty} c_{mdk} = \sum_{n=1}^{\infty} c_{mn} \sum_{\substack{d|n \\ d|[N]}} \mu(d)$$

$$= c_m + \sum_{\substack{n=N+1 \\ (n,[N])=1}}^{\infty} c_n . \tag{7}$$

By (4) we have now that

$$\Big| \sum_{\substack{n=N+1 \\ (n,[N])=1}}^{\infty} c_{mn} \Big| \leqq \sum_{n=N+1}^{\infty} |c_{mn}| \to 0 \qquad (N\to\infty).$$

This with (7) proves our Theorem II.

If we repeal the assumption (4) then in general the numbers c_m are not uniquely determined by the numbers A_n.

In this circumstance we have

THEOREM III. *There exist numbers* $c_1, c_2, \cdots, c_m, \cdots$ *which are not all equal to* 0 *and such that the series* (2) *are convergent and their sums* $A_n=0$ *for all* n.

To prove this we put $c_m=\mu(m)/m$. Verification of the statement of the theorem is based on the famous formula of Euler-von Mangoldt

$$\sum_{m=1}^{\infty}\frac{\mu(m)}{m}=0. \tag{8}$$

As is well known, the convergence of the series in (8) is quite a deep fact which is equivalent to the prime number theorem (cf. [9]). We have

$$A_n=\sum_{k=1}^{\infty}\frac{\mu(nk)}{nk}=\frac{\mu(n)}{n}\sum_{\substack{k=1\\(k,n)=1}}^{\infty}\frac{\mu(k)}{k}.$$

On the other hand we have

$$0=\prod_{p|n}\left(1-\frac{1}{p}\right)^{-1}\cdot\sum_{m=1}^{\infty}\frac{\mu(m)}{m}=\prod_{p|n}\left(1-\frac{1}{p}\right)^{-1}\cdot\prod_{p|n}\left(1-\frac{1}{p}\right)\cdot\sum_{\substack{k=1\\(k,n)=1}}^{\infty}\frac{\mu(k)}{k}$$
$$=\sum_{\substack{k=1\\(k,n)=1}}^{\infty}\frac{\mu(k)}{k}.$$

Therefore $A_n=0$ for all n, and among the numbers c_m there are infinitely many of them that are not equal to 0. This proves the theorem.

3. Let us use the results of §2 in the theory of "arithmetical means with displacements," i. e. in the theory of operators $L_nf(x)$ defined by (1).

THEOREM IV. *If for arbitrary* x *the formula*

$$f(x)=a_0+\sum_{m=1}^{\infty}(a_m\cos 2\pi mx+b_m\sin 2\pi mx)$$

is right, then we have

$$L_nf(x)-a_0=\sum_{k=1}^{\infty}(a_{nk}\cos 2\pi nkx+b_{nk}\sin 2\pi nkx)$$

for $n=1, 2, 3, \cdots$.

PROOF. We have by a simple calculation

$$L_nf(x)-a_0=\frac{1}{n}\sum_{k=0}^{n-1}\sum_{m=1}^{\infty}\left(a_m\cos 2\pi m\left(x+\frac{k}{n}\right)+b_m\sin 2\pi m\left(x+\frac{k}{n}\right)\right)$$

$$=\sum_{m\equiv 0\ (\mathrm{mod}\ n)}(a_m\cos 2\pi mx+b_m\sin 2\pi mx).$$

If we put

$$c_m=a_m\cos 2\pi mx+b_m\sin 2\pi mx$$

$$A_n=L_nf(x)-a_0\,,$$

then, on the basis of Theorem I, the statement of Theorem V follows from the formula (10).

THEOREM V. *If for arbitrary* $\varepsilon>0$ *the series*

$$\sum_{n=1}^{\infty}(|a_n|+|b_n|)n^{\varepsilon} \tag{11}$$

converges, then for $n=1,2,3,\cdots$ *the formulas*

$$a_n\cos 2\pi nx+b_n\sin 2\pi nx=\sum_{d=1}^{\infty}\mu(d)(L_{dn}f(x)-a_0) \tag{12}$$

are true. The series in (12) *are absolutely and uniformly convergent for all* x.

The proof is immediate from the estimate

$$2^{\nu(n)}\leqq\tau(n)=O(n^{\varepsilon})\quad\text{for any fixed } \varepsilon>0$$

and Theorem I. Here, $\tau(n)$ denotes the number of positive divisors of n.

Analogously, from Theorem II follows

THEOREM VI. *If*

$$\sum_{n=1}^{\infty}(|a_n|+|b_n|)<\infty \tag{13}$$

then we have for every x

$$a_n\cos 2\pi nx+b_n\sin 2\pi nx=\lim_{N\to\infty}\sum_{d|[N]}\mu(d)(L_{dn}f(x)-a_0) \tag{14}$$

uniformly for all n.

We note that the condition (11) of Theorem V is satisfied in the following two cases (cf. [15]):

1) $f(x)$ is a function of bounded variation in $[0,1]$ and belongs to $\mathrm{Lip}\,\alpha$, $\alpha>0$;

2) $f(x)$ belongs to $\mathrm{Lip}\,\alpha$, $\alpha>1/2$.

Sufficient conditions for (13) can be found in [5], [6], and [7].

If we substitute in (12) $-x$ for x, then we get

$$a_n \cos 2\pi nx = \sum_{d=1}^{\infty} \mu(d) L_{dn}\left(\frac{f(x)+f(-x)}{2} - a_0\right), \tag{15}$$

$$b_n \sin 2\pi nx = \sum_{d=1}^{\infty} \mu(d) L_{dn}\left(\frac{f(x)-f(-x)}{2}\right). \tag{16}$$

These formulas can be used in harmonic analysis. For instance, if we put $x=0$ in (15) we obtain

$$a_n = \frac{1}{n} \sum_{d=1}^{\infty} \frac{\mu(d)}{d} \left(\sum_{k=0}^{dn-1} f\left(\frac{k}{dn}\right) - \int_0^1 f(x)dx\right). \tag{17}$$

If we substitute in (14) $-x$ for x, then we find

$$a_n \cos 2\pi nx = \lim_{N\to\infty} \sum_{d\mid[N]} \mu(d) L_{dn}\left(\frac{f(x)+f(-x)}{2} - a_0\right), \tag{18}$$

$$b_n \sin 2\pi nx = \lim_{N\to\infty} \sum_{d\mid[N]} \mu(d) L_{dn}\left(\frac{f(x)-f(-x)}{2}\right). \tag{19}$$

Taking $x=0$ in (18) we obtain

$$a_n = \frac{1}{n} \lim_{N\to\infty} \sum_{d\mid[N]} \frac{\mu(d)}{d} \left(\sum_{k=0}^{dn-1} f\left(\frac{k}{dn}\right) - \int_0^1 f(x)dx\right). \tag{20}$$

The formulas (18), (19) and (20) hold true for every $f(x)\in L(0,1)$ with absolutely convergent Fourier series.

4. There exist some other systems of linear operators which are also connected with arithmetical inversion formulas. For instance consider for odd n

$$L_n^* f(x) = \frac{1}{n}\left(f(x) + 2\sum_{k=1}^{n-1} (-1)^k f\left(x+\frac{k}{n}\right) - f(x+1)\right). \tag{21}$$

We have

THEOREM VII. *If the function $f(x+t)-f(x+1-t)$ is expanded for $t\in[0,1]$ in the series*

$$f(x+t)-f(x+1-t) = \sum_{n=1}^{\infty} c_n(f) \cos n\pi t, \tag{22}$$

then

$$L_n^* f(x) = \sum_{k=1}^{\infty} c_{kn}(f) \qquad (n=1,3,5,\cdots). \tag{23}$$

We note that

$$c_n(f)=2\int_0^1 (f(x+t)-f(x+1-t))\cos n\pi t\,dt$$

$$=\begin{cases} 0 & \textit{for even } n, \\ 4\int_0^1 f(x+t)\cos n\pi t\,dt & \textit{for odd } n. \end{cases} \tag{24}$$

PROOF. Let us introduce a 2-periodical even function $g(t)$ such that

$$g(t)=f(x+t)-f(x+1-t) \qquad \text{for } t\in[0,1].$$

We have then

$$\sum_{k=0}^{n-1} g\left(\frac{2k}{n}\right)=\sum_{m=1}^{\infty} c_m(f)\sum_{k=0}^{n-1}\cos\frac{2\pi km}{n}=n\sum_{m\equiv 0\ (\mathrm{mod}\ n)} c_m(f).$$

On the other hand we have for each odd n

$$\sum_{k=0}^{n-1} g\left(\frac{2k}{n}\right)=g(0)+2\sum_{k=1}^{(n-1)/2} g\left(\frac{2k}{n}\right)$$

$$=f(x)-f(x+1)+2\sum_{k=1}^{(n-1)/2}\left(f\left(x+\frac{2k}{n}\right)-f\left(x+\frac{n-2k}{n}\right)\right)$$

$$=f(x)-f(x+1)+2\sum_{\nu=1}^{n-1}(-1)^{\nu} f\left(x+\frac{\nu}{n}\right)$$

$$=nL_n^* f(x). \tag{26}$$

Comparing (25) and (26) we obtain the formulas (23). The formulas (23) have the same structure as those in (2). Therefore, by the theorems in §2, we get formulas expressing $c_n(f)$ through $L_n^* f(x)$.

THEOREM VIII. *If for the function $f(x)$ we have the formulas* (23) *and if for some $\varepsilon>0$ the series*

$$\sum_{\substack{n=1 \\ n\ \mathrm{odd}}}^{\infty} |c_n(f)|n^{\varepsilon}<\infty \tag{27}$$

then we have for every odd n

$$c_n(f)=\sum_{\substack{d=1 \\ d\ \mathrm{odd}}}^{\infty} \mu(d)L_{nd}^* f(x), \tag{28}$$

and the series in (28) *is absolutely convergent.*

Replacing (27) by the assumption

$$\sum_{n=1}^{\infty} |c_n(f)|<\infty \tag{29}$$

we get

THEOREM IX. *If for the function* $f(x)$ (29) *is true, then the numbers* $c_n(f)$ *are uniquely determined by*

$$c_n(f)=\lim_{N\to\infty}\sum_{d\mid U_N}\mu(d)L_{nd}^{*}f(x), \tag{30}$$

where $U_N=3\cdot 5\cdot\cdots\cdot p$ *is the product of all odd prime numbers* $\leqq N$.

5. The analogues of Theorem III for $L_n f(x)$ and $L_n^* f(n)$ are given by

THEOREM X. *There exists a continuous function* $f_1(x)\neq 0$ *such that for* $x=0$ *we have*

$$L_n f_1(0)=0 \qquad (n=1, 2, 3, \cdots),$$

and

$$L_n^* f_1(0)=0 \qquad (n=1, 3, 5, \cdots).$$

For this function we take

$$f_1(x)=\sum_{m=1}^{\infty}\frac{\mu(m)}{m}\cos 2\pi mx. \tag{31}$$

It is evident that $f_1(x)\neq 0$. The uniform convergence of the series (31) follows from a result of H. Davenport [8].

By the theory of prime numbers [9] we know that

$$L_n f_1(0)=2\sum_{k=1}^{\infty}\frac{\mu(nk)}{nk}=\frac{\mu(n)}{n}\sum_{(k,n)=1}\frac{\mu(k)}{k}=0.$$

Using (23) and (24) we get

$$L_n^* f_1(0)=2\sum_{2k-1\equiv 0\ (\mathrm{mod}\ n)}\frac{\mu(2k-1)}{2k-1}=2\frac{\mu(n)}{n}\sum_{(m,2n)=1}\frac{\mu(m)}{m}=0.$$

This proves the theorem.

From this theorem it follows that an arbitrary continuous function $f(x)$ is not uniquely determined by the values of the operators $L_n f(x)$ and $L_n^* f(x)$ at the point $x=0$.

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