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A PRERADICAL WHICH SATISFIES THE PROPERTY THAT EVERY WEAKLY DIVISIBLE MODULE IS DIVISIBLE

By

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Recently M. Sato has studied a radical satisfying the property that every weakly codivisible module is codivisible in [6]. In this paper we study preradicals for which every weakly divisible module is divisible. We characterize an idempotent preradical with this property in Theorems 1.7 and 1.8. Moreover we characterize an idempotent preradical for which every weakly divisible module is injective in Proposition 1.10. Dually we consider a radical for which every weakly codivisible module is projective in Proposition 1.13.

In § 2 we study a preradical t which has the property that $t(E/K)=(t(E)+K)/K$ holds for any injective module E and any submodule K of E. We call this an injectively epi-preserving preradical and characterize in Theorem 2.1.

Dualizing this, we study a preradical which has the property that $t(K)=K\cap t(P)$ holds for any projective module P and any submodule K of P, and we have Theorem 2.4.

Last we give examples of these preradicals.

1. Weakly (co-) divisible modules and (co-) divisible modules.

Throughout this paper R is a ring with a unit element, every right R-module is unital and Mod-R is the category of right R-modules. A subfunctor of the identity functor of Mod-R is called a preradical. A preradical t is called idempotent (resp. radical) if $t(t(M))=t(M)$ (resp. $t(M/t(M))=0$) for any module M. For a preradical t we put $\mathcal{T}_t=\{M\in \text{Mod-}R\,;\, t(M)=M\}$ and $\mathcal{F}_t=\{M\in \text{Mod-}R\,;\, t(M)=0\}$ whose elements are said to be torsion and torsionfree modules respectively. We say that M is divisible (resp. weakly divisible) if $\text{Hom}_R(-, M)$ preserves the exactness for every exact sequence $0\to A\to B\to C\to 0$ with $C\in\mathcal{T}_t$ (resp. $B\in\mathcal{T}_t$). Dually we say that M is codivisible (resp. weakly codivisible) if $\text{Hom}_R(M, -)$ preserves the exactness for every exact sequence $0\to A\to B\to C\to 0$ with $A\in\mathcal{F}_t$ (resp. $B\in\mathcal{F}_t$).

To begin with we study a fundamental property of weakly divisible modules.

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LEMMA 1.1. *Let t be a preradical and A a weakly divisible module. If K is a submodule of A containing $t(A)$, then K is also a weakly divisible module.*

PROOF. Consider the following exact sequence $0 \to X \xrightarrow{f} Y$ with $Y \in \mathcal{T}_t$. Since A is weakly divisible, for any $g \in \mathrm{Hom}_R(X, K)$ there exists some $h \in \mathrm{Hom}_R(Y, A)$ such that $i \cdot g = f \cdot h$, where i is the inclusion map of K into A. Then we have $h(Y) = h(t(Y)) \subset t(A) \subset K$. Thus K is weakly divisible.

PROPOSITION 1.2. *Let t be an idempotent preradical and consider an exact sequence $0 \to N \to M \to M/N \to 0$ of right R-modules. Then we have*

(1) *If N is weakly divisible, then $t(N) = N \cap t(M)$.*

(2) *If N is divisible, then $t(M/N) = (t(M)+N)/N$.*

PROOF. (1) Suppose that N is weakly divisible. We consider the following exact sequence,

$$0 \to N \cap t(M) \to t(M) \to t(M)/(N \cap t(M)) \to 0\,.$$

Since $t(N) \subset N \cap t(M) \subset N$, $N \cap t(M)$ is weakly divisible by Lemma 1.1. Thus the above sequence splits, and so $N \cap t(M)$ is a torsion module. Hence $N \cap t(M) = t(N \cap t(M)) \subset t(N)$, and so we have $N \cap t(M) = t(N)$.

(2) Suppose that N is divisible. We put $t(M/N) = L/N$, where L is a submodule of M containing N. Then the sequence $0 \to N \to L \to L/N \to 0$ splits as N is divisible and L/N is torsion. So there exists a torsion submodule H of L such that $L = N \oplus H$. Thus $t(L) = t(N) \oplus H$ and we have $t(M)+N \supset t(L)+N = t(N)+H+N = L$. On the other hand, since $t(M/N) \supset (t(M)+N)/N$, we have $L \supset t(M)+N$. Thus $L = t(M)+N$, and so $t(M/N) = (t(M)+N)/N$.

Next we characterize weakly divisible modules for an idempotent preradical.

For a module M let $E(M)$ denote the injective hull of M.

LEMMA 1.3. *Let t be an idempotent preradical. Then an R-module A is weakly divisible if and only if A contains $t(E(A))$.*

PROOF. Let A be a weakly divisible module. Since $t(A) \subset A \cap t(E(A)) \subset A$, $A \cap t(E(A))$ is weakly divisible by Lemma 1.1. As $t(E(A))$ is torsion, the exact sequence $0 \to A \cap t(E(A)) \to t(E(A))$ splits. Thus there is some submodule K of $t(E(A))$ such that $t(E(A)) = (A \cap t(E(A))) \oplus K$. Then we have $A \cap K = 0$ as $(A \cap t(E(A))) \cap K = A \cap K$. Since A is an essential submodule of $E(A)$, we have $K = 0$. Hence $t(E(A)) = A \cap t(E(A)) \subset A$. The converse is easily verified.

The following lemma can be seen in Proposition 4.1 in [2] for a left exact radical.

LEMMA 1.4. *Let t be an idempotent preradical. Then an R-module M is divisible if and only if $E(M)/M$ is torsionfree.*

PROOF. Let $E(M)/M\in\mathcal{F}_t$. Consider an exact sequence of Mod-R $0\rightarrow X\xrightarrow{i}Y\xrightarrow{g}Y/X\rightarrow 0$ with $Y/X\in\mathcal{T}_t$. For any $f\in\mathrm{Hom}_R(X, M)$, there exists an $h\in\mathrm{Hom}_R(Y, E(M))$ such that $j\cdot f=h\cdot i$, where j is the inclusion map of M into $E(M)$. Then h induces $\tilde{h}\in\mathrm{Hom}_R(Y/X, E(M)/M)$ such that $\tilde{h}(y+X)=h(y)+M$ for any $y\in Y$. Since $Y/X\in\mathcal{T}_t$ and $E(M)/M\in\mathcal{F}_t$, we have $\tilde{h}=0$. This implies that $h(Y)\subset M$ and so M is divisible.

Conversely assume that M is divisible. Put $t(E(M)/M)=L/M$ where L is a submodule of $E(M)$ containing M, and consider the exact sequence $0\rightarrow M\rightarrow L\rightarrow L/M\rightarrow 0$. As M is divisible and L/M is torsion, the above sequence splits. Since L is an essential extension of M, we have $L/M=0$ and so $E(M)/M$ is torsionfree.

A preradical t is called a left exact (resp. an epi-preserving) preradical if $t(N)=N\cap t(M)$ (resp. $t(M/N)=(t(M)+N)/N$) for any $M\in$ Mod-R and any submodule N of M.

For a preradical t let $\sigma_t(M)$ denote $M\cap t(E(M))$ for any $M\in$ Mod-R. It is easily verified that $\sigma_t(M)$ is uniquely determined for any choice of $E(M)$.

LEMMA 1.5. *$\sigma_t(\_)$ is a left exact preradical for a preradical t.*

PROOF. It is easily verified that $\sigma_t(\_)$ is a preradical. Let $M\in$ Mod-R and N a submodule of M. We can find a submodule K of $E(M)$ such that $E(M)=E(N)\oplus K$. Hence we have $t(E(M))=t(E(N))\oplus t(K)$ and $E(N)\cap t(E(M))=t(E(N))\oplus(E(N)\cap t(K))=t(E(N))$. Thus $\sigma_t(N)=N\cap t(E(N))=N\cap E(N)\cap t(E(M))=N\cap t(E(M))=N\cap M\cap t(E(M))=N\cap\sigma_t(M)$.

LEMMA 1.6. *Let t be a preradical. Then $t(M)\supset M\cdot t(R)$ holds for any $M\in$ Mod-R.*

PROOF. Define $f_m\in\mathrm{Hom}_R(R, M)$ by $f_m(r)=mr$ for $m\in M$ and $r\in R$. Then we have $m\cdot t(R)=f_m(t(R))\subset t(M)$, since t is a preradical. Thus we have $M\cdot t(R)\subset t(M)$.

Now we shall study the following condition (*):

(*) Every weakly divisible module is divisible.

THEOREM 1.7. *Let t be a preradical. Consider the following conditions from* (1) *to* (8):

(1) *Every weakly divisible module is divisible.*

(2) *$A/t(A)$ has no nonzero torsion factor module for any weakly divisible module A.*

(3) *$M=K+\sigma_t(M)$ holds for any module M and any submodule K of M such that M/K is torsion.*

(4) *$M/\sigma_t(M)$ has no nonzero torsion factor module for every module M.*

(5) *Any direct sum of $R/\sigma_t(R)$ has no nonzero torsion factor module.*

(6) *$M/M\cdot\sigma_t(R)$ has no nonzero torsion factor module for every module M.*

(7) *$M=K+M\cdot\sigma_t(R)$ holds for any module M and any submodule K of M such that M/K is torsion.*

(8) *$M/t(M)$ has no nonzero torsion factor module for every module M such that $t(M)=\sigma_t(M)$.*

Then we have the following implications (1)→(2)→(3)→(4)→(5)→(6)→(7)→(3) *and* (4)→(8). *If further t is an idempotent preradical, all the conditions are equivalent.*

PROOF. (1)→(2): Consider the following exact sequence $0\to K\to A\to A/K\to 0$ where A is weakly divisible and K is a submodule of A containing $t(A)$ and A/K is torsion. By Lemma 1.1, K is weakly divisible and therefore divisible. Thus the above sequence splits, as A/K is torsion. So there exists a torsion submodule H of A such that $A=K\oplus H$. Since $K\supset t(A)\supset H$ and $0=K\cap H$, we have $H=0$, and so $A/K=0$ as desired.

(2)→(3): Let M/K be torsion for a submodule K of M. Since $\mathcal{T}_t$ is closed under taking factor modules and $M/(K+\sigma_t(M))\cong(M+t(E(M)))/(K+t(E(M)))$, $(M+t(E(M)))/(K+t(E(M)))$ is torsion. As $E(M)\supset M+t(E(M))\supset t(E(M))$, $M+t(E(M))$ is weakly divisible by Lemma 1.1. Since $K+t(E(M))$ contains $t(M+t(E(M)))$, we have $(M+t(E(M)))/(K+t(E(M)))=0$. Thus $M=K+\sigma_t(M)$ holds.

(3)→(4): For a module M, let K be a submodule of M containing $\sigma_t(M)$ such that M/K is torsion. Then we have $M=K+\sigma_t(M)=K$.

(4)→(5): Consider the following canonical epimorphism $g:\oplus R_i\to\oplus(R_i/\sigma_t(R_i))$, where $\oplus R_i$ is a direct sum of $R_i=R$. Clearly $\mathrm{Ker}(g)=\oplus\sigma_t(R_i)$. As the preradical $\sigma_t(\_)$ preserves direct sums, $\oplus\sigma_t(R_i)=\sigma_t(\oplus R_i)$. Thus $\oplus(R_i/\sigma_t(R_i))\cong(\oplus R_i)/\sigma_t(\oplus R_i)$, and (5) holds under the assumption.

(5)→(6): For a module M let N be a submodule of M containing $M\cdot\sigma_t(R)$ such that M/N is torsion. Then M/N is a right $R/\sigma_t(R)$-module. So M/N is a factor module of a direct sum of $R/\sigma_t(R)$, and so $M/N=0$.

(6)→(7): For a module M let K be a submodule of M such that M/K is torsion. As $\mathcal{T}_t$ is closed under taking factor modules, $M/(K+M\cdot\sigma_t(R))$ is torsion. But since $M/(M\cdot\sigma_t(R))$ has no nonzero factor module, we must have $M=$

$K+M\cdot\sigma_t(R)$.

(7)→(3): By Lemma 1.6, $M\cdot\sigma_t(R)\subset\sigma_t(M)$ holds for any module M and so (3) holds.

(4)→(8): It is clear.

Henceforth let t be an idempotent preradical. We shall prove the implications (8)→(2) and (2)→(1).

(8)→(2): For any weakly divisible module A, $t(A)=\sigma_t(A)$ holds by Lemma 1.3. Thus (2) holds.

(2)→(1): By Lemma 1.3 and Lemma 1.4, it is sufficient to prove that if $A\supset t(E(A))$, then $E(A)/A$ is torsionfree. We put $t(E(A)/A)=L/A$, where L is a submodule of $E(A)$ containing A. By Lemma 1.1, L is weakly divisible since $t(E(A))\subset L\subset E(A)$. Since $t(L)\subset A\subset L$ and $L/A\in\mathcal{T}_t$, we have $L/A=0$ and so $E(A)/A$ is torsionfree as desired.

For an idempotent preradical t, we have another characterization of the condition (*).

THEOREM 1.8. *For an idempotent preradical t, the following assertions are equivalent*:

(1) *Every weakly divisible module is divisible.*

(2) *For any weakly divisible mhdule A and any submodule B of A, $(t(A)+B)/B=t(A/B)$ holds.*

(3) *For any divisible module A and any submodule B of A, $(t(A)+B)/B=t(A/B)$ holds.*

(4) *For any injective module A and any submodule B of A, $(t(A)+B)/B=t(A/B)$ holds.*

PROOF. (1)→(2): Let A be a weakly divisible module and B a submodule of A. We put $t(A/B)=C/B$, C a submodule of A. Consider the sequence $0\to t(A)+B\to C\to C/(t(A)+B)\to 0$. By Lemma 1.1, $t(A)+B$ is weakly divisible, and so $t(A)+B$ is divisible by the assumption. Thus by (2) of Proposition 1.2, $t(C/(t(A)+B))=0$. But since $C/(t(A)+B)$ is torsion, we have $C=t(A)+B$ as desired.

(2)→(3)→(4): These are clear.

(4)→(1): By Lemmas 1.3 and 1.4, it is sufficient only to prove that for a module A if $A\supset t(E(A))$, then $t(E(A)/A)=0$. On the other hand, $t(E(A)/A)=(t(E(A))+A)/A$. Thus if $A\supset t(E(A))$, $t(E(A)/A)=0$ holds as desired.

COROLLARY 1.9. *If t is an epi-preserving and idempotent preradical, then every weakly divisible module is divisible.*

PROOF. It is clear by Theorem 1.8.

Next we consider the following condition (**):

(**) Any weakly divisible module is injective.

PROPOSITION 1.10. *Let t be a preradical. Consider the following conditions*:

(1) *Any weakly divisible module is injective.*

(2) *t has the following properties*:

(a) *Any factor module of $E/t(E)$ is torsionfree for any injective module E.*

(b) *For any module M, $E(M)/M$ is torsion.*

(3) *For any module M, $M+t(E(M))=E(M)$ holds.*

Then we have the implications (1)→(2)→(3). *If further t is an idempotent preradical, then* (3)→(1) *holds.*

PROOF. (1)→(2): For an injective module E, let L be a submodule of E containing $t(E)$. By Lemma 1.1, L is weakly divisible and so injective. Thus there exists a submodule H of E such that $E=L\oplus H$. Then $t(E)=t(L)\oplus t(H)$ and so $t(E)=L\cap t(E)=t(L)\oplus(L\cap t(H))=t(L)$. Thus we have $t(H)=0$ and so $t(E/L)=0$. So (a) holds. Next for a module M, we put $t(E(M)/M)=L/M$, where L is submodule of $E(M)$. By Lemma 1.1, L is weakly divisible and so injective. Thus L is a direct summand of $E(M)$. Since L is a large submodule of $E(M)$, $L=E(M)$ and so $E(M)/M$ is torsion. So (b) holds.

(2)→(3): For a module M, $t(E(M)/M)=E(M)/M$ by (b). By (a), $(E(M)/M)/((M+t(E(M)))/M)$ is torsionfree, and so $t(E(M)/M)=(M+t(E(M)))/M$. Thus $M+t(E(M))=E(M)$.

(3)→(1): By Lemma 1.3, it is sufficient to prove that if $M\supset t(E(M))$ for a module M, then $M=E(M)$. If $M\supset t(E(M))$, then $M=M+t(E(M))=E(M)$.

In the remainder of this section we treat weakly codivisible modules. At first in the case t is a radical, we shall characterize a weakly codivisible module with a projective cover.

LEMMA 1.11. *Let t be a preradical. If A is a weakly codivisible module and K is a submodule of $t(A)$, then A/K is also a weakly codivisible module.*

PROOF. See (1) of Lemma 11 of [6].

LEMMA 1.12. *Let t be a radical. Suppose that a module A has a projective cover $0\to K\to P\xrightarrow{h} A\to 0$. Then A is weakly codivisible if and only if $t(P)$ contains K.*

PROOF. Let A be a weakly codivisible module. Consider the sequence $0\to(K+t(P))/t(P)\to P/t(P)\to P/(K+t(P))\to 0$. By Lemma 1.11, $P/(K+t(P))$ is weakly codivisible as $P/(K+t(P))\cong A/h(t(P))$ and $t(A)\supset h(t(P))$. Since $P/t(P)$ is torsionfree, the above sequence splits. As $(K+t(P))/t(P)$ is a small submodule of $P/t(P)$, we have $(K+t(P))/t(P)=0$, and so $K\subset t(P)$. The converse is easily verified.

Next we consider the property that any weakly codivisible module is projective.

PROPOSITION 1.13. *Let t be a preradical and R a right perfect ring.* $(0\to K_M\to P_M\to M\to 0)$ *denotes the projective cover of a module M. Consider the following conditions*:

(1) *Any weakly codivisible module is projective.*

(2) *t has the following properties*:

(a) *Any submodule of $t(P)$ is torsion for any projective module P.*

(b) *For any module M, K_M is torsionfree.*

(3) *$K_M\cap t(P_M)=0$ holds for any module M.*

Then we have the implications (1)→(2)→(3). *If further t is a radical, then* (3)→(1) *holds.*

PROOF. (1)→(2): Let K be a submodule of $t(P)$ for a projective module P. Since P/K is weakly codivisible by Lemma 1.11, P/K is projective and so there exists a module S such that $P=K\oplus S$. Thus $t(P)=t(K)\oplus t(S)$ and so we have $K=K\cap t(P)=t(K)\oplus(K\cap t(S))=t(K)$. Thus (a) holds. By Lemma 1.11, $P_M/t(K_M)$ is weakly codivisible for a module M and so projective. Thus the sequence $0\to t(K_M)\to P_M\to P_M/t(K_M)\to 0$ splits. Since $t(K_M)$ is a small submodule of P_M, $t(K_M)=0$ and (b) holds.

(2)→(3): For a module M, $K_M\cap t(P_M)$ is torsion by (a). Thus $K_M\cap t(P_M)=t(K_M\cap t(P_M))\subset t(K_M)=0$ by (b) as desired.

(3)→(1): It is sufficient to prove that for a module M, if $t(P_M)\supset K_M$, then $K_M=0$ by Lemma 1.12. This is easy.

2. Injectively epi-preserving preradial and projectively left exact preradical.

In this section we study a preradical which satisfies (4) of Theorem 1.8. We call it an injectively epi-preserving preradical.

THEOREM 2.1. *Let t be a preradical. Then the following assertions are equivalent*:

(1) *$(\sigma_t(M)+N)/N\supset t(M/N)$ holds for any module M and any submodule N of M.*

(2) *For a module M such that $t(M)=\sigma_t(M)$, $t(M/N)=(t(M)+N)/N$ holds for*

any submodule N of M.

(3) *$t(E/K)=(t(E)+K)/K$ holds for any injective module E and any submodule K of E.*

(4) *Any factor module of $M/\sigma_t(M)$ is torsionfree for any module M.*

(5) *Any factor module of $E/t(E)$ is torsionfree for any injective module E.*

(6) *Any factor module of any direct sum of $R/\sigma_t(R)$ is torsionfree.*

(7) *Any factor module of $M/M\cdot\sigma_t(R)$ is torsionfree for any module M.*

(8) *$(M\cdot\sigma_t(R)+N)/N\supset t(M/N)$ holds for any module M and any submodule N of M.*

(9) *Any factor module of $P/\sigma_t(P)$ is torsionfree for any projective module P.*

(10) *For any injective module E, $t(L)=L\cdot t(R)$ holds for any factor module L of $E/t(E)$.*

PROOF. (1)→(2): For any module M and any submodule N of M, $t(M/N)\supset(t(M)+N)/N$ holds since t is a preradical. If $t(M)=\sigma_t(M)$, then $t(M/N)\subset(t(M)+N)/N$ holds, and so (2) holds.

(2)→(3)→(5) and (4)→(9): These are clear.

(5)→(4): Since $M/\sigma_t(M)\cong(M+t(E(M)))/t(E(M))$ for a module M, it is sufficient to prove that any factor module of $(M+t(E(M)))/t(E(M))$ is torsionfree for a module M. Let N be a submodule of $M+t(E(M))$ containing $t(E(M))$. Since $E(M)/N$ is torsionfree, $(M+t(E(M)))/N$ is torsionfree as desired.

(9)→(6)→(7): We can verify as in the proof of (4)→(5)→(6) of Theorem 1.7.

(7)→(8): Let M be a module and N a submodule of M. We put $M/N=X$ and $(M\cdot\sigma_t(R)+N)/N=Y$. Then X/Y is torsionfree, and so $0=t(X/Y)\supset(t(X)+Y)/Y$. Thus $t(X)\subset Y$ holds.

(8)→(1): It follows from the fact that $M\cdot\sigma_t(R)\subset\sigma_t(M)$ for any module M.

(3)→(10): Let E be an injective module and K a submodule of E containing $t(E)$. By the assumption we have $t(E/K)=0$ and $(E/K)\cdot t(R)=(E\cdot t(R)+K)/K\subset(t(E)+K)/K=0$, and so $t(E/K)=(E/K)\cdot t(R)$ holds as desired.

(10)→(5): Let E be an injective module and K a submodule of E containing $t(E)$. By the assumption, $t(E/K)=(E/K)\cdot t(R)$. Since $(E/K)\cdot t(R)\subset(t(E)+K)/K=0$, $t(E/K)=0$ as desired.

COROLLARY 2.2. *Let t be a preradical. If $t(E(R))\supset R$, then t is an injectively epi-preserving preradical and any injective module is torsion.*

PROOF. The first claim is clear by (6) of Theorem 2.1. If $t(E(R))\supset R$, then $\sigma_t(R)=R$ and so $\sigma_t(M)=M$ for any module M. Thus any injective module is torsion.

Next we dualize Theorem 2.1. We consider a preradical t which has the property that $t(K)=K\cap t(P)$ holds for any projective module P and any submodule K of P. We call it a projectively left exact preradical.

LEMMA 2.3. *Let t be a preradical. For any projective module P, $t(P)=P\cdot t(R)$ holds.*

PROOF. It is obvious.

(2) and (4) of the following theorem are seen in Theorem 13 of [6] for a radical t.

THEOREM 2.4. *Let t be a preradical. Then the following assertions are equivalent:*

(1) *$M\cdot t(R)\cap N\subset t(N)$ holds for any module M and any submodule N of M.*

(2) *For any module M such that $t(M)=M\cdot t(R)$, $t(N)=N\cap t(M)$ holds for any submodule N of M.*

(3) *For any projective module P, $t(K)=K\cap t(P)$ holds for any submodule K of P.*

(4) *Any submodule of $M\cdot t(R)$ is torsion for any module M.*

(5) *For any projective module P, any submodule of $t(P)$ is torsion.*

(6) *Any submodule of any direct sum of $t(R)$ is torsion.*

(7) *t has the following properties:*

(a) *$t(R)$ is torsion.*

(b) *For a projective module P and a torsion submodule L of P, any submodule of L is torsion.*

PROOF. (1)→(2): It is clear.

(2)→(3): This is clear by Lemma 2.3.

(3)→(5): Let P be a projective module and K a submodule of $t(P)$. By the assumption, $t(K)=K\cap t(P)=K$. Hence (5) holds.

(5)→(6): Let K be a submodule of $\oplus t(R)$. Since $\oplus R$ is projective and $t(\oplus R)=\oplus t(R)\supset K$, K is torsion.

(4)→(1): For a module M, let N be a submodule of M. Since $M\cdot t(R)\cap N\subset M\cdot t(R)$, $M\cdot t(R)\cap N=t(M\cdot t(R)\cap N)\subset t(N)$.

(7)→(5): (a) implies the fact that for any projective module P, $t(P)$ is torsion by Lemma 2.3. Thus (5) holds.

(3)→(7): (a) is easy. Let P be a projective module, L a torsion submodule of P and K a submodule of L. Then we have $t(K)=K\cap t(P)\supset K\cap t(L)=K\cap L=K$. Thus (b) holds.

(6)→(4): Let M be a module and K a submodule of $M \cdot t(R)$. The canonical epimorphism $g: R^{(M)} \to M \to 0$ induces $t(g): t(R)^{(M)} \to M \cdot t(R) \to 0$, as $-\otimes_R t(R)$ is right exact. Since $t(g)$ is an epimorphism, K is a factor module of some submodule of $t(R)^{(M)}$. Thus K is torsion.

COROLLARY 2.5. *If a preradical t is epi-preserving and projectively left exact, then t is left exact.*

PROOF. It is sufficient to prove that for any module M any submodule of $t(M)$ is torsion. Since t is epi-preserving, $t(M)=M \cdot t(R)$ for any module M by [1]. Thus by (4) of Theorem 2.4, any submodule of $t(M)$ is torsion.

PROYOSITION 2.6. *If t is an epi-preserving preradical satisfying* (a) *of* (7) *of Theorem* 2.4, *then t is idempotent.*

PROOF. Let M be a module. Since $t(M)=M \cdot t(R)$ and $t(t(M))=t(M) \cdot t(R)= M \cdot t(R) \cdot t(R)=M \cdot t(t(R))$, $t(M)=t(t(M))$.

Finally we give examples of these preradicals. They all are neither idempotent nor radical. Especially, the preradical in Example 1 is both injectively epi-preserving and projectively left exact. However the preradicals in Examples 2 and 3 are either injectively epi-preserving or projectively left exact.

EXAMPLE 1. Let Z be the ring of rational integers. For a module M_Z, we put $t(M)=J(M)+\mathrm{Soc}(M)$, where $J(M)$ denotes the intersections of all maximal submodules of M and $\mathrm{Soc}(M)$ denotes the sum of all minimal submodules of M. It is easily verified that t is a preradical on Mod-Z.

(1) t is not an idempotent preradical (hence not left exact), as $t(Z/8Z) \neq t(t(Z/8Z))$.

(2) t is not a radical (hence not epi-preserving), as $t((Z/8Z)/t(Z/8Z)) \neq 0$.

(3) t is a projectively left exact preradical as $t(Z)=0$.

(4) t is an injectively epi-preserving preradical by Corollary 2.2 as $t(E(Z)) = E(Z) \supset Z$.

(5) $\mathcal{T}_t$ is not closed under taking submodules as $t(E(Z))=E(Z)$ and $t(Z)=0$.

(6) $\mathcal{F}_t$ is not closed under taking factor modules as $t(Z)=0$ and $t(Z/2Z) \neq 0$.

EXAMPLE 2. Let Z be the ring of rational integers. For a module M_Z, we put $t(M)=J(M) \cap \mathrm{Soc}(M)$. It is easily verified that t is a preradical on Mod-Z. t is not an idempotent preradical, not a radical and a projectively left exact preradical as $t(Z/8Z) \neq t(t(Z/8Z))$, $t((Z/8Z)/t(Z/8Z)) \neq 0$ and $t(Z)=0$, respectively.

Suppose that t is an injectively epi-preserving preradical. By (6) of Theorem 2.1, any factor module of a direct sum of Z is torsionfree as $t(E(Z))=0$. Thus any Z-module is torsionfree. But $t(Z/8Z)\neq 0$, and so t is not an injectively epi-preserving preradical.

EXAMPLE 3. Let Z be the ring of rational integers. For a module M_Z, $t(M)=\mathrm{Soc}(M)+2M$, where $2M=\{2m : m\in M\}$. It is easily proved that t is a preradical on Mod-Z. t is not an idempotent preradical, not a radical and an injectively epi-preserving preradical as $t(Z/8Z)=t(t(Z/8Z))$, $t((Z/8Z)/t(Z/8Z))\neq 0$ and $t(E(Z))=E(Z)$, respectively. Suppose that t is a projectively left exact preradical. Then $t(Z)=2Z$ is torsion by (6) of Theorem 2.4. But $t(Z)\neq t(t(Z))$, and so t is not a projectively left exact preradical.

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Addendum

Recently the author has received a paper by M. Sato entitled "On pseudo-cohereditary sub-torsion theories and weakly divisible modules", where he has also studied the property that every weakly divisible module is divisible for an idempotent preradical.