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ON CONJUGATE POINTS OF A NILPOTENT LIE GROUP

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Let G be a Lie group with a left invariant metric. The properties of curvature of G has been investigated by Milnor [3]. In particular, it is known that if the Lie group G is nilpotent but not commutative then for any left invariant metric there exists a direction of strictly negative sectional curvature and a direction of strictly positive sectional curvature [3]. It is also well-known that there is no conjugate point if the sectional curvature is non-positive. Moreover O'Sullivan [6] has shown that if G is a nilpotent Lie group but not commutative then there exists a conjugate point of G for any left invariant metric. In this note we show that under the same assumption G has a conjugate point in the center. We consider the relations between first conjugate locus and cut locus for a simply connected 2-step nilpotent Lie group. Moreover in the case of Heisenberg group, we prove that first conjugate locus coincides with cut locus for any left invariant metric. The authors would like to express their thanks to the refree for his helpful suggestions.

§ 1. Preliminaries.

Let G be a Lie group of dimension n with a left invariant metric $\langle\ ,\ \rangle$ and $\mathfrak{g}$ the Lie algebra of left invariant vector fields of G. The Riemannian connection ∇ for $(G, \langle\ ,\ \rangle)$ is given by

$$\langle\nabla_X Y, Z\rangle=(1/2)\{\langle[X, Y], Z\rangle-\langle[Y, Z], X\rangle+\langle[Z, X], Y\rangle\} \tag{1.1}$$

for left invariant vector fields $X, Y, Z\in\mathfrak{g}$.

Take an orthonormal basis $\{X_1, \cdots, X_n\}$ of $\mathfrak{g}$ and let C_{ij}^k be the structure constants with respect to $\{X_1, \cdots, X_n\}$, that is,

$$[X_i, X_j]=\sum_{k=1}^{n} C_{ij}^k X_k\ . \tag{1.2}$$

Then we have

$$\nabla_{X_i}X_j=\frac{1}{2}\sum_{k=1}^{n}(C_{ij}^k-C_{jk}^i+C_{ki}^j)X_k \tag{1.3}$$

for $i, j=1, \cdots, n$ by (1.1).

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Consider a geodesic $\sigma(t)$ through $\sigma(0)=e\in G$. We can write

$$\dot{\sigma}(t)=\sum_{i=1}^{n} a_i(t)(X_i)_{\sigma(t)} \tag{1.4}$$

where $a_i(t): \boldsymbol{R}\rightarrow\boldsymbol{R}$ are C^∞ functions.

Now

$$\nabla_{\dot{\sigma}}\dot{\sigma}=\sum_j\Big(\frac{da_j}{dt}(X_j)_{\sigma(t)}+a_j(t)\sum_i a_i(t)(\nabla_{X_i X_j})_{\sigma(t)}\Big)$$

$$=\sum_k\Big\{\frac{da_k}{dt}+\sum_{i,j}a_i(t)a_j(t)\frac{1}{2}(C_{ij}^k-C_{jk}^i+C_{ki}^j)\Big\}(X_k)_{\sigma(t)}.$$

Noting that $C_{ij}^k=-C_{ji}^k$, we see that a curve $\sigma(t)$ is a geodesic of $(G, \langle\ ,\ \rangle)$ if and only if $a_k(t)$ satisfy the following equations:

$$\frac{da_k}{dt}=\sum_{i,j}a_i a_j C_{jk}^i \quad (k=1, \cdots, n). \tag{1.5}$$

REMARK 1. If the metric $\langle\ ,\ \rangle$ on G is bi-invariant, a curve $\sigma(t)$ is a geodesic if and only if a_k $(k=1, \cdots, n)$ are constant.

REMARK 2. It is known that a simply connected Lie group G admits a bi-invariant metric if and only if G is the product of a compact Lie group and a vector group [3].

For a Riemannian manifold M of n-dimension let R denote the curvature of M. A vector field $X(t)$ along a geodesic $\sigma(t)$ is called a *Jacobi field* if X satisfies the Jacobi equation

$$\nabla_{\dot{\sigma}}^2 X+R(X, \dot{\sigma})\dot{\sigma}=0. \tag{1.6}$$

It is known that the space of solutions of (1.6) is of $2n$-dimension and the subspace of solutions with $X(0)=0$ is of n-dimension.

Fix a point $p\in M$ and consider a geodesic $\sigma(t)$ parameterized by arc length through $p=\sigma(0)$. A point $q=\sigma(t_0)\in M$ is said to be *conjugate to* p *along a geodesic* $\sigma(t)$ if there exists a non-zero Jacobi field J along $\sigma(t)$ such that $J(0)=J(t_0)=0$. Furthermore let $d(p, q)$ denote the distance between p and q in M. A point $q=\sigma(t_1)\in M$ is called a *cut point along a geodesic* $\sigma(t)$ if $d(\sigma(0), \sigma(t))=t$ for $0\leqq t\leqq t_1$ and $d(\sigma(0), \sigma(t))<t$ for $t>t_1$.

§2. Existence of a conjugate point.

In this section we improve a theorem of O'Sullivan [6].

THEOREM 2.1. *Let G be a nilpotent Lie group with a left invariant metric. Let Z denote the center of G and $C(e)$ the first conjugate locus of the identity element $e\in G$. If G is not commutative,*

$$C(e)\cap Z\neq\emptyset .$$

PROOF.[1] Consider the desending central series $\{C^k(\mathfrak{g})\}_k$ of the Lie algebra $\mathfrak{g}$ of G:

$$\mathfrak{g}\supset C^1(\mathfrak{g})=[\mathfrak{g},\mathfrak{g}]\supset C^2(\mathfrak{g}=[\mathfrak{g}, C^1(\mathfrak{g})]\supset\cdots\supset C^k(\mathfrak{g})\supset\cdots$$

with $C^l(\mathfrak{g})\neq(0)$ and $C^{l+1}(\mathfrak{g})=(0)$. By the proof of Theorem 2.4 in [3] (p. 301), for each unit vector $X\in C^l(\mathfrak{g})$, the Ricci curvature $r(X)$ is positive. Since $C^l(\mathfrak{g})$ is contained in the center of $\mathfrak{g}$, the integral curve $C(t)$ through e of the vector field X is a geodesic contained in the center Z. Since the Riemannian metric is left invariant, the Ricci curvature $r(\dot{C}(t))$ is positive constant along the geodesic $C(t)$. By the proof of a Theorem of Myers ([5], see also [4] p. 104), we see that the geodesic $C(t)$ has a conjugate point to $e\in G$. q. e. d.

REMARK. In general a simply connected solvable Lie group does not have a conjugate point. See [3] example 1.7.

3. Cut locus and first conjugate locus.

In this section we exclusively consider a 2-step nilpotent Lie group. Let $\mathfrak{g}$ be a 2-step nilpotent Lie algebra with an inner product $\langle\ ,\ \rangle$ and $\mathfrak{z}$ the center of $\mathfrak{g}$. Fix an orthonormal basis $\{X_1,\cdots,X_n\}$ of $\mathfrak{g}$ such that $\{X_{k+1}, X_l, X_{l+1},\cdots, X\}$ is a basis of $\mathfrak{z}$ and $\{X_{l+1},\cdots,X_n\}$ is a basis of $[\mathfrak{g},\mathfrak{g}]$. The following ranges of indices will be used throughout this section.

$$1\leqq A, B, \cdots\leqq n$$

$$1\leqq i, j, \cdots\leqq k$$

$$k+1\leqq p, q, \cdots\leqq l$$

$$l+1\leqq \alpha, \beta, \cdots\leqq n .$$

Let G be the simply connected Lie group with the Lie algebra $\mathfrak{g}$. From now on let $\sigma(t)$ be a geodesic of G parameterized by arc length through $\sigma(0)=e\in G$. By (1.5), we have

1) This proof is due to the refree.

$$(3.1)\qquad \begin{cases} \dfrac{da_\alpha}{dt}=0 \\ \dfrac{da_p}{dt}=0 \\ \dfrac{da_i}{dt}=\sum_{j,\alpha} a_j a_\alpha C_{ji}^{\alpha}\,. \end{cases}$$

Since $\exp: \mathfrak{g}\to G$ is diffeomorphism, we can define coordinates x_A on G (with respect to the basis $\{X_1, \cdots, X_n\}$) by

$$\exp\left(\sum_A x_A(g)X_A\right)=g \quad (g\in G)\,.$$

Since $\exp X \exp Y=\exp(X+Y+(1/2)[X, Y])$ for $X, Y\in\mathfrak{g}$, we have

$$(3.2)\qquad \begin{cases} x_i(gh)=x_i(g)+x_i(h) \\ x_p(gh)=x_p(g)+x_p(h) \\ x_\alpha(gh)=x_\alpha(g)+x_\alpha(h)+\dfrac{1}{2}\sum_{j,k} x_j(g)x_k(h)C_{jk}^{\alpha} \end{cases}$$

for $g, h\in G$. Since X_A ($A=1, \cdots, n$) are left invariant, we get

$$(3.3)\qquad \begin{cases} X_i=\dfrac{\partial}{\partial x_i}-\dfrac{1}{2}\sum_{\beta,j} C_{ij}^{\beta}x_j\dfrac{\partial}{\partial x_\beta} \\ X_p=\dfrac{\partial}{\partial x_p} \\ X_\alpha=\dfrac{\partial}{\partial x_\alpha} \end{cases}$$

Let $\sigma(t)=(x_A(t))$ be a geodesic through e. By (1.4) and (3.3), we get

$$(3.4)\qquad \begin{cases} \dfrac{dx_i}{dt}=a_i(t) \\ \dfrac{dx_p}{dt}=a_p \\ \dfrac{dx_\alpha}{dt}=a_\alpha-\dfrac{1}{2}\sum_{i,j}C_{ij}^{\alpha}a_i(t)x_j(t)\,. \end{cases}$$

LEMMA 3.1. *Suppose that* $x_i(t_0)=0$ *for all* i *with* $t_0>0$. *Then we have*

$$(3.5)\qquad \sum_p x_p^2(t_0)+\sum_\alpha x_\alpha^2(t_0)\geqq t_0^2$$

and the equality holds if and only if the geodesic $\sigma(t)$ *is contained in the center* *of G.*

PROOF. Since

$$x_\alpha(t)=a_\alpha t-\frac{1}{2}\sum_{i,j}\int_0^t C_{ij}^\alpha a_i(s)x_j(s)ds\,,$$

$$\sum_\alpha x_\alpha^2(t)=\Big(\sum_\alpha a_\alpha^2\Big)t^2-\sum_{\alpha,i,j}a_\alpha t\int_0^t C_{ij}^\alpha a_i(s)x_j(s)ds+\frac{1}{4}\sum_\alpha\Big(\sum_{i,j}\int_0^t C_{ij}^\alpha a_i(s)x_j(s)ds\Big)^2.$$

By (3.1), we see that

$$\sum_{\alpha,i,j}\int_0^t a_\alpha a_i(s)C_{ij}^\alpha x_j(s)ds=\sum_j\int_0^t\frac{da_j}{dt}(s)x_j(s)ds\,.$$

By integrating in part, we have

$$\sum_j\int_0^t\frac{da_j}{dt}(s)x_j(s)ds=\Big(\sum_j a_j(t)x_j(t)\Big)-\sum_j\int_0^t a_j(s)\frac{dx_j}{dt}(s)ds$$

$$=\Big(\sum_j a_j(t)x_j(t)\Big)-\sum_j\int_0^t a_j^2(s)ds$$

by (3.4). Since $x_i(t_0)=0$ by assumption and

$$\sum_j a_j^2(s)=1-\sum_\alpha a_\alpha^2-\sum_p a_p^2\,,$$

we get

$$\sum_\alpha x_\alpha^2(t_0)=\Big(\sum_\alpha a_\alpha^2\Big)t_0^2+\Big\{1-\Big(\sum_\alpha a_\alpha^2+\sum_p a_p^2\Big)\Big\}t_0^2+\frac{1}{4}\sum_\alpha\Big(\sum_{i,j}\int_0^{t_0}C_{ij}^\alpha a_i(s)x_j(s)ds\Big)^2$$

$$\geqq\Big(1-\sum_p a_p^2\Big)t_0^2\,.$$

Moreover the equality holds if and only if

$$\sum_{i,j}\int_0^{t_0}C_{ij}^\alpha a_i(s)x_j(s)ds=0 \qquad \text{for all } \alpha$$

and thus $x_\alpha(t_0)=a_\alpha t_0$ for all α. This implies that $\sum_\alpha a_\alpha^2+\sum_p a_p^2=1$ and $a_i(t)=0$ for all i, and hence the geodesic $\sigma(t)$ is contained in the center of G. q. e. d.

COROLLARY 3.2. *Fix a point $x\in G$ which is contained in the center of G. If $\sigma(t)$, $C(t)$ are geodesics parameterized by are length through $\sigma(0)=C(0)=e\in G$, $\sigma(t_0)=C(t_1)=x$ and $C(t)$ is contained in the center of G but $\sigma(t)$ is not contained in the center of G, then we have $t_1>t_0$, that is, length of $C(t)$ from e to x is longer than the one of $\sigma(t)$.*

COROLLARY 3.3. *There is no closed geodesic in a simply connected 2-step nilpotent Lie group G with a left invariant metric.*

PROOF. If there is a closed geodesic $\sigma(t)$ in G, then there exists $t_0>0$ such that $\sigma(t_0)=e\in G$. In particular, $x_A(t_0)=0$ for all A. But this is impossible by (3.5).

Now we consider the cut locus of G. Since G has a conjugate point and G acts itself as an isometry group, the cut locus of each point $g\in G$ is not empty.

THEOREM 3.4. *Let G be a simply connected 2-step nilpotent Lie group with a left invariant metric. If $x\in G$ is a first conjugate point to $e\in G$ along a geodesic $C(t)$ contained in the center of G, it is also a cut point of $C(t)$ with respect to $e\in G$.*

PROOF. If $x\in G$ is not a cut point of $C(t)$ with respect to e, there is a point $y\in G$ such that $y\neq x$ and y is a cut point of $C(t)$ with respect to e. Since y is not a first conjugate point to e along $C(t)$, it is known that there exist at least two minimizing geodesics ([1], [2]). Since $C(t)$ is the unique geodesic from e to y which is contained in the center of G, there exists a minimizing geodesic $\sigma(t)$ from e to y which is not contained in the center of G and has the same length as $C(t)$. This contradicts Corollary 3.2. q. e. d.

§ 4. Heisenberg group H of dimension 3.

In this section we consider the Heisenberg group of dimension 3, that is,

$$H=\left\{\begin{pmatrix} 1 & y & z \\ 0 & 1 & x \\ 0 & 0 & 1 \end{pmatrix} \,\middle|\, x, y, z\in \boldsymbol{R}\right\}.$$

Let $\mathfrak{h}$ be the Lie algebra of H and fix a left invariant metric on H. Note that $\mathfrak{h}\supset C^1(\mathfrak{h})=[\mathfrak{h}, \mathfrak{h}]\supset C^2(\mathfrak{h})=(0)$. Take an orthonormal basis $\{X_1, X_2, X_3\}$ of $\mathfrak{h}$ in such a way that

$$[X_1, X_2]=C_{12}^3 X_3,\ C_{12}^3\neq 0, \qquad [X_1, X_3]=[X_2, X_3]=0.$$

From now on put $C=C_{12}^3$ for simplicity. Note that $\mathfrak{h}$ is a 2-step nilpotent Lie algebra. By (3.2) and (3.3), we have

$$\begin{cases} x_1(gh)=x_1(g)+x_1(h) \\ x_2(gh)=x_2(g)+x_2(h) \\ x_3(gh)=x_3(g)+x_3(h)+(1/2)C(x_1(g)x_1(h)-x_1(h)x_2(g)) \end{cases}$$

for $g,\ h\in H$ and

$$\begin{cases} X_1=\dfrac{\partial}{\partial x_1}-(1/2)Cx_2\dfrac{\partial}{\partial x_3} \\ X_2=\dfrac{\partial}{\partial x_2}+(1/2)Cx_1\dfrac{\partial}{\partial x_3} \\ X_3=\dfrac{\partial}{\partial x_3}\,. \end{cases} \tag{4.1}$$

Let $\sigma(t)$ be a geodesic through $e\in H$. By (1.5) we have

$$\begin{cases} \dfrac{da_1}{dt}=-a_3a_2C \\ \dfrac{da_2}{dt}=a_3a_1C \\ \dfrac{da_3}{dt}=0\,. \end{cases} \tag{4.2}$$

Since a_3 is constant and

$$\begin{pmatrix} \dfrac{da_1}{dt} \\ \dfrac{da_2}{dt} \end{pmatrix}=a_3C\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \end{pmatrix},$$

$$\begin{bmatrix} a_1(t) \\ a_2(t) \end{bmatrix}=\begin{bmatrix} \cos a_3Ct & -\sin a_3Ct \\ \sin a_3Ct & \cos a_3Ct \end{bmatrix}\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \tag{4.3}$$

is the solution of (4.2) with the initial condition $a_1,\ a_2\in \boldsymbol{R}$. Put $\sigma(t)=(x_1(t),\ x_2(t),\ x_3(t))$. Since $\dot{\sigma}(t)=\sum_i a_i(t)(X_i)_{\sigma(t)}$, $x_i(t)$ $(i=1,\ 2,\ 3)$ are a solution of ordinary differential equations

$$\begin{cases} \dfrac{dx_1}{dt}=a_1(t) \\ \dfrac{dx_2}{dt}=a_2(t) \\ \dfrac{dx_3}{at}=a_3-(1/2)C(a_1(t)x_2-a_2(t)x_1) \end{cases} \tag{4.4}$$

by (4.1). It is easy to see that if $a_3\neq 0$

(4.5)
$$\begin{cases} x_1(t)=(a_1/a_3C)\sin a_3Ct+(a_2/a_3C)(\cos a_3Ct-1) \\ x_2(t)=(a_2/a_3C)\sin a_3Ct-(a_1/a_3C)(\cos a_3Ct-1) \\ x_3(t)=a_3t+(1/2a_3)(a_1^2+a_2^2)t-(1/2a_3^2C)(a_1^2+a_2^2)\sin a_3Ct \end{cases}$$

is the solution of (4.4) with the initial condition $\frac{dx_i}{dt}(0)=a_i$ $(i=1, 2, 3)$ and $x_i(0)=0$ $(i=1, 2, 3)$, and if $a_3=0$

(4.6)
$$\begin{cases} x_1(t)=a_1t \\ x_2(t)=a_2t \\ x_3(t)=0\,. \end{cases}$$

Now we consider Jacobi fields J along a geodesic $\sigma(t)$ such that $J(0)=0$ and $(\nabla_{\dot\sigma}J)(0)=w$. Let $T_e(H)$ denote the tangent space of H at the identity element $e\in H$ and $\mathrm{Exp}_e: T_e(H)\to H$ the exponential map. Consider the variation field of the 1-parameter family of geodesics $\mathrm{Exp}_e(t(\dot\sigma(0)+sw))$. By (4.5), for a geodesic $\sigma(t)$ with $\dot\sigma(0)=(a_1, a_2, a_3)$, $a_3\neq 0$, a basis of Jacobi fields along the geodesic $\sigma(t)$ is given by

(4.7)
$$\begin{cases} J_1=t\dot\sigma(t)=\left(t\dfrac{dx_1}{dt},\, t\dfrac{dx_2}{dt},\, t\dfrac{dx_3}{dt}\right) \\ J_2=\left(\dfrac{1}{a_3C}\sin a_3Ct,\, \dfrac{-1}{a_3C}(\cos a_3Ct-1),\, \dfrac{a_1}{a_3}t-\dfrac{a_1}{a_3^2C}\sin a_3Ct\right) \\ J_3=\left(\dfrac{1}{a_3C}(\cos a_3Ct-1),\, \dfrac{1}{a_3C}\sin a_3Ct,\, \dfrac{a_2}{a_3}t-\dfrac{a_2}{a_3^2C}\sin a_3Ct\right). \end{cases}$$

By (4.5) and (4.6), for a geodesic $\sigma(t)$ with $\dot\sigma(0)=(a_1, a_2, 0)$, a basis of Jacobi fields along the geodesic $\sigma(t)$ is given by

(4.8)
$$\begin{cases} J_1=(a_1t,\, a_2t,\, 0) \\ J_2=(-a_2t,\, a_1t,\, 0) \\ J_3=(0,\, 0,\, t)\,. \end{cases}$$

From (4.8) we see that there is no conjugate point along a geodesic $\sigma(t)=(a_1t, a_2t, 0)$, Now we compute the first conjugate point along a geodesic $\sigma(t)$ with $\dot\sigma(0)=(a_1, a_2, a_3)$, $a_3\neq 0$, where $a_1^2+a_2^2+a_3^2=1$.

We need the following Lemma.

LEMMA 4.1. *We define a function* $f: \boldsymbol{R}\to\boldsymbol{R}$ *by*

$$f(\theta)=\begin{vmatrix} \sin\theta & \cos\theta-1 & a_1\cos\theta-a_2\sin\theta \\ 1-\cos\theta & \sin\theta & a_1\sin\theta+a_2\cos\theta \\ a_1(\theta-\sin\theta) & a_2(\theta-\sin\theta) & \dfrac{1+a_3^2}{2}-\dfrac{1-a_3^2}{2}\cos\theta \end{vmatrix}$$

Then $f(\theta)=\{2(1-\cos\theta)-(1-a_3^2)\theta\sin\theta\}$, $f(\theta)=f(-\theta)$ *and* $f(\theta)>0$ *for* $0<|\theta|<2\pi$.

PROOF. Our first claim is a straightforward computation. Note that

$$f'(\theta)=(1+a_3^2)\sin\theta-(1-a_3^2)\theta\cos\theta$$

and

$$f''(\theta)-2a_2^2\cos\theta+(1-a_3^2)\theta\sin\theta\ .$$

For $0<\theta<\pi/2$ $f''(\theta)>0$ and hence $f'(\theta)>0$ for $0<\theta<\pi/2$ since $f'(0)=0$. Obviously $f'(\theta)>0$ for $\pi/2<\theta<\pi$ and hence $f(\theta)>0$ for $0<\theta<\pi$. It is clear that $f(\theta)>0$ for $\pi\leqq\theta<2\pi$. q. e. d.

From (4.7) and Lemma 4.1, by putting $\theta=a_3Ct$, we see that the first conjugate point along a geodesic $\sigma(t)$ with $\dot{\sigma}(0)=(a_1, a_2, a_3)$ $a_3\neq 0$ is given by $\sigma(2\pi/|a_3C|)$ $(0, 0, \pi(1+a_3^2)/|a_3C|a_3)$. Now we can summarize as follows.

THEOREM 4.2. *For a left invariant metric on the Heiseeberg group H, the first conjugate locus of the identity element of H is contained in the center of H and given by*

$$\{(0, 0, \pm\pi(1+a_3^2)/a_3^2|C|)\,;\,0<a_3^2\leqq 1\}=\{(0, 0, \pm s\pi/|C|\,;\,s\geqq 2\}\ .$$

REMARK. Let $\sigma(t)$ be a geodesic and $q=\sigma(t_0)$ a conjugate point to $\sigma(0)$. The dimension of Jacobi fields J such that $J(0)=J(t_0)=0$ is called the *order of the conjugate point q* of the geodesic $\sigma(t)$. The order of the first conjugate point of a geodesic $\sigma(t)$ in H with $\sigma(0)=(a_1, a_2, a_3)$ $a_3\neq 0$ is given as follows:

$$\begin{cases} 1 & \text{if} \quad a_3\neq 1 \\ 2 & \text{if} \quad a_3=1\,. \end{cases}$$

THEOREM 4.3. *Let H be the Heisenberg group with a left invariant metric. For each $g\in H$ the cut locus of g coincides with the first conjugate locus of g. Moreover the cut locus of the identity element of H is contained in the center of H.*

PROOF. Since H acts as an isometry group via left translations, it is enough to see our claim at the identity element $e \in H$. Let $C(t)$ be a geodesic through e which is contained in the center of H and let x be the first conjugate point to e along $C(t)$. Then the point x is also a cut point of $C(t)$ with respect to $e \in H$ by Theorem 3.4. Now we consider a geodesic $\sigma(t)$ with $\dot{\sigma}(0)=(a_1, a_2, a_3)$ through e which is not contained in the center of H and let x be the first conjugate point to e along $\sigma(t)$. We claim that $\sigma(t)$ realizes the distance between e and x. By Theorem 4.2, the first conjugate point x is contained in the center of H. Put $x=(0, 0, s\pi/|C|)$. We may assume that $1>a_3>0$. Then $s>2$. By (4.5) we have

$$a_3Ct_0=2\pi m \ (m \in \boldsymbol{Z}), \ s\pi/|C|=a_3t_0+\frac{1}{2a_3}(a_1^2+a_2^2)t_0=\frac{1}{2a_3}(1+a_3^2)t_0 .$$

Since x is the first conjugate point to e along $\sigma(t)$, that length of $\sigma(t)$ from e to x is given by $t_0=(\sqrt{s-1}2\pi)/|C|$. Since $\sqrt{s-1}2\pi/|C|<s\pi/|C|$, the length $\sigma(t)$ from e to x is less than that of the geodesic $C(t)=(0, 0, t)$ $(0 \leqq t \leqq (s/|C|)\pi)$ contained in the center of H. Let $\tau(t)$ be a minimal geodesic from e to x. Then x is also a conjugate point to e along $\tau(t)$ by (4.5) and Lemma 4.1. Since $\tau(t)$ is minimal, x is also the first conjugate point to e along $\tau(t)$. In this case we have $t_2=\sqrt{s-1}2\pi/|C|$ where $x=\tau(t_2)$ by the same computation as above and hence $\sigma(t)$ and $\tau(t)$ have the same length. This implies that if x is the first conjugate point to e along $\sigma(t)$ then $\sigma(t)$ realizes the distance $d(e, x)$, and hence the point x is also a cut point to e. q. e. d.

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