

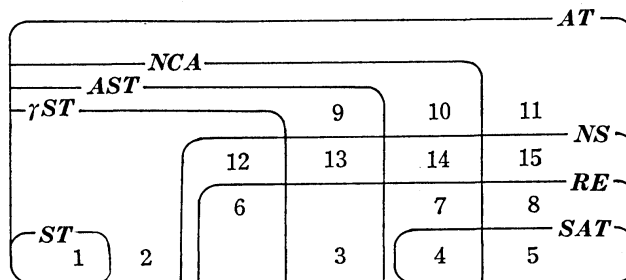
ON A CLASSIFICATION OF ARONSZAJN TREES II

By

Masazumi HANAZAWA

§ 1. Introduction.

In the former paper [3], we considered the classification of Aronszajn trees by the notions of Souslin trees, ω_1 -trees with property γ , almost-Souslin trees, ω_1 -trees with no club antichain, special Aronszajn trees and R -embeddable trees. As we remarked in its last section, there is another interesting notion. It is the notion of non-Souslin trees which had been introduced by Baumgartner [1]. The classification of Aronszajn trees by this notion together with the previous ones is shown by the following :



where **ST**=the class of Souslin trees,
 γ ST=the class of ω_1 -trees with property γ ,
AST=the class of almost-Souslin trees,
NCA=the class of ω_1 -trees with no club anti-chain,
SAT=the class of special Aronszajn tree,
RE=the class of R -embeddable ω_1 -trees,
NS=the class of non-Souslin trees,
AT=the class of Aronszajn trees.

Under ZFC alone, none of the categories but Category 5 can be proved to be non-void. In the former paper we proved that if $V=L$, Categories 1~11 are all non-void (note that the trees constructed in Theorems 9, 10 and 11 [3], are the elements of Categories 9, 10 and 11 respectively). In this paper we shall prove that if $V=L$, remaining Categories 12~15 are also non-void. It is shown as a

by-product that \diamond suffices for the existence of non-Souslin trees which are not \mathbf{R} -embeddable.

§ 2. Preliminaries.

Most of the notions and the notations which are used here are described in the former paper. It is assumed that the reader knows them. Let $T = \langle T, <_T \rangle$ be a tree. $\langle X, <_T \rangle$ is called a subtree of T if $X \subset T$. $\langle X, <_T \rangle$ is called a transitive subtree of T if it is a subtree of T such that $(\forall x \in X \forall y \in T)[y <_T x \rightarrow y \in X]$ (in the paper [3], we called a transitive subtree a subtree). When $X \subset T$, we use

\tilde{X} to denote $\langle X, <_T \rangle$,

$ht_X(x)$ to denote the height of x in \tilde{X} ,

\tilde{X}_α to denote the set $\{x \in X : ht_X(x) = \alpha\}$,

$\tilde{X} \upharpoonright \alpha$ to denote the set $\{x \in X : ht_X(x) < \alpha\}$.

But \tilde{T} , $ht_T(x)$, \tilde{T}_α , $\tilde{T} \upharpoonright \alpha$ will exceptionally be written as T , $ht(x)$, T_α , $T \upharpoonright \alpha$ respectively. If $S \subset \omega_1$, $T \upharpoonright S$ is the set $\{x \in T : ht(x) \in S\}$. Recall that Ω is the set of all limit ordinals $< \omega_1$. In this paper ω_1 -trees are assumed to have only one minimal element (a root).

Before introducing more special notions, we shall raise well-known facts.

LEMMA 1. *If T is an \mathbf{R} -embeddable tree with $ht(T) \leq \omega_1$, then the tree $\langle T \upharpoonright (\omega_1 \setminus \Omega), <_T \rangle$ is \mathbf{Q} -embeddable.*

PROOF. With each $x \in T \upharpoonright (\omega_1 \setminus \Omega)$, associate a $q \in \mathbf{Q}$ such that $e(x') < q < e(x)$, where $e : T \rightarrow \mathbf{R}$ is the embedding and x' means the immediate predecessor of x .

LEMMA 2. *If T is a \mathbf{Q} -embeddable uncountable tree, then T contains an uncountable anti-chain.*

PROOF. Let e embed T in \mathbf{Q} . Clearly $\{x \in T : e(x) = q\}$ is an anti-chain and is uncountable for some $q \in \mathbf{Q}$.

LEMMA 3. *Let T be an \mathbf{R} -embeddable tree. If X is an uncountable subset of T , then X contains an uncountable anti-chain of T .*

PROOF. \tilde{X} is clearly \mathbf{R} -embeddable. If \tilde{X}_α is uncountable for some α , then $\tilde{X}_\alpha (\subset X)$ is an uncountable anti-chain of \tilde{X} and hence an uncountable anti-chain of T . If \tilde{X}_α is countable for all α , then $\tilde{X} \upharpoonright (\omega_1 \setminus \Omega)$ is uncountable. Since

$\langle \tilde{X} \upharpoonright (\omega_1 \setminus \Omega), <_T \rangle$ is \mathbf{Q} -embeddable by Lemma 1, there is an uncountable anti-chain $\subset \tilde{X} \upharpoonright (\omega_1 \setminus \Omega) \subset X$ by Lemma 2.

LEMMA 4. *Let T be a tree with height ω_1 . If T_α is finite for uncountably many α , then T has a confinal branch.*

PROOF. Put $T^* = \{x \in T : x \text{ has an extension in every higher level } T_\alpha\}$. It is easy to see by the assumption that the transitive subtree $\langle T^*, <_T \rangle$ of T has height ω_1 . Pick a branch b of \hat{T}^* . We shall show that the order type, say λ , of b is ω_1 . Suppose $\lambda < \omega_1$. Pick $\alpha < \omega_1$ such that $\lambda < \alpha$ and T_α is finite. Put $Y_x = \{y \in T_\alpha : x <_T y\}$ for each $x \in b$. Then $\bigcap \{Y_x : x \in b\}$ is non-empty, since (1) $Y_x \neq \emptyset$, (2) $x <_T y \rightarrow Y_x \supseteq Y_y$ and (3) Y_x is finite. Pick $y \in \bigcap \{Y_x : x \in b\}$. Then $b \subset \hat{y}$, this contradicts the assumption that b is a branch (a maximal linearly ordered subset of T), q. e. d.

Now recall that \mathfrak{T} is the tree $\bigcup_{\alpha < \omega_1} R^{\alpha+1}$ with the ordering defined by $x <_T y \leftrightarrow x \subset y$ and that if $x \in \mathfrak{T}$, $m(x)$ is the real number $x(ht(x))$. When $x \in \mathfrak{T}$ and a limit ordinal λ is in $\text{dom}(x)$, we write $\lim_{\xi \rightarrow \lambda} x(\xi) = r$ instead of $(\forall q < r \exists \alpha < \lambda \forall \beta < \lambda [\alpha < \beta \rightarrow q < x(\beta) \leq r])$. Now we define a transitive subtree \mathfrak{T}_P of \mathfrak{T} as follows:

$$\mathfrak{T}_P = \{x \in \mathfrak{T} : P(x)\},$$

where $P(x)$ is the conjunction of the following three:

- (1) $x(\alpha) \geq 0$ for all $\alpha \in \text{dom}(x)$;
- (2) $x(\alpha) < x(\alpha+1)$ for all α with $\alpha+1 \in \text{dom}(x)$;
- (3) for all limit ordinals $\lambda \in \text{dom}(x)$,

$$(\forall r > 0) [\lim_{\xi \rightarrow \lambda} x(\xi) = r \leftrightarrow x(\lambda) = r].$$

For a transitive subtree T of \mathfrak{T}_P , we put

$$T^0 = \{x \in T : m(x) = 0\}.$$

We shall write

$$x \overset{\triangleright}{>}_T y \text{ instead of } x <_T y \ \& \ (x, y] \cap T^0 = \emptyset.$$

LEMMA 5. *Let T be a transitive subtree of \mathfrak{T}_P . Then for every $x, y \in T$:*

- (1) $m(x) \geq 0$;
- (2) $x \overset{\triangleright}{>}_T y \rightarrow m(x) < m(y)$;
- (3) *the function m increases monotonously on $[x, y]$ if $(x, y) \cap T^0 = \emptyset$;*
- (4) $m(x) > 0 \rightarrow \exists y [y \overset{\triangleright}{>}_T x]$;
- (5) $\lambda \in \Omega \ \& \ x, y \in T_\lambda \ \& \ \hat{x} = \hat{y} \rightarrow x = y$;
- (6) $m(x) = 0 \rightarrow ht(x) \in \Omega$;

(7) if $ht(y) \in \Omega$, then for every $r > 0$,

$$\lim_{z \rightarrow y} m(z) = r \quad \text{iff} \quad m(y) = r,$$

where $\lim_{z \rightarrow y} m(z) = r$ means

$$(\forall \varepsilon > 0)(\exists z <_T y)(\forall w \in [z, y])[m(w) > r - \varepsilon];$$

(8) $y <_T x$ & $x \in T^0$ & $q \in \mathbf{Q} \rightarrow (\exists z)[y \vec{<}_T z <_T x$ & $m(z) > q]$.

PROOF. The first seven statements are easily checked. To show the last one, suppose that $y <_T x \in T^0$ and $q \in \mathbf{Q}$. Let w be the least of those elements z that $y <_T z \leq_T x$ and $m(z) = 0$. By (3), the function m increases monotonously on $[y, w)$, since $(y, w) \cap T^0 = \emptyset$. Hence w increases monotonously on $[ht(y), ht(w))$. Hence $\lim_{\xi \rightarrow ht(w)} w(\xi) = \infty$ because of (7) and $w(ht(w)) = 0$. Pick ζ so that $ht(y) < \zeta < ht(w)$ and $w(\zeta) > q$. Put $z = w \upharpoonright (\zeta + 1)$. Then

$$y \vec{<}_T z <_T w \leq_T x \quad \& \quad m(z) = z(ht(z)) = w(\zeta) > q, \quad \text{q. e. d.}$$

If a transitive subtree of \mathfrak{T}_P is an ω_1 -tree, we call it a P -tree. Recall that an ω_1 -tree T is called a non-Souslin tree if every uncountable subset of T contains an uncountable anti-chain. By *NS*, we denote the class of all non-Souslin trees.

LEMMA 6. Let T be an Aronszajn P -tree. If

$$\{\alpha : T_\alpha \cap T^0 \text{ is finite}\}$$

is a stationary set, then

- (i) if X is an uncountable subset of T^0 , \tilde{X}_α is uncountable for some $\alpha < \omega_1$,
- (ii) $T \in \mathbf{NS}$.

PROOF. (i) Let X be an uncountable subset of T^0 and suppose that \tilde{X}_α is countable for all $\alpha < \omega_1$. Put:

$$C = \{\lambda \in \Omega : \tilde{X} \upharpoonright \lambda \subseteq T \upharpoonright \lambda\}.$$

C is a club set since $\tilde{X} \upharpoonright \alpha$ is countable for all $\alpha < \omega_1$. Hence by the assumption of the lemma, the set

$$E = \{\lambda \in C : T_\lambda \cap T^0 \text{ is finite}\}$$

is stationary and hence uncountable. Put:

$$Y = \{y \in T : y \leq_T x \text{ for some } x \in X\}.$$

CLAIM. If $\lambda \in E$, then \tilde{Y}_λ is a subset of $T^0 \cap T_\lambda$.

PROOF OF CLAIM. Since \tilde{Y} is a transitive subtree of T , $\tilde{Y}_\lambda \subseteq T_\lambda$. Let $y \in \tilde{Y}_\lambda$. Let x be a minimal element of $\{x \in X : y \leq_T x\}$. Then $ht_X(x) = \lambda$. (The reason: In general $ht_X(z) \leq ht(z)$. Hence by the minimality of x , $ht_X(x) \leq \lambda$. If $ht_X(x) < \lambda$ then $ht(x) < \lambda$ because $\lambda \in C$; this contradicts $y \leq_T x$). Now suppose $y \notin T^0$. Then we can pick $w \vec{z}_T y$ by Lemma 5-(4). Pick β so that $ht(w) < \beta < \lambda$ and pick $z \in X$ so that $ht_X(z) = \beta$ and $z <_T x$. Then $ht(w) < \beta = ht_X(z) \leq ht(z) < ht(y) = \lambda$ and so $w <_T z <_T y \leq_T x$. Thus $z \in (w, y] \cap T^0$, a contradiction. Claim is thus proved. Thus \tilde{Y}_λ is finite for all $\lambda \in E$. By Lemma 4, \tilde{Y} has a cofinal branch which is also a cofinal branch of T . This is absurd since $T \in \mathbf{AT}$.

(ii) Let X be an uncountable subset of T . For each $z \in T^0$, put:

$$X_{(z)} = \{x \in X : z \vec{z}_T x\},$$

$$Z = \{z \in T^0 : X_{(z)} \neq \emptyset\}.$$

Case 1. Z is uncountable. By (i), we can find an uncountable subset Y (i.e. \tilde{Z}_α for some α) of Z such that Y is an anti-chain of T . With each $y \in Y$ associate an element, say $x(y)$, of $X_{(y)}$. Then the subset $\{x(y) : y \in Y\}$ of X is clearly an uncountable anti-chain of T .

Case 2. Z is countable. Since the uncountable set X is the union of $\{X_{(z)} : z \in Z\}$, we can find $z \in Z$ such that $X_{(z)}$ is uncountable. Note that $\tilde{X}_{(z)}$ is an \mathbf{R} -embeddable tree by Lemma 5-(3). By Lemma 3, $\tilde{X}_{(z)}$ contains an uncountable anti-chain which is also an antichain of T and is contained in X . Lemma 6 is thus proved.

COROLLARY 7. Let T be an Aronszajn P -tree. If the set

$$\{\alpha < \omega_1 : m(x) > 0 \text{ for all } x \in T_\alpha\}$$

is stationary, then $T \in \mathbf{NS}$.

Though this corollary assumes a rather strong condition, it suffices for our purpose. In this sense Lemma 6 is redundant. Lemma 6 stands because of its own interest.

Recall that a \diamond_x -sequence $\langle Z_\alpha : \alpha < \omega_1 \rangle$ has the following properties: If T is an ω_1 -tree and is a transitive subtree of \mathfrak{T} , then

(1) if X is a subset of T , then the set

$$\{\alpha < \omega_1 : X \cap T \upharpoonright \alpha = Z_\alpha\} \text{ is stationary;}$$

(2) if e is a function which embeds T in \mathbf{R} , then

$$\{\alpha < \omega_1 : e \upharpoonright (T \upharpoonright \alpha) = Z_\alpha\} \text{ is a stationary set.}$$

Recall that a $\diamond_{\mathfrak{x}}^*$ -sequence $\langle \{W_i^\alpha : i \in \omega\} : \alpha < \omega_1 \rangle$ has the following properties: If T is an ω_1 -tree and is a transitive subtree of \mathfrak{T} , then

(1) if X is a subset of T , then

$\{\alpha < \omega_1 : X \cap T \upharpoonright \alpha = W_i^\alpha \text{ for some } i < \omega\}$ contains a club set.

(2) if e is a function which embeds T in \mathbf{R} , then

$\{\alpha < \omega_1 : e \upharpoonright (T \upharpoonright \alpha) = W_i^\alpha \text{ for some } i < \omega\}$ contains a club set.

LEMMA 8. (1) (\diamond) There exists a $\diamond_{\mathfrak{x}}$ -sequence.

(2) (\diamond^*) There exists a $\diamond_{\mathfrak{x}}^*$ -sequence.

LEMMA 9. Let T be a P -tree and $\langle Z_\alpha : \alpha < \omega_1 \rangle$ a $\diamond_{\mathfrak{x}}$ -sequence. If for every $\lambda \in \mathcal{Q}$ $(\forall x \in T_\lambda)[Z_\lambda \neq \hat{x}]$ holds, then $T \in \mathbf{AT}$.

PROOF. Suppose that X were a cofinal branch of T . Then there is a $\lambda \in \mathcal{Q}$ such that $Z_\lambda = X \cap T \upharpoonright \lambda$. Let x be the unique element of $X \cap T_\lambda$. Then $Z_\lambda = X \cap T \upharpoonright \lambda = \hat{x}$, a contradiction.

LEMMA 10. Let T be a P -tree and $\langle Z_\alpha : \alpha < \omega_1 \rangle$ a $\diamond_{\mathfrak{x}}$ -sequence. Let T satisfy the following condition:

(1) if $\lambda \in \mathcal{Q}$ and Z_λ is a function which embeds $T \upharpoonright \lambda$ in $[0, 1)$, then there is an $x \in T_\lambda$ such that

$$(*) \quad (\forall n \exists y <_T x)[Z_\lambda^{\text{sup}}(y) - 1/n < Z_\lambda(y)],$$

where $Z_\lambda^{\text{sup}}(y) = \sup\{Z_\lambda(z) : y <_T z \in T \upharpoonright \lambda\}$.

Then T is not \mathbf{R} -embeddable.

PROOF. Let e embed T in \mathbf{R} . We may assume $\text{ran}(e) \subset [0, 1)$. Put:

$$C = \{\lambda \in \mathcal{Q} : (\exists y \in T)[x <_T y \ \& \ q < e(y)] \rightarrow (\exists y \in T \upharpoonright \lambda)[x <_T y \ \& \ q < e(y)]\}$$

for every $q \in \mathcal{Q}$ and every $x \in T \upharpoonright \lambda$.

Clearly C is club and hence we can pick $\lambda \in C$ such that $e \upharpoonright (T \upharpoonright \lambda) = Z_\lambda$. Then Z_λ embeds $T \upharpoonright \lambda$ in $[0, 1)$. So, by the assumption, we can take $x \in T_\lambda$ which satisfies (*). Let x' be one of the immediate successors of x . Pick n so that $1/n < e(x') - e(x)$. Pick $y <_T x$ so that $Z_\lambda^{\text{sup}}(y) - 1/n < Z_\lambda(y)$. Since $\lambda \in C$, $e(x') \leq \sup\{e(z) : y <_T z \in T \upharpoonright \lambda\} = \sup\{e(z) : y <_T z \in T \upharpoonright \lambda\} = Z_\lambda^{\text{sup}}(y)$. It follows that $1/n < e(x') - e(x) < Z_\lambda^{\text{sup}}(y) - e(y) < 1/n$, a contradiction.

LEMMA 11. Let $\langle \{W_i^\alpha : i < \omega\} : \alpha < \omega_1 \rangle$ be a $\diamond_{\mathfrak{x}}^*$ -sequence. Let T be a P -tree which satisfies the following condition:

(1) whenever $\lambda \in \Omega$ and W_i^λ is an anti-chain of $T \upharpoonright \lambda$ and $x \in T_\lambda$, one of the following conditions holds:

- (a) $(\exists y <_T x)(\forall z \in T \upharpoonright \lambda)[y <_T z \rightarrow z \in W_i^\lambda]$,
- (b) $(\exists y \overset{\lambda}{\succ}_T x \exists q > 0)[m(x) \leq m(y) + q \ \& \ (\forall z \in W_i^\lambda)[y \overset{\lambda}{\succ}_T z \rightarrow m(z) \geq m(y) + 2q]]$.

Then T has property γ .

The proof of this lemma is given separately in a later section, since it is rather long.

Finally we define for two ω_1 -trees $(T, <_0)$ and $(T', <_1)$ an ω_1 -tree $T \dot{+} T'$ as follows: The field of $T \dot{+} T'$ is $T \times \{0\} \cup T' \times \{1\} \setminus \{\langle 0_1, 1 \rangle\}$, where $0_0, 0_1$ are the roots of T, T' respectively; The ordering $<_T$ of $T \dot{+} T'$ is defined by

$$\begin{aligned} \langle x, 0 \rangle <_T \langle y, 0 \rangle & \quad \text{if } x, y \in T \text{ and } x <_0 y, \\ \langle x, 1 \rangle <_T \langle y, 1 \rangle & \quad \text{if } x, y \in T' \setminus \{0_1\} \text{ and } x <_1 y, \\ \langle 0_0, 0 \rangle <_T \langle y, 1 \rangle & \quad \text{if } y \in T' \setminus \{0_1\}. \end{aligned}$$

§ 3. Theorems.

THEOREM 12 (\diamond^*). $(NS \setminus RE) \cap \gamma ST \neq \emptyset$.

PROOF. Let $\langle Z_\alpha : \alpha < \omega_1 \rangle$ be a \diamond_x -sequence and $\langle \{W_i^\alpha : i < \omega\} : \alpha < \omega_1 \rangle$ a \diamond_x^* -sequence. We define a P -tree T by induction on levels so that T satisfies the following:

- (1) if $\alpha < \beta < \omega_1$ and $x \in T_\alpha$ and $q \in \mathbf{Q} \cap (m(x), \infty)$, there is a $y \in T_\beta$ such that $x \overset{\lambda}{\succ}_T y$ and $m(y) < q$.

Set $T_0 = \{0_T\}$;

$$T_{\alpha+1} = \{x \cup \{\langle q, \alpha+1 \rangle\} : x \in T_\alpha \ \& \ m(x) < q \in \mathbf{Q}\}.$$

Let $\lambda \in \Omega$ and suppose $T \upharpoonright \lambda$ has been defined so that (1) holds. Fix an increasing sequence $\langle \lambda_n : n < \omega \rangle$ such that $\lim_{n \in \omega} \lambda_n = \lambda$. For each $x \in T \upharpoonright \lambda$ and each positive rational q , we define $t_\lambda(x, q)$ as follows: Let $x \in T \upharpoonright \lambda$ and $0 < q \in \mathbf{Q}$. We pick $x_n, x_n^* \in T \upharpoonright \lambda, q_n, q_n^* > 0$ inductively so that:

- (a) $x_0 = x$ and $q_0 = q$;
- (b) if W_n^λ is an anti-chain of $T \upharpoonright \lambda$ and

$$(\exists z \in T \upharpoonright \lambda)[x_n \overset{\lambda}{\succ}_T z \in W_n^\lambda \ \& \ m(z) < m(x_n) + q_n],$$

then

$$x_n \overset{\lambda}{\succ}_T x_n^* \in W_n^\lambda \ \& \ m(x_n^*) < m(x_n) + q_n \ \text{and} \ q_n^* = m(x_n) + q_n - m(x_n^*);$$

$$\text{otherwise, } x_n^* = x_n \ \text{and} \ q_n^* = q_n/2;$$

- (c) $x_n^* \not\rightarrow_T x_{n+1}$ & $ht(x_{n+1}) > \lambda_n$ & $m(x_{n+1}) < m(x_n^*) + q_n^*$ (this is possible by (1));
 (d) $q_{n+1} = m(x_n^*) + q_n^* - m(x_{n+1})$.

Put:
$$t_\lambda(x, q) = \bigcup_{n < \omega} x_n \cup \{\langle \sup_{n < \omega} m(x_n), \lambda \rangle\}.$$

Notice that $x \rightarrow_T t_\lambda(x, q)$ and $0 < m(t_\lambda(x, q)) \leq m(x) + q$. Now, we shall define T_λ .

Case 1. Z_λ is a cofinal branch of $T \upharpoonright \lambda$. For each $x \in T \upharpoonright \lambda$ and each positive rational q , pick $x^* \in T \upharpoonright \lambda$ and q^* so that:

$$x <_T x^* \in Z_\lambda, ht(x^*) = ht(x) + 1, m(x^*) < m(x) + q \text{ and } q^* = m(x) + q - m(x^*).$$

And put:
$$u_\lambda(x, q) = t_\lambda(x^*, q^*).$$

We set:
$$T_\lambda = \{u_\lambda(x, q) : x \in T \upharpoonright \lambda, 0 < q \in \mathbf{Q}\}.$$

Note that if $u = u_\lambda(x, q)$, then $Z_\lambda \neq \hat{u}$, $x \rightarrow_T u$ and $0 < m(u) \leq m(x) + q$.

Case 2. Z_λ is a function which embeds $T \upharpoonright \lambda$ in $[0, 1)$. Pick $y_n, y_n^* \in T \upharpoonright \lambda$ inductively as follows:

- (a) $y_0 = 0_T$;
 (b) if W_n^λ is an anti-chain of $T \upharpoonright \lambda$ and

$$(\exists z \in W_n^\lambda)[y_n <_{Tz} \text{ & } Z_\lambda^{\text{sup}}(y_n) - 1/(n+1) < Z_\lambda(z)],$$

then

$$y_n <_T y_n^* \in W_n^\lambda \text{ and } Z_\lambda^{\text{sup}}(y_n) - 1/(n+1) < Z_\lambda(y_n^*);$$

$$\text{otherwise, } y_n <_T y_n^* \in T \upharpoonright \lambda \text{ and } Z_\lambda^{\text{sup}}(y_n) - 1/(n+1) < Z_\lambda(y_n^*);$$

- (c) $y_{n+1} >_T y_n^*$ & $ht(y_{n+1}) > \lambda_n$;

(see Lemma 10 for the definition of $Z_\lambda^{\text{sup}}(y_n)$).

Put:
$$s_\lambda = \bigcup_{n < \omega} y_n \cup \{\langle r, \lambda \rangle\},$$

where the real r is taken so that $s_\lambda \in \mathfrak{X}_P$ (such an r is unique).

We set:
$$T_\lambda = \{s_\lambda\} \cup \{t_\lambda(x, q) : x \in T \upharpoonright \lambda, 0 < q \in \mathbf{Q}\}.$$

Case 3. Otherwise. We set:

$$T_\lambda = \{t_\lambda(x, q) : x \in T \upharpoonright \lambda, 0 < q \in \mathbf{Q}\}.$$

T_λ is thus defined. Now set:

$$T = \bigcup_{\alpha < \omega_1} T_\alpha.$$

T is clearly a P -tree. We can easily check that $T \in \mathbf{AT}$ by Lemma 9, $T \in \mathbf{RE}$ by Lemma 10, $T \in \mathbf{NS}$ by Corollary 7, $T \in \gamma\mathbf{ST}$ by Lemma 11, using the following facts:

- (a) even when Z_λ is a cofinal branch of $T \upharpoonright \lambda$, $Z_\lambda \neq \hat{x}$ for every $x \in T_\lambda$;

(b) if Z_λ is a function which embeds $T \upharpoonright \lambda$ in $[0, 1)$, then for every $n < \omega$, $y_n <_T s_\lambda \in T_\lambda$ and $Z_\lambda^{\text{sup}}(y_n) - 1/n \leq Z_\lambda^{\text{sup}}(y_{n-1}) - 1/n < Z_\lambda(y_n)$;

(c) stationarily many ordinals $\in \Omega$ are put in Case 3 and for every such ordinal λ it holds that $(\forall x \in T_\lambda)[m(x) > 0]$;

(d) if W_n^λ is an anti-chain of $T \upharpoonright \lambda$ and $t = t_\lambda(x, q)$, then the one of the following holds:

1°. $x_n^* \in W_n^\lambda \cap \dot{t}$ and $(\forall z \in T \upharpoonright \lambda)[x_n^* <_T z \rightarrow z \in W_n^\lambda]$;

2°. $m(t) \leq m(x_n^*) + q_n^*$ & $\forall z[x_n^* \vec{>}_T z \in W_n^\lambda \rightarrow m(z) \geq m(x_n) + 2q_n^*]$;

(e) if W_n^λ is an anti-chain of $T \upharpoonright \lambda$, Z_λ is a function which embeds $T \upharpoonright \lambda$ in $[0, 1)$ and $t = s_\lambda$, then one of the following holds:

1°. $y_n^* \in W_n^\lambda \cap \dot{t}$ and hence $(\forall z \in T \upharpoonright \lambda)[y_n^* <_T z \rightarrow z \in W_n^\lambda]$;

2°. $Z_\lambda^{\text{sup}}(y_n) - 1/(n+1) < Z_\lambda(y_{n+1})$, $\forall z[y_n <_T z \text{ \& } Z_\lambda^{\text{sup}}(y_n) - 1/(n+1) < Z_\lambda(z) \rightarrow z \in W_n^\lambda]$ and hence $\forall z[y_{n+1} <_T z \rightarrow z \in W_n^\lambda]$.

Theorem 12 is thus proved.

THEOREM 13 (\diamond^*). $(NS \setminus RE) \cap (AST \setminus \gamma ST) \neq \emptyset$.

PROOF. Assume \diamond^* . We can take $T \in (NS \setminus RE) \cap \gamma ST$ (Theorem 12) and $T' \in RE \cap (AST \setminus \gamma ST)$ (Devlin and Shelah [2, Theorem 4.4]). Then clearly $T \dot{+} T' \in (NS \setminus RE) \cap (AST \setminus \gamma ST)$.

THEOREM 14 (\diamond). $(NS \setminus RE) \cap (NCA \setminus AST) \neq \emptyset$.

PROOF. Let $\langle Z_\alpha : \alpha < \omega_1 \rangle$ be a \diamond_x -sequence. To define a P -tree, we construct each level T_α by induction on α ensuring that the following holds:

(1) if $\alpha < \beta < \omega_1$ & $x \in T_\alpha$ & $m(x) < q \in \mathbf{Q}$, there is a $y \in T_\beta$ such that $x \vec{>}_T y$ & $m(y) < q$, and additionally if β is a successor ordinal, there is a $y' \in T_\beta$ such that $x \vec{>}_T y'$ & $m(y') = q$.

Set: $T_0 = \{0_T\}$;

$$T_{\alpha+1} = \{x \cup \{ \langle q, \alpha+1 \rangle \} : x \in T_\alpha, m(x) < q \in \mathbf{Q}\}.$$

Let $\lambda \in \Omega$ and suppose that $T \upharpoonright \lambda$ has been defined. Fix an increasing sequence $\langle \lambda_n : n < \omega \rangle$ such that $\sup_{n < \omega} \lambda_n = \lambda$. For each $x \in T \upharpoonright \lambda$ and each rational $q < m(x)$, we shall define $t_\lambda(x, q)$ as follows: First take an increasing sequence $\langle q_n : n < \omega \rangle$ such that $\lim_{n < \omega} q_n = q$ and $m(x) < q_0$. Pick x_n for every $n < \omega$ by induction so that:

$$x_0 = x;$$

$$x_n \vec{>}_T x_{n+1} \text{ \& } ht(x_{n+1}) > \lambda_n \text{ \& } m(x_{n+1}) = q_n,$$

(this is possible by (1)). We set :

$$t_\lambda(x, q) = \bigcup_{n < \omega} x_n \cup \{\langle q, \lambda \rangle\}.$$

Notice that $x \overset{\rightarrow}{\leq}_T t_\lambda(x, q)$ and $m(t_\lambda(x, q)) = q$.

Now we shall define T_λ .

Case 1. Z_λ is a cofinal branch of $T \upharpoonright \lambda$. For each $x \in T \upharpoonright \lambda$ and each rational $q > m(x)$, pick x^* so that $ht(x^*) = ht(x) + 1$, $x <_T x^*$, $m(x^*) < q$ and $x^* \in Z_\lambda$. Put :

$$s_\lambda(x, q) = t_\lambda(x^*, q).$$

We set : $T_\lambda = \{s_\lambda(x, q) : x \in T \upharpoonright \lambda, m(x) < q \in \mathbf{Q}\}$.

Clearly $Z_\lambda \neq \{y \in T \upharpoonright \lambda : y <_T s_\lambda(x, q)\}$.

Case 2. Z_λ is an anti-chain of $T \upharpoonright \lambda$. For each $x \in T \upharpoonright \lambda$ and each rational $q > m(x)$, pick x^* and $q^* \in \mathbf{Q}$ so that :

(a) if $(\exists w \in T \upharpoonright \lambda)[x \overset{\rightarrow}{\leq}_T w \in Z_\lambda \ \& \ m(w) < q]$, then

$$x \overset{\rightarrow}{\leq}_T x^* \in Z_\lambda, \ m(x^*) < q \ \text{and} \ m(x^*) < q^* < q;$$

(b) otherwise, $x^* = x$ and $m(x) < q^* < q$.

Put : $u_\lambda(x, q) = t_\lambda(x^*, q^*)$.

We set : $T_\lambda = \{u_\lambda(x, q) : x \in T \upharpoonright \lambda, m(x) < q \in \mathbf{Q}\}$.

Case 3. Z_λ is a function which embeds $T \upharpoonright \lambda$ in $[0, 1)$. Pick y_n for each $n < \omega$ by induction so that :

$$y_0 = 0_T;$$

$$y_{n+1} >_T y_n \ \& \ ht(y_{n+1}) > \lambda_n \ \& \ Z_\lambda^{\text{sup}}(y_n) - 1/(n+1) < Z_\lambda(y_{n+1}).$$

Put : $v_\lambda = \bigcup_{n < \omega} y_n \cup \{\langle r, \lambda \rangle\}$,

where $r \in \mathbf{R}$ is taken so that $v_\lambda \in \mathfrak{F}_P$. We set :

$$T_\lambda = \{v_\lambda\} \cup \{t_\lambda(x, q) : x \in T \upharpoonright \lambda, m(x) < q \in \mathbf{Q}\}.$$

Case 4. Otherwise. We set :

$$T_\lambda = \{t_\lambda(x, q) : x \in T \upharpoonright \lambda, m(x) < q \in \mathbf{Q}\}.$$

T_λ is thus defined. Now we set :

$$T = \bigcup_{\alpha < \omega_1} T_\alpha.$$

Clearly T is P -tree. We can easily check that $T \in \mathbf{AT}$ by Lemma 9, $T \in \mathbf{RE}$ by Lemma 10 and $T \in \mathbf{NS}$ by Corollary 7, using the following :

(a) $Z_\lambda \neq \hat{x}$ for every $x \in T_\lambda$, even if Z_λ is a cofinal branch of $T \upharpoonright \lambda$;

(b) if Z_λ embeds $T \upharpoonright \lambda$ in $[0, 1)$, then $y_n < v_\lambda$ and

$$Z_\lambda^{\text{sup}}(y_n) - 1/n \leq Z_\lambda^{\text{sup}}(y_{n-1}) - 1/n < Z_\lambda(y_n);$$

(c) stationarily many limit ordinals are put in Case 4, and for such an ordinal λ $m(x) > 0$ for all $x \in T_\lambda$.

To see that $T \in \mathbf{NCA}$, suppose that there were a club anti-chain X of T . Put:
 $C_1 = \{\lambda \in \Omega : (\forall x \in T \upharpoonright \lambda \forall q \in \mathbf{Q}) [(\exists w \in T) R(x, w, X, q) \rightarrow (\exists w \in T \upharpoonright \lambda) R(x, w, X, q)]\}$,
 where $R(x, w, X, q)$ stands for $x \succ_T w \in X$ & $m(w) < q$. Clearly C_1 is a club set.
 Hence so is $C = C_1 \cap \{ht(x) : x \in X\}$. So we can pick $\lambda \in \Omega$ so that $\lambda \in C \cap \{\alpha < \omega_1 : X \cap T \upharpoonright \alpha = Z_\alpha\}$. Then we can pick $t \in X \cap T_\lambda$ since $\lambda \in \{ht(t) : t \in X\}$. Since $X \cap T \upharpoonright \lambda = Z_\lambda$, Z_λ is an anti-chain of $T \upharpoonright \lambda$ and so we can take $x \in T \upharpoonright \lambda$ and $q \in \mathbf{Q}$ so that $t = u_\lambda(x, q)$. Then $m(t) = q^* < q$. Thus $R(x, t, X, q)$ and hence $(\exists w \in T \upharpoonright \lambda) R(x, w, X, q)$ because $\lambda \in C_1$. Since $X \cap T \upharpoonright \lambda = Z_\lambda$, this implies that $x^* \in Z_\lambda$. Thus, $x^*, t \in X$ and $x^* <_T t$. This is absurd since X is an anti-chain. $T \in \mathbf{NCA}$ is thus shown.

On the other hand, it can be easily checked that the set $\{t_\lambda(0_T, 1) : \lambda \text{ is a limit ordinal ordinal put in Case 4}\}$ is a stationary anti-chain and hence $T \notin \mathbf{AST}$, q. e. d.

THEOREM 15 (\diamond). $(\mathbf{NS} \setminus \mathbf{RE}) \setminus \mathbf{NCA} \neq \emptyset$.

PROOF. Assume \diamond . We can take $T \in (\mathbf{NS} \setminus \mathbf{RE}) \cap (\mathbf{NCA} \setminus \mathbf{AST})$ (by Theorem 14) and $T' \in \mathbf{SAT} \setminus \mathbf{NCA}$ ([3, Theorem 5]). Then clearly $T \dot{+} T' \in (\mathbf{NS} \setminus \mathbf{RE}) \setminus \mathbf{NCA}$.

§4. Proof of Lemma 11.

Let X be an uncountable anti-chain of T . Put:

$$C_0 = \{\alpha < \omega_1 : X \cap T \upharpoonright \alpha = W_i^\alpha \text{ for some } i \in \omega\},$$

$$C_1 = \{\lambda \in \Omega : (\forall y \in T \upharpoonright \lambda) [(\exists x \in X) [y <_T x] \rightarrow (\exists z \in X \cap T \upharpoonright \lambda) [y <_T z]]\},$$

$$C_2 = \{\lambda \in \Omega : (\exists x \in X) [y \succ_T x \text{ \& } m(x) < q] \rightarrow (\exists z \in X \cap T \upharpoonright \lambda) [y \succ_T z \text{ \& } m(z) < q]\},$$

for every $y \in T \upharpoonright \lambda$ and every $q \in \mathbf{Q}$.

Let C be a club set such that $C \subseteq C_0 \cap C_1 \cap C_2$.

CLAIM 1. $X \cap T_\lambda = \emptyset$ for every $\lambda \in C$.

PROOF. Suppose $\lambda \in C$ and $x \in X \cap T_\lambda$. Pick $i \in \omega$ so that $X \cap T \upharpoonright \lambda = W_i^\lambda$. Since X is an anti-chain, W_i^λ is an anti-chain of $T \upharpoonright \lambda$. Hence by the assumption of the lemma, (a) or (b) must hold.

Case 1. (a) holds. Pick $y <_T x$ so that $(\forall z \in T \upharpoonright \lambda)[y <_T z \rightarrow z \in W_i^\lambda]$. Since $W_i^\lambda = X \cap T \upharpoonright \lambda$ and $\lambda \in C_1$, $\neg(\exists x \in X)[y <_T x]$. This contradicts " $y <_T x \in X$ ".

Case 2. (b) holds. Pick $y \succ_T x$ and $q > 0$ so that:

$$m(x) \leq m(y) + q \quad \text{and} \quad (\forall z \in W_i^\lambda)[y \succ_T z \rightarrow m(z) \geq m(y) + 2q].$$

Since $X \cap T \upharpoonright \lambda = W_i^\lambda$ and $\lambda \in C_2$, $\neg(\exists x \in X)[y \succ_T x \ \& \ m(x) < m(y) + 2q]$. This contradicts " $x \in X \ \& \ y \succ_T x \ \& \ m(x) \leq m(y) + q$ ". Claim 1 is thus proved.

Let $\langle \lambda_\xi : \xi < \omega_1 \rangle$ be the monotone enumeration of $C \cup \{0\}$. Let $\langle x_n^\xi : n < \omega \rangle$ be an enumeration of $X \cap T \upharpoonright (\lambda_{\xi+1} \setminus \lambda_\xi)$ such that $x_n^\xi \neq x_m^\xi$ if $n \neq m$, for each $\xi < \omega_1$.

We shall define w_n^ξ for each $\xi < \omega_1$ and each $n < \omega$.

Case 1. $ht(x_n^\xi) \in \Omega$. w_n^ξ is taken so that $(w_n^\xi, x_n^\xi]$ is a singleton set, i.e. w_n^ξ is the immediate predecessor of x_n^ξ .

Case 2. $ht(x_n^\xi) \in \Omega$. First note that there is a $y <_T x_n^\xi$ such that $(y, x_n^\xi] \cap \bigcup_{j < n} \hat{x}_j^\xi = \emptyset$. (To see this, suppose not. Then $\hat{x}_n^\xi \subset \bigcup_{j < n} \hat{x}_j^\xi$ and hence $\hat{x}_n^\xi \subset \hat{x}_j^\xi$ for some $j < n$, which implies $x_n^\xi \leq_T x_j^\xi$ (Lemma 5-(5)). But it is absurd since $x_n^\xi \neq x_j^\xi$ and X is an anti-chain).

Subcase 2.1. $m(x_n^\xi) = 0$. Take y_n^ξ so that:

$$\lambda_\xi < ht(y_n^\xi), \quad y_n^\xi <_T x_n^\xi \quad \text{and} \quad (y_n^\xi, x_n^\xi] \cap \bigcup_{j < n} \hat{x}_j^\xi = \emptyset.$$

w_n^ξ is taken so that

$$y_n^\xi \succ_T w_n^\xi <_T x_n^\xi \quad \text{and} \quad m(w_n^\xi) > m(y_n^\xi) + 1,$$

(this is possible by Lemma 5-(8)).

Subcase 2.2. $m(x_n^\xi) > 0$. We can take y_n^ξ so that:

$$ht(y_n^\xi) > \lambda_\xi, \quad (y_n^\xi, x_n^\xi] \cap \bigcup_{j < n} \hat{x}_j^\xi = \emptyset \quad \text{and} \quad y_n^\xi \succ_T x_n^\xi.$$

Then w_n^ξ is taken so that

$$y_n^\xi \succ_T w_n^\xi \succ_T x_n^\xi \quad \text{and} \quad m(x_n^\xi) - m(w_n^\xi) < m(w_n^\xi) - m(y_n^\xi),$$

(this is possible by Lemma 5-(7)).

w_n^ξ is thus defined. Now put:

$$U = \bigcup \{(w_n^\xi, x_n^\xi] : \xi < \omega_1, n < \omega\}.$$

This is a nbd of X .

Finally we shall define a nbd V of $T \upharpoonright C$ such that $U \cap V = \emptyset$. For this purpose, we shall define v^* for every $v \in T \upharpoonright C$. Let $v \in T \upharpoonright C$ and put $\lambda = ht(v)$. Let i be the number such that $W_i^\lambda = X \cap T \upharpoonright \lambda$. W_i^λ is clearly an anti-chain of $T \upharpoonright \lambda$. So by the assumption of the lemma, Condition (a) or (b) must hold for v (substituted for x).

Case 1. (a) holds. Then we can take $v^* <_T v$ so that

$$(\forall z \in T \upharpoonright \lambda)[v^* <_T z \rightarrow z \in W_i^{\lambda}].$$

Case 2. (b) holds. Take $u \in T \upharpoonright \lambda$ and $q > 0$ so that $u \succ_T v$ and

$$m(v) \leq m(u) + q \quad \text{and} \quad (\forall z \in W_i^{\lambda})[u \succ_T z \rightarrow m(z) \geq m(u) + 2q].$$

We may assume that $m(u) > m(v) - 1$. (If not so, by Lemma 5-(7), there is u' such that $u \succ_T u' \succ_T v$ and $m(u') > m(v) - 1$. Then take u' and $m(u) + q - m(u')$ instead of u and q .)

CLAIM 2. For at most only one pair $\langle \xi, n \rangle$, $(u, v] \cap (w_n^\xi, x_n^\xi] \neq \emptyset$.

PROOF. We show first that $(u, v] \cap (w_n^\xi, x_n^\xi] \neq \emptyset$ implies that (1) $ht(x_n^\xi) < \lambda$ and (2) $ht(x_n^\xi) \in \mathcal{Q}$. To show (1), suppose not. Then by choice of w_n^ξ , $ht(x_n^\xi) > ht(w_n^\xi) \geq \lambda_\xi \geq \lambda$. Hence $(u, v] \cap (w_n^\xi, x_n^\xi] = \emptyset$ which contradicts the assumption. To show (2), suppose not. Then $(w_n^\xi, x_n^\xi] = \{x_n^\xi\}$. Hence $x_n^\xi \in (u, v]$, so $u \succ_T x_n^\xi$. Note $x_n^\xi \in W_i^{\lambda}$. (For, $x_n^\xi \in X \cap T \upharpoonright \lambda$ by (1) and $W_i^{\lambda} = X \cap T \upharpoonright \lambda$ by choice of i .) Hence by the property of q , $m(x_n^\xi) \geq m(u) + 2q > m(u) + q \geq m(v)$. This is absurd since $x_n^\xi \leq_T v$. Next we show that $(u, v] \cap (w_n^\xi, x_n^\xi] \neq \emptyset$ implies that $u \in (y_n^\xi, x_n^\xi]$, where y_n^ξ is as given in the definition of w_n^ξ . (Note that $ht(x_n^\xi) \in \mathcal{Q}$ by the above.) Suppose that there is $t \in (u, v] \cap (w_n^\xi, x_n^\xi]$. Then $u \succ_T t \succ_T v$ and $y_n^\xi \leq_T w_n^\xi <_T t$. So, u and y_n^ξ are comparable. It suffices to show that $y_n^\xi <_T u$. If $m(x_n^\xi) = 0$, then $m(y_n^\xi) < m(w_n^\xi) - 1 < m(t) - 1 < m(v) - 1 < m(u)$ and so $y_n^\xi <_T u$. If $m(x_n^\xi) > 0$, then $y_n^\xi \succ_T w_n^\xi \succ_T x_n^\xi$. Hence $u \succ_T x_n^\xi$, since $u \succ_T t \succ_T v$ and $w_n^\xi \succ_T t \leq_T x_n^\xi$. So, by the property of q , $m(x_n^\xi) \geq m(u) + 2q$ and hence

$$\begin{aligned} m(t) - m(y_n^\xi) &> m(w_n^\xi) - m(y_n^\xi) > m(x_n^\xi) - m(w_n^\xi) > m(x_n^\xi) - m(v) \\ &> (2q + m(u)) - (m(u) + q) = q \geq m(v) - m(u) > m(t) - m(u), \end{aligned}$$

which mean $y_n^\xi <_T u$. In both cases, $y_n^\xi <_T u$. Thus $(u, v] \cap (w_n^\xi, x_n^\xi] \neq \emptyset$ implies $u \in (y_n^\xi, x_n^\xi]$. The claim follows from this immediately. For, there is at most only one pair $\langle \xi, n \rangle$ which satisfies $u \in (y_n^\xi, x_n^\xi]$, since the intervals $(y_n^\xi, x_n^\xi]$, $\xi < \omega_1$ and $n < \omega$, have been taken so as to be mutually disjoint. Claim 2 is thus proved.

By this claim we can take v^* so that:

$$v^* <_T v \quad \text{and} \quad (\forall \xi < \omega_1, \forall n < \omega)[(v^*, v] \cap (w_n^\xi, x_n^\xi] = \emptyset].$$

v^* is thus defined for all $v \in T \upharpoonright C$. Clearly $(v^*, v] \cap U = \emptyset$. We set:

$$V = \bigcup \{(v^*, v] : v \in T \upharpoonright C\}.$$

Then V is a nbd of $T \upharpoonright C$ such that $U \cap V = \emptyset$. This completes the proof of

Lemma 11.

§ 5. Remark on Lemma 6.

First note that every ω_1 -tree is isomorphic to some P -tree. Concerning Lemma 6 and Corollary 7, it would be natural to ask whether the former is essentially more general than the latter: i. e. whether the following condition (C1) is strictly weaker than (C2) for Aronszajn trees T :

(C1) there is a P -tree T' isomorphic to T such that

$$\{\alpha < \omega_1 : T'_\alpha \cap (T')^0 \text{ is finite}\} \text{ is stationary};$$

(C2) there is a P -tree T'' isomorphic to T such that

$$\{\alpha < \omega_1 : T''_\alpha \cap (T'')^0 = \emptyset\} \text{ is stationary.}$$

The answer is affirmative: i. e. the following holds:

PROPOSITION (\diamond^*). *There is an Aronszajn tree which satisfies (C1) but does not (C2).*

PROOF. Let $\langle \{W_i^\alpha : i < \omega\} : \alpha < \omega_1 \rangle$ be a \diamond_x^* -sequence and $\langle Z_\alpha : \alpha < \omega_1 \rangle$ be a \diamond_x -sequence. We construct a P -tree T such that $T_\lambda \cap T^0$ has at most one element for every $\lambda \in \Omega$ but (C2) does not hold. We define T_α for $\alpha < \omega_1$ inductively ensuring that:

(1) if $\alpha < \beta < \omega_1$ and $x \in T_\alpha$ and $m(x) < q \in \mathbb{Q}$, then there is a $y \in T_\beta$ such that $x \vec{>}_T y$ and $m(y) < q$.

Put $T_0 = \{0_T\}$ and $T_{\alpha+1} = \{x \cup \langle q, \alpha+1 \rangle\} : x \in T_\alpha, m(x) < q \in \mathbb{Q}\}$. Let $\lambda \in \Omega$ and suppose that $T \upharpoonright \lambda$ has been defined. Let $\langle \lambda_n : n < \omega \rangle$ be a sequence such that $\lim_{n < \omega} \lambda_n = \lambda$. For each $x \in T \upharpoonright \lambda$ and each rational $q > m(x)$, we pick x_n inductively so that:

(a) if Z_λ is a cofinal branch of $T \upharpoonright \lambda$, then $x \vec{>}_T x_0$ and $x_0 \in Z_\lambda$ and $m(x_0) < q$; otherwise, $x_0 = x$;

(b) $x_{n+1} >_T x_n$, $ht(x_{n+1}) > \lambda_n$ and $m(x_{n+1}) < q$.

Put:

$$t_\lambda(x, q) = \bigcup_{n < \omega} x_n \cup \langle \sup_{n < \omega} m(x_n), \lambda \rangle.$$

Let $K(n)$ mean the number such that $n = 2^m(2 \cdot K(n) + 1) - 1$ for some $m \in \omega$. Now, we shall define y_n by induction as follows:

I. if Z_λ is a cofinal branch of $T \upharpoonright \lambda$ then y_0 is taken so that $y_0 \in Z_\lambda$; other-

wise, $y_0=0_T$;

II. (a) if $W_{K(n)}^\lambda$ is a function from $T \upharpoonright \lambda$ to $[0, \infty)$, then y_{n+1} is taken so that $y_{n+1} >_T y_n$ and $ht(y_{n+1}) > \lambda_n$ and one of the following holds:

1°. $W_{K(n)}^\lambda(y_{n+1}) \geq n$,

2°. $W_{K(n)}^\lambda(y_{n+1}) > \sup\{W_{K(n)}^\lambda(y) : y_n <_T y \in T \upharpoonright \lambda, ht(y) > \lambda_n\} - 1/n$;

(b) otherwise, y_{n+1} is taken so that $y_{n+1} >_T y_n$ and $ht(y_{n+1}) > \lambda_n$.

Put:

$$u_\lambda = \bigcup_{n < \omega} y_n \cup \{r, \lambda\},$$

where r is taken so that $u_\lambda \in \mathfrak{I}_P$. We set:

$$T_\lambda = \{u_\lambda\} \cup \{t_\lambda(x, q) : x \in T \upharpoonright \lambda, m(x) < q \in \mathbf{Q}\}.$$

Then the tree $T = \bigcup_{\alpha < \omega_1} T_\alpha$ is as required. To see that (C2) is false, take arbitrarily an isomorphic P -tree T' and an isomorphism f from T to T' . Define a function $e : T \rightarrow \mathbf{R}$ by $e(x) = m(f(x))$. Take club sets $C_0 = \{\lambda \in \mathcal{Q} : (\forall x \in T \upharpoonright \lambda \forall q \in \mathbf{Q})[(\exists y \in T)[x <_T y \ \& \ e(y) > q] \rightarrow (\exists y \in T \upharpoonright \lambda)[x <_T y \ \& \ e(y) > q]]\}$ and $C_1 \subseteq \{\lambda \in \mathcal{Q} : W^\lambda_i = e \upharpoonright (T \upharpoonright \lambda)$ for some $i\}$.

CLAIM. $e(u_\lambda) = 0$ for every $\lambda \in C_0 \cap C_1$.

PROOF. Suppose $e(u_\lambda) > 0$ and $\lambda \in C_0 \cap C_1$. Pick $i \in \omega$ so that $W^\lambda_i = e \upharpoonright (T \upharpoonright \lambda)$. Then we can take a $v <_T u_\lambda$ such that $f(v) \vec{>}_{T'} f(u_\lambda)$. Let t be an immediate successor of u_λ . Pick $n \in \omega$ so that: $\lambda_n > ht(v)$, $n > e(u_\lambda)$, $e(t) - e(u_\lambda) > 1/n$ and $K(n) = i$. Recall y_{n+1} in the definition of u_λ . Then 1° or 2° must hold. First notice that $f(v) \vec{>}_{T'} f(y_{n+1}) \vec{>}_{T'} f(u_\lambda)$, since $ht(v) < \lambda_n < ht(y_{n+1})$, $f(v) \vec{>}_{T'} f(u_\lambda)$ and $f(y_{n+1}) <_{T'} f(u_\lambda)$. And so $e(v) < e(y_{n+1}) < e(u_\lambda) < n$.

Case 1. 1° holds. Then $e(y_{n+1}) = W_{K(n)}^\lambda(y_{n+1}) \geq n$. This is absurd by the above notice.

Case 2. 2° holds. By $\lambda \in C_0$, $e(u_\lambda) > e(y_{n+1}) > \sup\{e(y) : y_n < y \in T \upharpoonright \lambda\} - 1/n = \sup\{e(y) : y_n < y \in T\} - 1/n \geq e(t) - 1/n$. This is absurd since $e(t) - e(u_\lambda) > 1/n$. Claim is thus proved.

It is obvious by the claim that T does not have property (C2). Proposition is thus proved.

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Department of Mathematics
Faculty of Science
Saitama University
Urawa, Saitama
338 Japan