TSUKUBA J. MATH.
Vol 5 No. 1 (1981). 67—83

A STUDY OF RINGS WITH TRIVIAL PRERADICAL IDEALS

(Dedicated to Professor Goro Azumaya for the celebration of his sixtieth birthday)

By

Hisao KATAYAMA

0. Introduction.

Our purpose is to study those rings without non-trivial preradical ideals of idempotent preradicals (or exact radicals), supplying the cases of idempotent radicals by [2], of left exact preradicals by [1, 6, 14, 17] and of left exact radicals by [2, 6]. In Theorem 3.1, we shall show that a ring R has no non-trivial idempotent preradical ideals if and only if every nonzero left ideal is a generator for R-mod (left G-ring). Generalizing this, we consider those rings whose nonzero finitely generated (or cyclic, essential) left ideals are generators. We shall give several examples which distinguish those rings to be refered.

1. Preliminaries.

This section consists of a list of definitions and properties of some type of preradicals treated in this paper. In particular, we shall give the bijections of those preradicals for Morita equivalent rings.

Let R be a ring with identity and R-mod the category of all unital left R-modules. A functor $\sigma: R\text{-mod}\rightarrow R\text{-mod}$ is called a *preradical* if $\sigma(M)$ is a submodule of M for each $M\in R$-mod and $\sigma(M)\alpha\subseteq\sigma(N)$ for each morphism $\alpha: M\rightarrow N$ in R-mod. A preradical σ is called an *idempotent preradical* (resp. a *radical*) if $\sigma(\sigma(M))=\sigma(M)$ (resp. $\sigma(M/\sigma(M))=0$) for all $M\in R$-mod. A preradical is called *left exact* (resp. *cohereditary*) if it is kernel preserving (resp. epi-preserving). Every left exact (resp. cohereditary) preradical is idempotent (resp. a radical). A preradical is called a *cotorsion radical* (resp. an *exact radical*) if it is an idempotent cohereditary radical (resp. a left exact cohereditary radical). For preradicals σ_1 and σ_2, $\sigma_1\leq\sigma_2$ means that $\sigma_1(M)\subseteq\sigma_2(M)$ for all $M\in R$-mod.

To a preradical σ for R-mod, we associate the pair $(\mathcal{T}_\sigma, \mathcal{F}_\sigma)$ of classes of

Received September 8, 1980.

modules in R-mod given by

$$\mathcal{T}_{\sigma}=\{{}_{R}X \mid \sigma(X)=X\} \quad \text{and} \quad \mathcal{F}_{\sigma}=\{{}_{R}X \mid \sigma(X)=0\}.$$

A class $\mathcal{C}$ of modules is called a *pretorsion class* if it is closed under quotients and direct sums, and is called a *pretorsion-free class* if it is closed under submodules and direct products. It is known that the assignment $\sigma\mapsto\mathcal{T}_{\sigma}$ is a bijection between idempotent preradicals for R-mod and pretorsion classes of modules, under which left exact preradicals correspond to pretorsion classes closed under submodules ([16, Chap. 6]). Dually, the assignment $\sigma\mapsto\mathcal{F}_{\sigma}$ is a bijection between radicals for R-mod and pretorsion-free classes of modules, under which coherereditary radicals correspond to pretorsion-free classes closed under quotients.

A class $\mathcal{T}$ of modules is called a *torsion class* if it is a pretorsion class closed under extentions. A class $\mathcal{F}$ of modules is called a *torsion-free class* if it is a pretorsion-free class closed under extensions. If σ is an idempotent radical for R-mod, then $\mathcal{T}_{\sigma}$ is a torsion class and $\mathcal{F}_{\sigma}$ is a torsion-free class. Moreover the pair $(\mathcal{T}_{\sigma}, \mathcal{F}_{\sigma})$ forms a torsion theory for R-mod in the sense of [3]. It is well known that, under the above assignments, we have bijective correspondences between: (1) idempotent radicals for R-mod, (2) torsion classes and (3) torsion-free classes. Finally, we remark that the assignment $\sigma\mapsto\mathcal{F}_{\sigma}$ is a bijection between cotorsion radicals for R-mod and torsion torsion-free (TTF-) classes, under which exact radicals correspond to TTF-classes closed under injective hulls.

For a class $\mathcal{C}$ of modules in R-mod, we define an idempotent preradical $t_{\mathcal{C}}$ for R-mod by

$$t_{\mathcal{C}}(M)=\sum\{\mathrm{Im}\,(\alpha) \mid \alpha\in\mathrm{Hom}_{R}(Q, M),\ Q\in\mathcal{C}\}$$

for each $M\in R$-mod. In general $\mathcal{C}$ is not a set. An accurate treatment of $t_{\mathcal{C}}(M)$ was given by K. Ohtake. Put $\mathcal{S}=\{\mathcal{C}'\subseteq\mathcal{C} \mid \mathcal{C}' \text{ is a set}\}$. Let $\mathcal{T}=\{t_{\mathcal{C}'}(M) \mid \mathcal{C}'\in\mathcal{S}\}$. Then $\mathcal{T}$ is a set and so $t_{\mathcal{C}}(M)$ is defined via $\Sigma\{t_{\mathcal{C}'}(M) \mid t_{\mathcal{C}'}(M)\in\mathcal{T}\}$. $t_{\mathcal{C}}$ is a unique minimal one of those preradicals t for R-mod satisfying $t(Q)=Q$ for all $Q\in\mathcal{C}$. If $\mathcal{C}=\{Q\}$ is a singleton, we write t_{Q} for $t_{\mathcal{C}}$. Some basic properties of t_{Q} are discussed in [8]. Dually, for a class $\mathcal{D}$ of modules in R-mod, we define a radical $k_{\mathcal{D}}$ for R-mod by

$$k_{\mathcal{D}}(M)=\bigcap\{\mathrm{Ker}\,(\alpha) \mid \alpha\in\mathrm{Hom}_{R}(M, Q),\ Q\in\mathcal{D}\}$$

for each $M\in R$-mod. $k_{\mathcal{D}}$ is a unique maximal one of those preradicals k for R-mod satisfying $k(Q)=0$ for all $Q\in\mathcal{D}$. If $\mathcal{D}=\{Q\}$ is a singleton, we write k_{Q} for $k_{\mathcal{D}}$. Some bacic properties of k_{Q} are discussed in [9].

LEMMA 1.1. *If t is an idempotent preradical for* R-mod, *then there exists a* *class C of modules in R-mod such that $t=t_C$. Dually, if k is a radical for R-mod, then there exists a class $\mathcal{D}$ of modules in R-mod such that $k=k_{\mathcal{D}}$.*

PROOF. Put $C=\mathcal{T}_t$ and $\mathcal{D}=\mathcal{F}_k$.

Now we assume that ${}_RP$ is a progenerator (=a finitely generated projective generator) in R-mod. We put $S=\mathrm{End}_R(P)$ and $P^*=\mathrm{Hom}_R(P, R)$. For a preradical σ for R-mod, we associate the pair $(\mathcal{T}, \mathcal{F})$ of classes of modules in S-mod defined by

$$\mathcal{T}=\{{}_SY \mid P\otimes_S Y\in\mathcal{T}_\sigma\} \quad \text{and} \quad \mathcal{F}=\{{}_SY \mid P\otimes_S Y\in\mathcal{F}_\sigma\}.$$

Since P_S is also a progenerator in mod-S, $P\otimes_S(\ )$: S-mod$\to R$-mod is an exact functor that commutes with direct sums and direct products of modules. Thus, if σ is an idempotent preradical for R-mod, then $\mathcal{T}=\mathcal{T}_\tau$ for some idempotent preradical τ for S-mod. Dually, if σ is a radical for R-mod, then $\mathcal{F}=\mathcal{F}_\tau$ for some radical τ for S-mod. Using $P\otimes_S P^*\cong R$, we obtain the following propositions.

PROPOSITION 1.2. *The asssgnment $\sigma\mapsto\tau$ where $\mathcal{T}_\tau=\{{}_SY|P\otimes_S Y\in\mathcal{T}_\sigma\}$ is an order preserving bijection between idempotent preradicals for R-mod and those for S-mod, under which left exact preradicals for R-mod correspond to those for S-mod.*

PROPOSITION 1.3. *The assignment $\sigma\mapsto\tau$ where $\mathcal{F}_\tau=\{{}_SY|P\otimes_S Y\in\mathcal{F}_\sigma\}$ is an order preserving bijection between radicals for R-mod and those for S-mod, under which cohereditary radicals for R-mod correspond to those for S-mod.*

It is easy to verify that, if σ is an idempotent radical for R-mod, then the pair $(\mathcal{T}_\tau, \mathcal{F}_\tau)$ of classes of modules forms a torsion theory for S-mod, where $\mathcal{T}_\tau=\{{}_SY|P\otimes_S Y\in\mathcal{T}_\sigma\}$ and $\mathcal{F}_\tau=\{{}_SY|P\otimes_S Y\in\mathcal{F}_\sigma\}$. Hence we have the coincidence of assignments $\sigma\mapsto\mathcal{T}_\tau\mapsto\tau$ and $\sigma\mapsto\mathcal{F}_\tau\mapsto\tau$.

PROPOSITION 1.4. *The assignment $\sigma\mapsto\tau$ is an order preserving bijection between idempotent radicals for R-mod and those for S-mod, under which left exact radicals for R-mod correspond to those for S-mod, cotorsion radicals for R-mod correspond to those for S-mod and exact radicals for R-mod correspond to those for S-mod.*

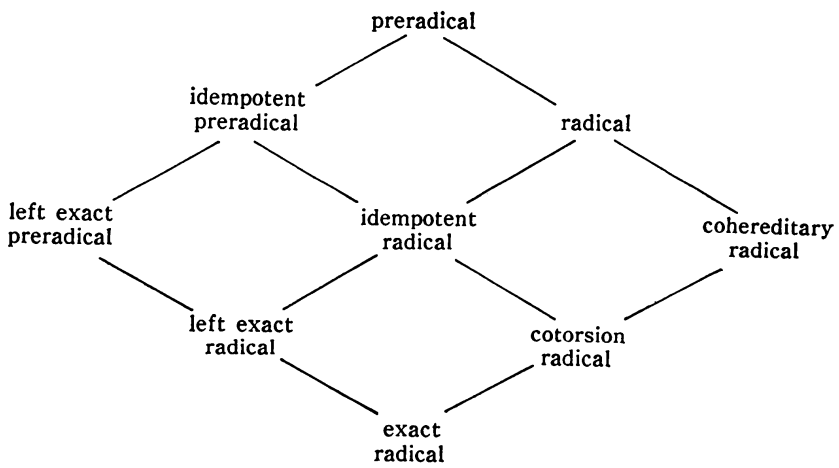
Recall that the assignment $\sigma\mapsto\sigma(R)$ is a bijection between coherediatry radicals for R-mod and ideals of R, under which cotorsion radicals for R-mod correspond to idempotent ideals of R. In [7], it is proved that if σ is a cotorsion radical for R-mod with associated idempotent ideal L of R, and if τ is the corresponding cotorsion radical for S-mod with associated idempotent ideal J of S, then $J=\{s\in S \mid Ps\subseteq LP\}$.

We refer to [4, Chap. 2] some Morita invariant properties around left exact radicals. In particular it is shown in [4, Prop. 9.4] that, if σ is a left exact radical for R-mod such that $\sigma(R)=0$, and if τ is the corresponding left exact radical for S-mod, then $\tau(S)=0$. The argument of the proof is valid for proving the first part of the next proposition.

PROPOSITION 1.5. *If σ is a radical for* R-mod *such that $\sigma(R)=0$, and if τ is the corresponding radical for* S-mod, *then $\tau(S)=0$. The same holds for an idempotent preradical.*

PROOF. Let σ be an idempotent preradical for R-mod such that $\sigma(R)=0$, and τ the corresponding idempotent preradical for S-mod. Assume $\tau(S)\neq 0$. Then we have a nonzero homomorphism $h: P\otimes_S\tau(S)\to P\otimes_S S\cong P$. Since ${}_RP$ is torsionless, for any nonzero $u\in \mathrm{Im}(h)$, there exists a $g\in P^*$ satisfying $(u)g\neq 0$. Thus $0\neq (u)g\in \mathrm{Im}(h\circ g)$. Since $P\otimes_S\tau(S)\in \mathcal{T}_\sigma$, we have $(u)g\in\sigma(R)$, which is impossible because $\sigma(R)=0$.

A table of preradicals



2. Simple rings.

We call an ideal I of R a *preradical ideal* if there exists a preradical σ for R-mod such that $\sigma(R)=I$. A preradical ideal of a left exact preradical (resp. a left exact radical) is nothing but a pretorsion ideal (resp. a torsion ideal) in the sense of [6]. From now on, we shall study the rings which have no non-trivial preradical ideals $\sigma(R)$, where we take σ as an idempotent preradical (or an exact radical, etc) for R-mod, and give several characterizations of those rings. Note that, for a preradical σ for R-mod, $\sigma(R)=R$ if and only if $\sigma=1$, where 1 stands for the identity functor for R-mod. Hence we may rephrase our question as: When the preradical ideals $\sigma(R)$ vanish for various types of preradicals $\sigma\neq 1$ for R-mod? To begin with, we have

PROPOSITION 2.1. *The following properties are equivalent for a ring R:*

(1) *$\sigma(R)=0$ for every preradical $\sigma\neq 1$ for R-mod.*

(2) *$\sigma(R)=0$ for every radical $\sigma\neq 1$ for R-mod.*

(3) *$\sigma(R)=0$ for every cohereditary radical $\sigma\neq 1$ for R-mod.*

(4) *There exist only two cohereditary radicals for R-mod.*

(5) *R is a simple ring (i.e. it has exactly two ideals).*

(6) *Every nonzero (cyclic) left R-module is faithful.*

(7) *$RK=R$ for every nonzero right ideal K of R.*

PROOF. Noting that each ideal of R is a preradical ideal of a cohereditary radical for R-mod, we have the implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) $\Rightarrow$ (5).

(5) $\Rightarrow$ (6). Let M ($\neq 0$) be a left R-module. Since $\mathrm{Ann}_R(M)$ is a proper ideal of R, we have $\mathrm{Ann}_R(M)=0$.

(6) $\Rightarrow$ (7). Assume $R\neq RK$ for some right ideal K of R. For any $a\in K$, we have $aR\subseteq K\subseteq RK$. Therefore $a\in\mathrm{Ann}_R(R/RK)=0$ by the assumption. Hence we obtain $K=0$.

(7) $\Rightarrow$ (1). Assume $\sigma(R)\neq 0$ for some preradical σ for R-mod. Then we have $R\sigma(R)=R$ by (7). Hence $\sigma(R)=R$, and so $\sigma=1$ as desired.

The vanishing of the preradical ideals $\sigma(R)$ of idempotent radicals (or left exact preradicals, left exact radicals) $\sigma\neq 1$ for R-mod has been characterized by several authors. We briefly summarize these results.

DEFINITION AND THEOREM A ([2, Prop. 1.10]). *The following properties are equivalent for a ring R:*

(1) *R is a left R-ring, i.e. $\sigma(R)=0$ for every idempotent radical $\sigma\neq 1$ for R-mod.*

(2) $\mathrm{Hom}_R(I, R/I)\neq 0$ *for every non-trivial left ideal I of R.*

(3) $\mathrm{Hom}_R(I, M)\neq 0$ *for every nonzero left ideal I of R and nonzero $M\in R$-mod.*

DEFINITION AND THEOREM B ([6, p2], [14, Theorem 1.7], [17, Theorem 2.1] and [1, Prop. 3.2]). *The following properties are equivalent for a ring R:*

(1) *R is a left SP-ring, i.e.* $\sigma(R)=0$ *for every left exact preradical* $\sigma\neq 1$ *for* R-mod.

(2) *Every nonzero left ideal of R is cofaithful.*

(3) *Every nonzero left ideal of R generates* $E({}_RR)$.

(4) *R is a left non-singular prime ring, and every non-singular quasi-injective left R-module is injective.*

DEFINITION AND THEOREM C ([2, Theorem 2.4] and [6, p91]). *The following properties are equivalent for a ring R:*

(1) *R is a left CTF-ring, i.e.* $\sigma(R)=0$ *for every left exact radical* $\sigma\neq 1$ *for* R-mod.

(2) *For every non-trivial left ideal I of R, there exist* $x\in I$, $y\in R\backslash I$ *such that* $(0:x)\subseteq(I:y)$.

(3) *Every nonzero injective left R-module is faithful.*

3. Left G-rings.

Remark that if a module ${}_RM$ is a generator for R-mod, then the dual $\mathrm{Hom}_R(M, R)\neq 0$. The next theorem deals with a ring R satisfying the converse statement.

THEOREM 3.1. *The following properties are equivalent for a ring R:*

(1) $\sigma(R)=0$ *for every idempotent preradical* $\sigma\neq 1$ *for* R-mod.

(2) *Every left R-module with nonzero dual is a generator for* R-mod.

(3) *Every nonzero torsionless left R-module is a generator for* R-mod.

(4) *Every nonzero submodule of a projective left R-module is a generator for* R-mod.

(5) *Every nonzero left ideal of R is a generator for* R-mod.

(6) *Every nonzero ideal of R is a generator for* R-mod.

PROOF. For a module ${}_RQ$, one can verify that $t_Q=1$ if and only if ${}_RQ$ is a generator for R-mod.

(1) $\Rightarrow$ (2). If a module ${}_RQ$ is not a generator for R-mod, then the idempotent preradical $t_Q\neq 1$. Therefore $t_Q(R)=0$ and so $\mathrm{Hom}_R(Q, R)=0$.

(2) $\Rightarrow$ (3). If ${}_RQ$ is a nonzero torsionless module, then $\mathrm{Hom}_R(Q, R)\neq 0$ and so ${}_RQ$ is a generator for R-mod.

(3) $\Rightarrow$ (4). This is clear from the facts that every projective module is torsionless and every submodule of a torsionless module is torsionless.

(4) $\Rightarrow$ (5) $\Rightarrow$ (6). Clear.

(6) $\Rightarrow$ (1). Let σ be an idempotent preradical for R-mod. Assume $\sigma(R)\neq 0$. Since $\sigma(R)$ is a generator for R-mod, we have $t_{\sigma(R)}=1$. Recall that $t_{\sigma(R)}$ is a unique minimal one of those preradicals t such that $t(\sigma(R))=\sigma(R)$. Hence we obtain $t_{\sigma(R)}\leq\sigma$, and so $\sigma=1$.

DEFINITION 3.2. A ring which satisfies one of the equivalent conditions of Theorem 3.1 is called a *left G-ring*.

COROLLARY 3.3. *A property that a ring is a left G-ring is Morita invariant.*

PROOF. This is clear by (4) of Theorem 3.1 or (1) of Theorem 3.1 combined with Proposition 1.5.

In [11, Theorem 4.15], it is proved that, for a ring R, every nonzero (simple) left R-module is a generator for R-mod if and only if R is simple artinian.

COROLLARY 3.4. *R is a left G-ring with nonzero (left) socle if and only if R is simple artinian.*

PROOF. Let R be a left G-ring with socle $S\neq 0$. Then S generates ${}_RR$, and so R is completely reducible. But since R is prime by Theorem B, R is simple artinian.

In [13, Theorem 1.2], it is proved that, every nonzero left ideal of R is a progenerator for R-mod if and only if R is left hereditary left noetherian prime ring without non-trivial idempotent ideals.

COROLLARY 3.5. *If R is left hereditary, R is a left G-ring if and only if every nonzero projective left R-module is a generator for* R-mod.

PROOF. This is clear by (4) of Theorem 3.1.

COROLLARY 3.6. *Every left G-ring is a left SP-ring. The converse holds if R is left self-injective. Also every left G-ring is a left R-ring.*

PROOF. This is clear by Theorem 3.1 and Theorems A and B.

PROPOSITION 3.7. *If R is a left G-ring, then the maximal left ring of quotients $Q_{\max}$ of R is simple and left self-injective. In particular $Q_{\max}$ is also a left G-ring. If R is a left G-ring and the classical left ring of quotients Q_{cl} of R exists, then Q_{cl} is also a left G-ring.*

PROOF. The first part follows from the fact that $Q_{\max}$ is simple and left self-injective if R is a left SP-ring [6, Prop. 6.2].

Now let A be a nonzero left ideal of Q_{cl}. Since $A\cap R$ is a nonzero left ideal of R, $A\cap R$ generates ${}_RR$. Thus there exist R-homomorphisms $f_i: A\cap R\to R$ $(i=1, \cdots, m)$ such that $\sum_{i=1}^{m} f_i : \bigoplus_{i=1}^{m}(A\cap R)\to R$ is an R-epimorphism. For each i, f_i induces a Q_{cl}-homomorphism $g_i: Q_{cl}(A\cap R)\to Q_{cl}$ defined by $(\sum_{j=1}^{n} q_jx_j)g_i=\sum_{j=1}^{n} q_j(x_j)f_i$ where $q_j\in Q_{cl}$ and $x_j\in A\cap R$. Since $Q_{cl}(A\cap R)=A$, we have a Q_{cl}-epimorphism $\sum_{i=1}^{m} g_i : \bigoplus_{i=1}^{m} A\to \sum_{i=1}^{m} Q_{cl}\operatorname{Im}(f_i)=Q_{cl}(\sum_{i=1}^{m}\operatorname{Im}(f_i))=Q_{cl}R=Q_{cl}$. Hence A is a generator for Q_{cl}-mod.

EXAMPLE 3.8. Every simple ring is a left G-ring, but the converse is not true. For a counter example, we may take the ring Z of integers. The ring Z_n of $n\times n$ matrices over Z is a left and right G-ring by using Corollary 3.3, which is not simple.

EXAMPLE 3.9. Every left G-ring is a left R-ring, but the converse is not true. For a counter example, we may take the ring $R=Z/(p^n)$, where p is a prime and n is an integer greater than 1. To prove R is a (left) R-ring, we verify that R satisfies (2) of Theorem A. For any non-trivial ideal $I=(p^i)/(p^n)$ where $i=1, \cdots, n-1$, we define $j=0$ if $2i-n\leq 0$ and $j=2i-n$ if $2i-n>0$. Then the correspondence $p^i+(p^n)\mapsto(p^j+(p^n))+I$ is a nonzero R-homomorphism from I to R/I. Now remark that R has the nonzero socle $(p^{n-1})/(p^n)$ but R is not simple artinian. Hence R is not a (left) G-ring by Corollary 3.4. We remark, by this example, a factor ring of a left G-ring need not be a left G-ring.

4. Some generalizations of left G-rings.

LEMMA 4.1. *Let $I=\sum_{i=1}^{m} Ra_i$ be a finitely generated left ideal of R. Then I is a generator for R-mod if and only if there exist subsets $\{b_{ij}\mid i=1, \cdots, m;\ j=1, \cdots, n\}$ and $\{s_{ij}\mid i=1, \cdots, m;\ j=1, \cdots, n\}$ of R such that (1) $\sum_{i=1}^{m} r_ia_i=0$ implies $\sum_{i=1}^{m} r_ib_{ij}=0$ for $\{r_i\mid i=1, \cdots, m\}\subseteq R$ and $j=1, \cdots, n$ and (2) $\sum_{j=1}^{n}\sum_{i=1}^{m} s_{ij}b_{ij}=1_R$.*

PROOF. Put $K=\{(b_1, \cdots, b_m)\in R^m \mid \sum_{i=1}^{m} r_ia_i=0$ implies $\sum_{i=1}^{m} r_ib_i=0$ for $r_1, \cdots, r_m\in R\}$. Then the correspondence $f\mapsto((a_1)f, \cdots, (a_m)f)$ is a bijection between $\operatorname{Hom}_R(I, R)$ and K. Now ${}_RI$ is a generator for R-mod if and only if there exists a subset $\{f_j \mid j=1, \cdots, n\}$ of $\mathrm{Hom}_R(I, R)$ such that $\sum_{j=1}^{n}(I)f_j=\sum_{j=1}^{n}\sum_{i=1}^{m}R(a_i)f_j=R$. This is equivalent to the existence of a subset $\{(b_{1j}, \cdots, b_{mj}) \mid j=1, \cdots, n\}$ of K such that $\sum_{j=1}^{n}\sum_{i=1}^{m}Rb_{ij}=R$.

PROPOSITION 4.2. *The following properties are equivalent for a ring* R:

(1) *Every nonzero finitely generated left ideal of* R *is a generator for* R-mod.

(2) *Every finitely generated left* R*-module with nonzero dual is a generator for* R-mod.

(3) *Every nonzero finitely generated torsionless left* R*-module is a generator for* R-mod.

(4) *Every nonzero finitely generated submodule of a projective left* R*-module is a generator for* R-mod.

PROOF. (1) $\Rightarrow$ (2). Let M be a finitely generated left R-module with an $f(\neq 0)\in \mathrm{Hom}_R(M, R)$. Then the finitely generated left ideal $\mathrm{Im}(f)$ generates ${}_RR$, and so M generates ${}_RR$.

(2) $\Rightarrow$ (3) $\Rightarrow$ (4) $\Rightarrow$ (1). Clear.

DEFINITION 4.3. A ring which satisfies one of the equivalent conditions of Proposition 4.2 is called a *left FGG-ring*.

COROLLARY 4.4. *If* R *is left semihereditary,* R *is a left FGG-ring if and only if every nonzero finitely generated projective left* R*-module is a generator for* R-mod.

PROPOSITION 4.5. *The following properties are equivalent for a ring* R:

(1) *Every nonzero cyclic left ideal of* R *is a generator for* R-mod.

(2) $Ra^{lr}=R$ *for every nonzero* $a\in R$, *where* $a^{lr}=\mathrm{Ann}_R^r(\mathrm{Ann}_R^l(a))$.

(3) $RK=R$ *for every nonzero annihilator right ideal* K (i.e. $K=\mathrm{Ann}_R^r(X)$ *for some subset* X *of* R) *of* R.

(4) *Every cyclic left* R*-module with nonzero dual is a generator for* R-mod.

(5) *Every nonzero cyclic torsionless left* R*-module is a generator for* R-mod.

(6) *Every nonzero cyclic submodule of a projective left* R*-module is a generator for* R-mod.

PROOF. (1) $\Rightarrow$ (2). By using Lemma 4.1, for a nonzero cyclic left ideal $I=Ra$ of R, I is a generator for R-mod if and only if there exist subsets $\{b_1, \cdots, b_n\}$ of a^{lr} and $\{s_1, \cdots, s_n\}$ of R such that $\sum_{j=1}^{n}s_jb_j=1_R$, or equivalently, $Ra^{lr}=R$.

(2) $\Rightarrow$ (3). Let K be a nonzero annihilator right ideal of R. For a nonzero $a \in K$, $a^{lr} \subseteq K$ implies $RK = K$ by the assumption (2).

(3) $\Rightarrow$ (4). Let L be a left ideal of R such that $\mathrm{Hom}_R(R/L, R) \neq 0$. Then we have $L^r \neq 0$. For every $c \in L^r$, we define $\xi_c \in \mathrm{Hom}_R(R/L, R)$ as $\xi_c(x+L) = xc$. Now consider an R-homomorphism $\xi = \sum \xi_c : \bigoplus_{c \in L^r} {}_R(R/L) \to R$. Then $\mathrm{Im}(\xi) = RL^r = R$ by (3). Thus ${}_R(R/L)$ is a generator for R-mod.

(4) $\Rightarrow$ (5) $\Rightarrow$ (6) $\Rightarrow$ (1). Clear.

DEFINITION 4.6. A ring which satisfies one of the equivalent conditions of Proposition 4.5 is called a *left CG-ring*.

PROPOSITION 4.7. *The following properties are equivalent for a ring R:*

(1) *R is a left FGG-ring.*

(2) *For each positive integer n, the ring R_n of $n \times n$ matrices over R is a left CG-ring.*

PROOF. (1) $\Rightarrow$ (2). By Proposition 4.2, we see that a property "left FGG" is Morita invariant.

(2) $\Rightarrow$ (1). Let I be any nonzero finitely generated left ideal of R, say $I = Ra_1 + \cdots + Ra_n$. In R_n we put $\omega = (a_{ij})$ where $a_{i1} = a_i$ and all other entries are zero. By the assumption, for some k, we have an R_n-epimorphism $\bigoplus_{i=1}^{k} R_n\omega \to R_n$, which is in fact an R-epimorphism under the change of rings $R \to R_n$ (canonical map). Since $R_n\omega \cong \bigoplus_{j=1}^{n} I$, there exists an R-epimorphism $\bigoplus_{i=1}^{k} \bigoplus_{j=1}^{n} I \to R_n$. Combining this with an R-epimorphism ${}_R(R_n) \to {}_RR$ ($(c_{ij}) \mapsto c_{11}$), we obtain a desired R-epimorphism $\bigoplus_{i=1}^{k} \bigoplus_{j=1}^{n} I \to R$.

Now we consider another generalization of left G-rings. Recall that a semiprime ring is characterized as a ring whose essential left ideals are faithful. In [5], D. Handelman studied the structure of left strongly semiprime (SSP-)rings (i.e. rings whose essential left ideals are cofaithful). Among others, a ring R is left SSP if and only if R is a finite subdirect product of left SP-rings. So we shall consider a ring R whose essential left ideals are generators for R-mod.

PROPOSITION 4.8. *The following properties are equivalent for a ring R:*

(1) *Every essential left ideal of R is a generator for* R-mod.

(2) *Every ideal which is essential in R as a left ideal is a generator for* R-mod.

(3) *Every module $_RQ$ satisfying that $t_Q(R)$ is an essential left ideal is a generator for R-mod.*

PROOF. (1) $\Rightarrow$ (2). Clear.

(2) $\Rightarrow$ (3). Assume, for a module $_RQ$, $t_Q(R)$ is essential in R as a left ideal. Since Q generates $t_Q(R)$ and $t_Q(R)$ generates $_RR$, Q is a generator for R-mod.

(3) $\Rightarrow$ (1). Let K be an essential left ideal of R. Clearly we have $K \subseteq t_K(R) \subseteq R$, and so $t_K(R)$ is essential in R as a left ideal. By (3), K is a generator for R-mod.

DEFINITION 4.9. A ring which satisfies one of the equivalent conditions of Proposition 4.8 is called a *left EG-ring.*

PROPOSITION 4.10. (1) *Every ring direct summand of a left EG-ring is a left EG-ring.*

(2) *Every finite direct sum of left EG-rings is a left EG-rmg.*

PROOF. (1). Let $T=R\oplus S$ be a ring decomposition of a left EG-ring T. To prove R is a left EG-ring, let A be any essential left ideal of R. It is easy to verify that $_T(A\oplus S)$ is essential in T. Hence for some n, there exist a T-epimorphism $\bigoplus_{i=1}^{n}(A\oplus S)\rightarrow T$, which is also an R-epimorphism. The projection map $T\rightarrow R$ is also an R-epimorphism. Composing these epimorphisms, we have an R-epimorphism $\bigoplus_{i=1}^{n}(A\oplus S)\rightarrow T\rightarrow R$. Now remark that every R-homomorphism f: $A\oplus S\rightarrow R$ vanishes S, becanse $R((0, S)f)=(R(0, S))f=(0, 0)f=0$. Hence we have an R-epimorphism $\bigoplus_{i=1}^{n}A\rightarrow R$.

(2). Let $R=\bigoplus_{i=1}^{n}R_i$ be a direct sum of left EG-rings R_i $(i=1, \cdots, n)$. For each i, we regard the projection map $\pi_i: R\rightarrow R_i$ as R-homomorphism. Then for any essential left ideal I of R, $(I)\pi_i$ is an essential left ideal and also is an R-submodule of R_i. By the assumption, for some k_i, there exists an R_i-epimorphism $\bigoplus_{j=1}^{k_i}(I)\pi_i\rightarrow R_i$ which is also an R-epimorphism. Combining this with an R-epimorphism $I\rightarrow(I)\pi_i$, we have an R-epimorphism $\bigoplus_{j=1}^{k_i}I\rightarrow R_i$. Hence we have a desired R-epimorphism $\bigoplus_{i=1}^{n}\bigoplus_{j=1}^{k_i}I\rightarrow\bigoplus_{i=1}^{n}R_i=R$.

Note that an infinite direct product of left EG-rings need not be a left EG-ring. For example, let K be a field and $R=\prod_{i=1}^{\infty}K$. Then $I=\bigoplus_{i=1}^{\infty}K$ is an essential ideal of R, but is not cofaithful, and so I is not a generator for R-mod.

PROPOSITION 4.11. *If R is a left EG-ring and the classical left ring of quotients Q_{cl} of R exists, then Q_{cl} is also a left EG-ring.*

PROOF. Let A be an essential left ideal of Q_{cl}. It is easy to verify that $A\cap R$ is an essential left ideal of R. Hence $A\cap R$ is a generator for R-mod, and so A is a generator for Q_{cl}-mod by the same argument of the proof of Proposition 3.7.

EXAMPLE 4.12. Every left G-ring is a left FGG-ring. Every left FGG-ring is a left CG-ring, but the converse is not true. In fact, we shall give an example of a left CG-ring R having a finitely generated essential left ideal which is not a generator for R-mod. Let $R=K[x, y]$ be a polynomial ring over a field K. Since R is a domain, every nonzero cyclic (left) ideal of R is isomorphic to R. Thus R is a (left) CG-ring. Now let $I=(x, y)$ be an ideal generated by x and y. Remark that I is essential in R. We claim, for every R-homomorphism $f: I\rightarrow R$, there exists an $r^*\in R$ such that $(z)f=zr^*$ for all $z\in I$. Put $(x)f=r$ and $(y)f=s$. Then $(xy)f=xs=yr$, and so there exists an $r^*\in R$ such that $r=xr^*$ and $s=yr^*$. Thus for every $z=ux+vy\in I$ where u and v are elements of R, $(z)f=ur+vs=(ux+vy)r^*=zr^*$ as desired. Hence for every $f\in \mathrm{Hom}_R(I, R)$, we have $\mathrm{Im}(f)\subseteq I$, proving that I is not a generator for R-mod.

EXAMPLE 4.13. Every left CG-ring is a left SP-ring, but the converse is not true. Let $D=Z_2[x_1, x_2, x_3, \cdots]$ be the free non-commuting Z_2-algebra on x_i $(i=1, 2, 3, \cdots)$. Let I be the two-sided ideal in D generated by monomials of the form $x_ix_jx_k$ with $i<j<k$. As is shown in [6, p9], $R=D/I$ is a left SP-ring. Now we show that a cyclic left ideal $A=(Dx_3+I)/I$ of R is not a generator for R-mod. For every R-homomorphism $f: {}_RA\rightarrow {}_RR$, we put $(x_3+I)f=m+I$, where $m\in D$. Let $m=m_1+\cdots+m_p$ be a sum of (distint) monomials in D. Since $x_1x_2x_3\in I$, we have $x_1x_2m=x_1x_2(m_1+\cdots+m_p)\in I$, and so $x_1x_2m_i\in I$ for $i=1, \cdots, p$. We may assume that each monomial $m_i=x_{j_1}x_{j_2}\cdots x_{j_k}\notin I$. Hence j_1 must be greater than 2, and so $m\in \sum_{n=3}^{\infty} x_nD$. Therefore $\mathrm{Im}(f)\subseteq(\sum_{n=3}^{\infty} Dx_nD+I)/I\neq R$, proving that A is not a generator for R-mod.

EXAMPLE 4.14. Every left G-ring is a left EG-ring, but the converse is not true. In fact, $R=Z\oplus Z$ is a (left) EG-ring by Proposition 4.10, but R is not prime. One may expect that, if R is a left EG-ring, then every (essential submodule of a) projective left R-module is a generator for R-mod. But this is not true. Once again let $R=Z\oplus Z$, and put $I=(Z, 0)$ be an ideal of R. Clearly ${}_RI$ is projective, but an easy verification shows that $t_I(R)=I$, which means ${}_RI$ is not a generator for R-mod.

PROPOSITION 4.15. *A ring R is left G if and only if R is both left EG and left R.*

PROOF. ($\Rightarrow$): This was done in Example 4.14 and Corollary 3.6. ($\Leftarrow$): For every idempotent preradical σ for R-mod satisfying $\sigma(R)\neq 0$, we shall show that $\sigma=1$. Put $\bar{\sigma}$ be the smallest radical larger than σ ([16, p137]). Since R is a left R-ring, we have $\bar{\sigma}=1$. We claim that $\sigma(R)$ is essential in ${}_RR$. Let A be a left ideal such that $A\cap\sigma(R)=0$. Then $\sigma(A)\subseteq A\cap\sigma(R)=0$, and so $A\in\mathcal{F}_\sigma=\mathcal{F}_{\bar{\sigma}}=\{0\}$ because $\bar{\sigma}=1$. Now since R is a left EG-ring, $\sigma(R)$ generates ${}_RR$. Thus for some n and a module ${}_RN$, $\bigoplus_{i=1}^{n}\sigma(R)=R\oplus N$. Since σ is idempotent, we also have $\bigoplus_{i=1}^{n}\sigma(R)=\sigma(R)\oplus\sigma(N)$. Therefore $\sigma(R)=R$ which means $\sigma=1$ as desired.

5. Rings without non-trivial left strongly idempotent ideals (left E2-rings).

PROPOSITION 5.1. *The following properties are equivalent for a ring R:*

(1) *$\sigma(R)=0$ for every cotorsion radical $\sigma\neq 1$ for R-mod.*

(2) *There exist only two cotorsion radicals for R-mod.*

(3) *R has no non-trivial idempotent ideals.*

PROOF. Clear.

By using Proposition 1.4, we observe that the above property is Morita invariant.

EXAMPLE 5.2. Every left R-ring has no non-trivial idempotent ideals, but the converse is not true. For a counter example, consider $S=Z\times Q$, where Z is the ring of integers and Q the field of rational numbers. Define the addition on S by component wise and the multiplication on S by

$$(z_1,\ q_1)*(z_2,\ q_2)=(z_1z_2,\ z_1q_2+z_2q_1).$$

Then S becomes a commutative ring without non-trivial idempotent ideals, but as is shown in [2, Example 1.16] S is not an R-ring.

DEFINITION 5.3. We shall call that an ideal I of a ring R is *left strongly idempotent*, if $J=IJ$ holds for every left ideal $J\subseteq I$.

Clearly every left strongly idempotent ideal is idempotent, but the converse is not true. For a counter example, let R be the ring of 2×2 upper triangular matrices over a field K. One can check that $\begin{pmatrix} 0 & K \\ 0 & K \end{pmatrix}$ is idempotent but not left strongly idempotent. On the other hand, $\begin{pmatrix} K & K \\ 0 & 0 \end{pmatrix}$ is left strongly idempotent.

THEOREM 5.4. *The following properties are equivalent for a ring* R:

(1) $\sigma(R)=0$ *for every exact radical* $\sigma\neq 1$ *for* R-mod.

(2) *There exist only two exact radicals for* R-mod.

(3) *If a nonzero injective module* ${}_RE$ *satisfies the condition that, for a left ideal* K, $\mathrm{Hom}_R(R/K, E)=0$ *implies* $K+\mathrm{Ann}_R(E)=R$, *then* E *is faithful.*

(4) *There are no non-trivial ideals* I *such that* $IN=N\cap IM$ *for each* ${}_RN\subseteq {}_RM$.

(5) *There are no non-trivial (idempotent) ideals* I *such that* $(R/I)_R$ *are flat.*

(6) R *has no non-trivial left strongly idempotent ideals.*

PROOF. (1) $\Leftrightarrow$ (2). Clear.

(1) $\Leftrightarrow$ (3). For a left exact radical σ for R-mod, there exists an injective module ${}_RE$ such that $\mathcal{T}_\sigma=\{{}_RM \mid \mathrm{Hom}_R(M, E)=0\}$. In this case, σ is an exact radical if and only if, for a left ideal K, $\mathrm{Hom}_R(R/K, E)=0$ implies $K+\sigma(R)=R$ ([15, Prop. 2.1]). Thus the equivalence of (1) and (3) is clear by noticing that $\sigma(R)=\mathrm{Ann}_R(E)$.

(2) $\Leftrightarrow$ (4). Clear.

(2) $\Leftrightarrow$ (5). Let σ be a cotorsion radical for R-mod. It is well known (for example [12]) that σ is left exact if and only if $(R/\sigma(R))_R$ is flat. Thus we have an equivalence of (1) and (5).

(5) $\Leftrightarrow$ (6). For an ideal I, $(R/I)_R$ is flat if and only if I is a left strongly idempotent ideal ([10, Theorem 2]).

DEFINITION 5.5. A ring which satisfies one of the equivalent conditions of Theorem 5.4 is called a *left E2-ring.*

COROLLARY 5.6. *The property that a ring is a left E2-ring is Morita invariant.*

PROOF. This is clear by (2) of Theorem 5.4 combined with Proposition 1.4.

COROLLARY 5.7. *If* R *is a left weakly regular ring (i.e. a ring whose left ideals are idempotent), then the following conditions are equivalent:*

(1) R *is a simple ring.*

(2) R *is a left E2-ring.*

PROOF. (1) $\Rightarrow$ (2). Clear.

(2) $\Rightarrow$ (1). Let I be a proper ideal of R. Since every left ideal is idempotent, I is a left strongly idempotent ideal. Hence $I=0$ and so R is simple.

EXAMPLE 5.8. There is a right $E2$-ring which is not a left $E2$-ring. Let $D=F[x, y]$ be the free non-commuting algebra on $\{x, y\}$ over a field F. Then $DxD=\bigoplus_{i=0}^{\infty} y^i xD \cong \bigoplus_{i=0}^{\infty} D_D$. The ring $R=\mathrm{End}\,(DxD_D)$ is right SP (cf. [6, Example 13.2]) and so is right $E2$. But R contains a non-trivial left strongly idempotent ideal $K=\bigoplus_{i=0}^{\infty} e_i R$, where e_i denotes the matrix with 1 in the (i, i) position, 0 elsewhere.

EXAMPLE 5.9. If R is a left CTF-ring, then every nonzero flat *right* R-module is faithful ([6, Prop. 13.9]). If R has this property, then R is left $E2$ by (5) of Theorem 5.4. But the converse is not true. Let

$$R=\left\{\begin{pmatrix} a & 0 & 0 \\ b & c & 0 \\ d & e & a \end{pmatrix} \mid a, b, c, d, e \in K\right\},$$

where K is a field. One can check that there are only two non-trivial idempotent ideals

$$I_1=\left\{\begin{pmatrix} a & 0 & 0 \\ b & 0 & 0 \\ d & e & a \end{pmatrix} \mid a, b, d, e \in K\right\} \text{ and } I_2=\begin{pmatrix} 0 & 0 & 0 \\ K & K & 0 \\ K & K & 0 \end{pmatrix}.$$

Put $J_1=\begin{pmatrix} 0 & 0 & 0 \\ K & 0 & 0 \\ K & K & 0 \end{pmatrix} \subseteq I_1$ and $J_2=\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ K & 0 & 0 \end{pmatrix} \subseteq I_2$. Then $J_i \neq I_i J_i$ $(i=1, 2)$. Thus I_i $(i=1, 2)$ are not left strongly idempotent ideals. This gives an example of left $E2$-ring having non-trivial idempotent ideals. The same argument shows R is also a right $E2$-ring. Now put

$$A=\begin{pmatrix} 0 & 0 & 0 \\ K & K & 0 \\ 0 & 0 & 0 \end{pmatrix} \text{ and } B=\left\{\begin{pmatrix} a & 0 & 0 \\ 0 & 0 & 0 \\ d & e & a \end{pmatrix} \mid a, d, e \in K\right\}.$$

Then $R=A \oplus B$, and so A_R is flat. But since $\mathrm{Ann}_R^r(A)=\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ K & K & 0 \end{pmatrix} \neq 0$, A_R is not faithful. Finally, we remark that R is not semiprime.

EXAMPLE 5.10. We give an example of a prime ring which is not left $E2$. Let V_D be an infinite dimensional vector space over a division ring D. Put $R=$

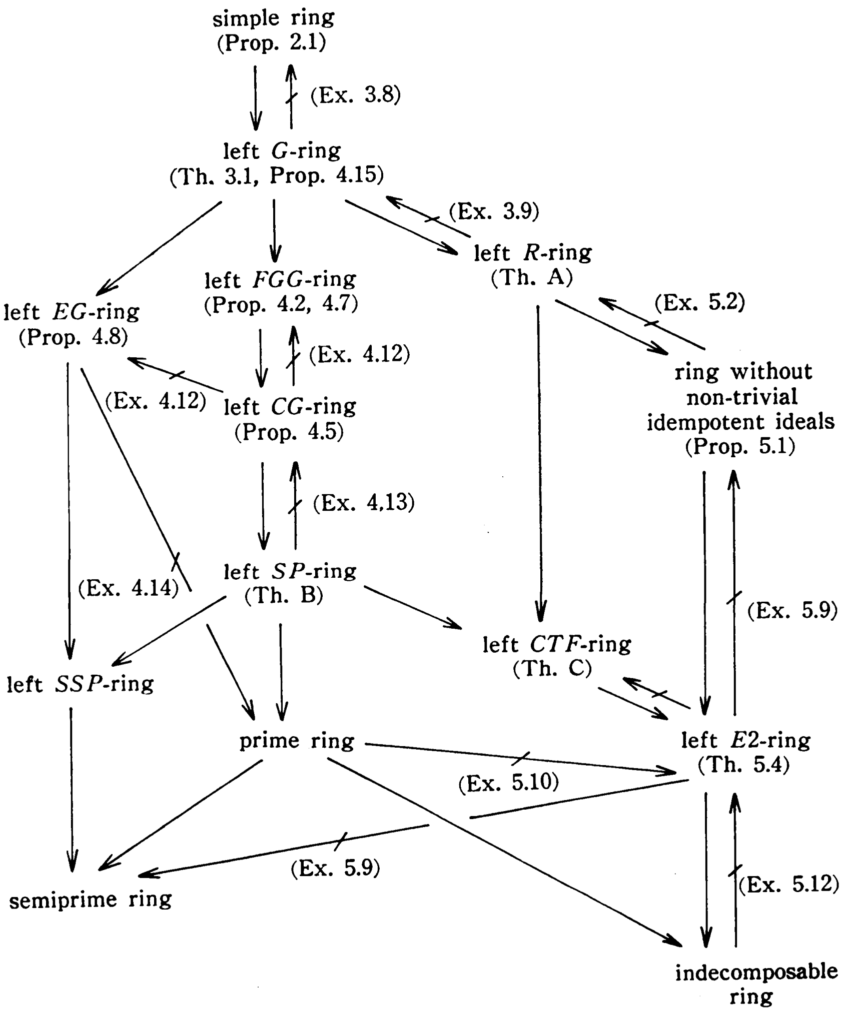
End(V_D). Then R is a regular and prime ring. Put $I=\mathrm{soc}(R)$, then I consists of $f\in R$ such that $\mathrm{Im}(f)_D$ is finite dimensional. Thus I is a non-trivial (left) strongly idempotent ideal. One may remark that $_RI$ is not cofaithful, and so R is not a left SSP-ring.

PROPOSITION 5.11. *If R is a left $E2$-ring, then no non-trivial ideals of R are direct summand as a right ideal.*

PROOF. Let I be a proper ideal and K a right ideal such that $R=I\oplus K$. For every left ideal $J\subseteq I$, we have $KJ\subseteq KI\subseteq I\cap K=0$, and so $IJ=(I\oplus K)J=RJ=J$. By the assumption, $I=0$ as desired.

EXAMPLE 5.12. By Proposition 5.11, every left $E2$-ring is indecomposable as a ring, but the converse is not true. For a counter example, we may take the ring of 2×2 upper triangular matrices over a field.

A table of rings



Acknowledgement

The author expresses his heartly thanks to Professor Y. Kurata for his various suggestions.

References

[1] Beachy, J.A. and Blair, W.D., Rings whose faithful left ideals are cofaithful. Pacific J. Math. **58** (1975), 1-13.
[2] Bican, L., Jambor, P., Kepka, T. and Němec, P., On rings with trivial torsion parts. Bull. Austral. Math. Soc. **9** (1973), 275-290.
[3] Dickson, S.E., A torsion theory for abelian categories. Trans. Amer. Math. Soc. **121** (1966), 223-235.
[4] Golan, J.S., Localization of noncommutative rings. Marcel Dekker, Inc., New York (1975).
[5] Handelman, D., Strongly semiprime rings. Pacific J. Math. **60** (1975), 115-122.
[6] Handelman, D., Goodearl, K.R. and Lawrence, J., Strongly prime and completely torsionfree rings. Carleton Mathematical Series **109**, Carleton University (1974).
[7] Izawa, T., A remark on idempotent ideals of Morita equivalent rings. Rep. Fac. Sci., Shizuoka Univ. **13** (1979), 15-16.
[8] Katayama, H., On the cotorsion radicals. J. Fac. Liberal Arts, Yamaguchi Univ. **8** (1974), 239-243.
[9] Kurata, Y. and Katayama, H., On a generalization of $QF-3'$ rings. Osaka J. Math. **13** (1976), 407-418.
[10] Kiełpiński, R., Direct sums of torsions. Bull. Acad. Polon. Sci. **19** (1971), 281-286.
[11] Mbuntum, F.F. and Varadarajan, K., Half-exact pre-radicals. Comm. Alg. **5** (1977), 555-590.
[12] Ramamurthi, V.S., On splitting cotorsion radicals. Proc. Amer. Math. Soc. **39** (1973), 457-461.
[13] Robson, J.C., A note on Dedekind prime rings. Bull. London Math. Soc. **3** (1971), 42-46.
[14] Rubin, R.A., Absolutely torsion-free rings. Pacific J. Math. **46** (1973), 503-514.
[15] Rubin, R.A., Semi-simplicity relative to kernel functors. Canad. J. Math. **26** (1974), 1405-1411.
[16] Stenström, B., Rings of quotients. Die Grundlehren der mathematischen Wissenschaften in Einzeldarstellungen **217**, Springer-Verlag, Berlin Heidelberg New York (1975).
[17] Viola-Prioli, J., On absolutely torsion-free rings. Pacific J. Math. **56** (1975), 275-283.

Department of Mathematics
Yamaguchi University
Yamaguchi, 753 Japan