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ON SOME OSCILLATING PROPERTIES OF THE SOLUTIONS OF A CLASS OF FUNCTIONAL —DIFFERENTIAL EQUATIONS

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A great number of papers is dedicated to the study of the oscillating properties of the solutions of functional-differential equations with retarded argument. The case when the transformed argument depends on the unknown function is of special interest since such a dependence is natural for many real systems [1], [2].

The oscillating properties of the solutions of a class of differential equations of retarded type with a delay depending on the unknoun function are studied in the present paper.

Consider the following equation of n-th ordes $(n>1)$

$$y^{(n)}(t)+(-1)^{n+1}\sum_{i=1}^{p}q_{i}(t)y(\Delta_{i}(t, y(t)))=0, \qquad (1)$$

where $q_{i}(t):[t_{o}, \infty)\rightarrow \boldsymbol{R}_{+}$, $\Delta_{i}(t, u):[t_{o}, \infty)\times \boldsymbol{R}\rightarrow \boldsymbol{R}$ for $i=\overline{1, p}$; $t_{o}\geqq 0$. $\boldsymbol{R}_{+}[0, +\infty)$ and $\boldsymbol{R}=(-\infty, +\infty)$.

Assume that for $i=\overline{1, p}$ the following conditions (A) hold.

A1. The function $\Delta_{i}(t, u)$ is continuous with respect to the set of arguments;

A2. $\Delta_{i}(t, u)<t$ for every fixed u, $|u|\leqq D$, where D is a positive constant;

A3. $\lim_{t\rightarrow\infty}\Delta_{i}(t, u)=+\infty$ for every fixed u, $|u|\leqq D$;

A4. $\Delta_{i}(t, u_{1})\leqq\Delta_{i}(t, u_{2})$ for $u_{1}\leqq u_{2}<0$, $\Delta_{i}(t, u_{1})\geqq\Delta_{i}(t, u_{2})$ for $0<u_{1}\leqq u_{2}$ and $\Delta_{i}(t_{1}, u)\leqq\Delta_{i}(t_{2}, u)$ for $t_{o}\leqq t_{1}\leqq t_{2}$;

A5. The functions $q_{i}(t)\in \boldsymbol{C}[[t_{o}, \infty), \boldsymbol{R}_{+}]$ and there exists an index $i_{o}\in\{1, 2, \cdots, p\}$ such that $q_{i_{o}}(t)>0$ for $t\geqq t_{o}$.

DEFINITION 1 ([3], p. 53). A real valued function $\Psi(t)$ will be considered a solution of the equation (1) if it is defined on the semiaxis $t\geqq t_{o}$, has n derivatives on it and satisfies the following conditions:

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$$\Psi^{(n)}(t)+(-1)^{n+1}\sum_{i=1}^{p} q_i(t)\Psi(\Delta_i(t,\ \Psi(t)))=0\,,$$

$$\Psi^{(k)}(t)=\Phi_k(t)\in C[E_{t_0},\ \boldsymbol{R}]\,,\quad k=\overline{0,\ n-1}\,,$$

$$\Phi_k(t_0)=\Psi^{(k)}(t_0+0)\,,$$

where $\Phi_k(t_0)$ are functions defined on the initial set E_{t_0}

$$E_{t_0}=\{t\geqq t_0:\Delta_i(t,\ u)\leqq t_0 \text{ for } |u|\leqq D,\ i=\overline{1,\ p}\}\cup\{t_0\}\,.$$

Consider the following set of solutions of the equation (1):

$$S=\{y(t):y(t)\not\equiv 0 \text{ for } t\geqq T\geqq t_0,\ T<\infty\}\,.$$

We shall assume that the functions $q_i(t)$ and $\Delta_i(t,\ u)$ are such that there exist solutions of the equation (1) belonging to S.

DEFINITION 2 ([3], p. 45). We shall say that the solution of the equation (1) belonging to the set S is oscillating with respect to the solution $y(t)\equiv 0$ if there exists an infinite set of values $t=t_i$ such that $y(t_i)=0$ and besides $\lim_{i\to\infty} t_i=+\infty$.

DEFINITION 3 ([3], p. 45). We shall say that the solution $y(t)$ of the equation (1) belonging to the set S is non-oscillating with respect to $y(t)\equiv 0$ if there exists $\bar{t}\geqq t_0$ such that $y(t)\neq 0$ for $t\geqq\bar{t}$.

THEOREM. *Assume that*

1. *The conditions* (*A*) *hold.*

2. $$\limsup_{t\to+\infty}\sum_{i=1}^{p}\int_{\Delta^*(t,u)}^{t}[\Delta_i(t,\ u)-\Delta_i(s,\ u)]^{n-1}q_i(s)ds>(n-1)! \tag{2}$$

for every fixed u, $|u|\leqq D$, *where* $\Delta^*(t,\ u)=\max_{1\leq i\leq p}\Delta_i(t,\ u)$.

Then every bounded solution $|y(t)|\leqq D$ *of the equation* (1) *is an oscillating one with respect to the solution* $y(t)\equiv 0$.

In order to prove this theorem we shall need the following

LEMMA ([4]). *Let* $u(t)$ *be a continuous positive function on the segment* $[t_0,\ \infty)$ *with absolutely continuous derivatives up to* $n-1$*-th order preserving their sign on this interval. If for almost every* $t\in[t_0,\ \infty)$

$$u^{(n)}(t)\leqq 0 \quad (\text{resp. } \geqq 0)\,,$$

then there exist numbers $\bar{t}\in[t_0,\ \infty)$ *and* $l\in\{0,\ 1,\ \cdots,\ n\}$ *such that* $l+n$ *is odd* (*resp. even*) *and the inequalities*

$$u^{(i)}(t)\geqq 0, \quad i=\overline{0, l-1} \tag{3}$$

$$(1)^{i+l}u^{(i)}(t)\geqq 0, \quad i=\overline{l, n} \tag{4}$$

hold for $t\geqq\bar{t}$.

PROOF OF THEOREM. Assume the contrary, i. e. that there exists a bounded non-oscillating solution $y(t)$ of the equation (1). We could assume without loss of generality that $y(t)>0$ for $t\geqq t_1$, where $t_0\leqq t_1<\infty$ since the case $y(t)<0$ could be reduced to the considered one by means of the substituting $y=-w$.

It follows from the conditions A1 and A3 that there exists a number $t_2\in[t_1, \infty)$ such that

$$\Delta_i(t, y(t))\geqq t_1, \quad t\geqq t_2, \quad i=\overline{1, p}. \tag{5}$$

Rewrite the equation (1) in the following form:

$$(-1)^n y^{(n)}(t)=\sum_{i=1}^{p} q_i(t)y(\Delta_i(t, y(t))). \tag{6}$$

Taking into account the conditions A5, the inequality (5) and the positivity of the solution $y(t)$ on the interval $[t_1, \infty)$, we could conclude that the equation (6) implies the inequality

$$(-1)^n y^{(n)}(t)>0 \quad (t\geqq t_2). \tag{7}$$

From this inequality it follows immediately that the derivatives $y^{(i)}(t)$ $(i=\overline{0, n-1})$ preserve their sign for $t\geqq t_3$, where $t_3\in[t_2, \infty)$.

Let $n=2\lambda$. Then $y^{(n)}(t)>0$ and according to Lemma there exist numbers $t_4\geqq t_3$ and $l\in\{0, 2, \cdots, 2\mu, \cdots, n\}$ such that $l+n$ is even and that the inequalities (3) and (4) hold for $t\geqq t_4$.

First assume that $l\geqq 2$. Then $y''(t)>0$ for $t\geqq t_4$, i. e. the function $y'(t)$ is a monotone non-decreasing one and its values are positive only. This contradicts the assumption that the function $y(t)$ is bounded. The contradiction obtained shows that $l\overline{\in}\{2, 4, \cdots, n\}$, i. e. $l=0$. Therefore, the inequality

$$(-1)^i y^{(i)}(t)>0 \tag{8}$$

holds for $t\geqq t_4$ and $i=\overline{0, n}$.

Applying the Taylor formula to some arbitrary u, v $(u\leqq v)$ and taking into account the inequality (8), we obtain

$$y(u)-y(v)=\frac{u-v}{1!}y'(v)+\frac{(u-v)^2}{2!}y''(v)+\cdots+\frac{(u-v)^{n-1}}{(n-1)!}y^{(n-1)}(\xi)$$

$$=\frac{v-u}{1!}(-1)^1y'(v)+\frac{(v-u)^2}{2!}(-1)^2y''(v)+\cdots$$

$$+\frac{(v-u)^{n-1}}{(n-1)!}(-1)^{n-1}y^{(n-1)}(\xi)>\frac{(-1)^{n-1}y^{(n-1)}(\xi)}{(n-1)!}(v-u)^{n-1}$$

$$>\frac{(-1)^{n-1}y^{(n-1)}(v)}{(n-1)!}(v-u)^{n-1},$$

where $\xi\in(u, v)$.

It follows from the conditions A3 and A4 that there exists $t_5\geqq t_4$ such that

$$y(\Delta_i(s, y(s)))-y(\Delta_i(t, y(t)))$$
$$=\frac{(-1)^{n-1}y^{(n-1)}(\Delta_i(t, y(t)))}{(n-1)!}[\Delta_i(t, y(t))-\Delta_i(s, y(s))]^{n-1} \tag{9}$$

for any $t\geqq s\geqq t_5$.

Multiplying (9) by $q_i(s)$ and summing up with respect to i from 1 to p, taking into account the condition A5, we obtain

$$\sum_{i=1}^{p} y(\Delta_i(s, y(s)))q_i(s)-\sum_{i=1}^{p} y(\Delta_i(t, y(t)))q_i(s)$$
$$>\frac{(-1)^{n-1}}{(n-1)!}\sum_{i=1}^{p} y^{(n-1)}(\Delta_i(t, y(t)))[\Delta_i(t, y(t))-\Delta_i(s, y(s))]^{n-1}q_i(s). \tag{10}$$

Then in virtue of (6) and (10) we have

$$(-1)^n y^{(n)}(s)>\sum_{i=1}^{p} y(\Delta_i(t, y(t)))q_i(s)$$
$$+\frac{(-1)^{n-1}}{(n-1)!}\sum_{i=1}^{p} y^{(n-1)}(\Delta_i(t, y(t)))[\Delta_i(t, y(t))-\Delta_i(s, y(s))]^{n-1}q_i(s). \tag{11}$$

Integrating (11) with respect to s from $\Delta^*(t, y(t))$ to t, we obtain

$$(-1)^n y^{(n-1)}(t)-(-1)^n y^{(n-1)}(\Delta^*(t, y(t)))$$
$$>\sum_{i=1}^{p} y(\Delta_i(t, y(t)))\int_{\Delta^*(t,y(t))}^{t} q_i(s)ds \tag{12}$$
$$-\frac{1}{(n-1)!}\sum_{i=1}^{p} y^{(n-1)}(\Delta_i(t, y(t)))\int_{\Delta^*(t,y(t))}^{t}[\Delta_i(t, y(t))-\Delta_i(s, y(s))]^{n-1}q_i(s)ds.$$

The inequality (7) implies that the function $y^{(n-1)}(t)$ is a non-decreasing one. Then taking into account the definition of $\Delta^*(t, y(t))$ and (7), we shall obtain the estimate

$$y^{(n-1)}(\Delta_i(t, y(t)))\leqq y^{(n-1)}(\Delta^*(t, y(t))) \quad (i=\overline{1, p})$$

and therefore

$$(-1)^n y^{(n-1)}(t)$$
$$>\frac{y^{(n-1)}(\Delta^*(t, y(t)))}{(n-1)!}\left[(n-1)!-\sum_{i=1}^{p}\int_{\Delta^*(t,y(t))}^{t}[\Delta_i(t, y(t))-\Delta_i(s, y(s))]^{n-1}q_i(s)ds\right]$$

$$+\sum_{i=1}^{p} y(\Delta_i(t,\ y(t)))\int_{\Delta^*(t,\ y(t))}^{t} q_i(s)ds \tag{13}$$

$$>\frac{y^{(n-1)}(\Delta^*(t,\ y(t)))}{(n-1)!}\left[(n-1)!-\sum_{i=1}^{p}\int_{\Delta^*(t,\ y(t))}^{t}[\Delta_i(t,\ y(t))-\Delta_i(s,\ y(s))]^{n-1}q_i(s)ds\right],$$

since $y(\Delta_i(t,\ y(t)))>0$, the condition A5 holds and $\Delta^*(t,\ y(t))<t$.

Taking into consideration the condition A4 and the assumption on the positivity of the solution $y(t)$ for the expression $[\Delta_i(t,\ y(t))-\Delta_i(s,\ y(s))]^{n-1}$, we obtain the following estimate:

$$\begin{aligned}&[\Delta_i(t,\ y(t))-\Delta_i(s,\ y(s))]^{n-1}\\ &=\{[\Delta_i(t,\ y(t))-\Delta_i(s,\ y(t))]+[\Delta_i(s,\ y(t))-\Delta_i(s,\ y(s))]\}^{n-1}\\ &\geqq[\Delta_i(t,\ y(t))-\Delta_i(s,\ y(t))]^{n-1}+[\Delta_i(s,\ y(t))-\Delta_i(s,\ y(s))]^{n-1}.\end{aligned} \tag{14}$$

In virtue of the condition (2) we obtain from (14)

$$\begin{aligned}&\sum_{i=1}^{p}\int_{\Delta^*(t,\ y(t))}^{t}[\Delta_i(t,\ y(t))-\Delta_i(s,\ y(s))]^{n-1}q_i(s)ds\\ &\geqq\sum_{i=1}^{p}\int_{\Delta^*(t,\ y(t))}^{t}[\Delta_i(t,\ y(t))-\Delta_i(s,\ y(t))]^{n-1}q_i(s)ds\\ &\quad+\sum_{i=1}^{p}\int_{\Delta^*(t,\ y(t))}^{t}[\Delta_i(s,\ y(t))-\Delta_i(s,\ y(s))]^{n-1}q_i(s)ds\\ &>\sum_{i=1}^{p}\int_{\Delta^*(t,\ y(t))}^{t}[\Delta_i(s,\ y(t))-\Delta_i(s,\ y(s))]^{n-1}q_i(s)ds)+(n-1)!\end{aligned} \tag{15}$$

for $t\geqq t_6\geqq t_5$.

Since $y(t)$ is a non-decreasing function and the condition A4 is satisfied, the last inequality could be rewritten as follows:

$$\sum_{i=1}^{p}\int_{\Delta^*(t,\ y(t))}^{t}[\Delta_i(t,\ y(t))-\Delta_i(s,\ y(s))]^{n-1}q_i(s)ds>(n-1)!\quad(t\geqq t_6). \tag{16}$$

Hence

$$(n-1)!-\sum_{i=1}^{p}\int_{\Delta^*(t,\ y(t))}^{t}[\Delta_i(t,\ y(t))-\Delta_i(s,\ y(s))]^{n-1}q_i(s)ds<0\quad(t\geqq t_6). \tag{17}$$

From (8) we obtain

$$y^{(n-1)}(\Delta^*(t,\ y(t)))<0\quad(t\geqq t_6) \tag{18}$$

for $i=n-1$.

Taking into account (17) and (18), we could rewrite the inequality (13) in the form:

$$(-1)^n y^{(n-1)}(t)>0$$

and this contradicts the inequality (8) for $i=n-1$.

Therefore, every bounded solution of the equation (1) is an oscillating one.

Let $n=2\lambda+1$. The conditions of Lemma are satisfied in this case as well as the case $n=2\lambda$. Therefore, there exist numbers $t_4 \geqq t_3$ and $l \in \{1, 2, \cdots, n\}$ such that $l+n$ is even and that the inequalities (3) and (4) hold for $t \geqq t_4$. It is clear that $l=2\mu+1 \geqq 1$. If $l>1$, it follows from (3) that the bounded function $y(t)$ should diverge to infinity, which is impossible. Thus, $l=1$ and the condition (4) could be rewritten as follows

$$(-1)^{i+1} y^{(i)}(t)>0 \quad (t \geqq t_4 ; i=\overline{1, n}) \tag{19}$$

Applying the Taylor formula and taking into account (19), we obtain

$$y(v)-y(u) \geqq \frac{(v-u)^{n-1}}{(n-1)!}(-1)^n y^{(n-1)}(v). \tag{20}$$

Arguing as in the proof of the theorem for $n=2\lambda$, we get to a contradiction with the inequality (19). Thus, the theorem has been proved.

Point out that the behaviour of the solutions of the equation (1) in the case the delay depends only on the argument has been considered in [5]-[7].

Next we shall give some examples.

EXAMPLE 1. Consider the second order equation

$$y''(t)-M(1+\cos t) y\left(t-\frac{\pi}{2}\right)-N y(t-\pi-y^2(t))=0, \tag{21}$$

where M and N are positive constants and satisfy the condition

$$\left(\frac{\pi^2}{8}+\frac{\pi}{2}+2\right) M+\frac{\pi^2}{8} N>1. \tag{22}$$

The functions $\Delta_1(t, u)=t-\frac{\pi}{2}$, $\Delta_2(t, u)=t-\pi-u^2$ are continuous on the set $[t_0, \infty) \times[-D, D]$, where $t_0>\pi+D$ and $D=const>0$. They satisfy the conditions A2—A4. The functions $q_1(t)=M(1+\cos t) \geqq 0$ and $q_2(t)=N>0$ are defined and continuous on the interval $[t_0, \infty)$. Hence they satisfy the condition A5. The condition (2) holds. Actually, by (22)

$$\limsup_{t \to \infty}\left\{\int_{\Delta^*(t, u)}^t \sum_{i=1}^2[\Delta_i(t, u)-\Delta_i(s, u)] q_i(s) d s\right\}$$

$$=\limsup_{t_+\infty}\left\{(M+N) \int_{t-\pi / 2}^t(t-s) d s+M \int_{t-\pi / 2}^t(t-s) \cos s d s\right\}$$

$$=\frac{(M+N) \pi^2}{8}+\limsup_{t \to \infty}\left\{M[\sin t-\cos t]+\frac{M \pi}{2} \cos t\right\} \geqq \frac{M+N}{8} \pi^2$$

$$+2M+\frac{M\pi}{2}>1.$$

Therefore, in virtue of Theorem, every bounded solution of the equation (21) is oscillating.

EXAMPLE 2. Consider now the third order equation

$$y'''(t)+y(t-2-\sqrt[3]{6}\,y^2(t))=0. \tag{23}$$

The conditions (A) and (2) hold for it and, therefore, all the bounded solutions of this equation are oscillating.

In fact, the functions $q(t)=1$, $\Delta(t, u)=t-2-\sqrt[3]{6}\,u^2$ are continuous and take positive values for $|u|\leq D$ and $t\geqq t_0=2+\sqrt[3]{6}\,D^2$, where $D=const>0$. It could be easily verified that the conditions A2-A5 are satisfied. Let us verify the condition (2):

$$\limsup_{t\to\infty}\int_{\Delta^*(t,u)}^{t}[\Delta(t, u)-\Delta(s, u)]^2 q(s)ds$$

$$=\limsup_{t\to\infty}\int_{t-2-\sqrt[3]{6}u^2}^{t}(t-s)^2 ds$$

$$=\limsup_{t\to\infty}\frac{(2+\sqrt[3]{6}\,u^2)^3}{3}=\frac{8}{3}+4u^2\sqrt[3]{6}+2u^4\sqrt[3]{36}+2u^6>\frac{8}{3}>2.$$

Therefore, according to Theorem, every bounded solution of the equation (23) is oscillating.

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